Seminar

"Theory of fundamental interactions"

Relativistic quantum mechanics of twisted Dirac particles in uniform magnetic and electric fields

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Twisted wave in vacuum:

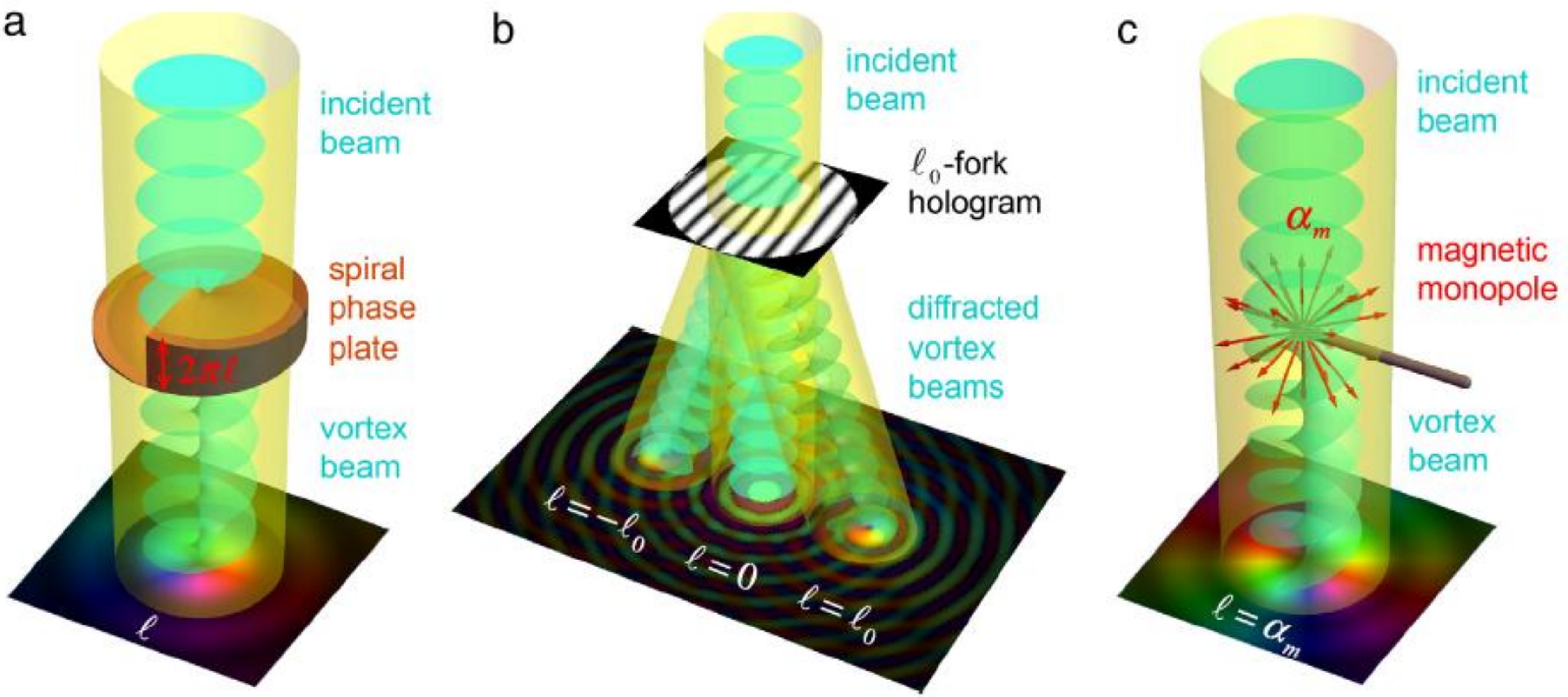
$$w(z) = w_0\sqrt{1+\frac{z^2}{z_R^2}}, \quad R(z) = z + \frac{z_R^2}{z}, \quad z_R = \frac{kw_0^2}{2},$$

$$\Phi_G(z) = N \arctan\left(\frac{z}{z_R}\right), \quad N = 2n + |\ell| + 1,$$

the real functions $\mathbb{A}$ and Φ define the amplitude and phase, k is the beam wavenumber, w_0 is the beam waist (minimum beam width), $R(z)$ is the radius of curvature of the wavefront, $\Phi_G(z)$ is the Gouy phase, z_R is the Rayleigh diffraction length, $L_n^{|\ell|}$ is the generalized Laguerre polynomial, and $n = 0, 1, 2, \ldots$ is the radial quantum number. The spin function η is an eigenfunction of the Pauli operator σ_z: $\sigma_z\eta^{\pm} = \pm\eta^{\pm}$, $\eta^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\eta^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

New quantum-mechanical results for a Dirac particle in a uniform magnetic field

K.Y. Bliokh et al. / Physics Reports 690 (2017) 1–70



In a uniform magnetic field:

$$\Psi = \mathcal{A}\exp(i\ell\phi)\exp(ip_z z), \qquad \int \Psi^\dagger \Psi \, r\mathrm{d}r\mathrm{d}\phi = 1,$$

$$\mathcal{A} = \frac{C_{n\ell}}{w_m}\left(\frac{\sqrt{2}r}{w_m}\right)^{|\ell|} L_n^{|\ell|}\left(\frac{2r^2}{w_m^2}\right)\exp\left(-\frac{r^2}{w_m^2}\right)\eta,$$

$$C_{n\ell} = \sqrt{\frac{2n!}{\pi(n+|\ell|)!}}, \qquad w_m = \frac{2}{\sqrt{|e|B}}.$$

The Landau beam is defined by

$$\psi = \mathcal{A}\exp(i\ell\phi)\exp(ip_z z), \qquad \int \psi^\dagger \psi\, r\, dr\, d\phi = 1,$$

$$\mathcal{A} = \frac{C_{n\ell}}{w_m}\left(\frac{\sqrt{2}r}{w_m}\right)^{|\ell|} L_n^{|\ell|}\left(\frac{2r^2}{w_m^2}\right)\exp\left(-\frac{r^2}{w_m^2}\right)\eta,$$

$$C_{n\ell} = \sqrt{\frac{2n!}{\pi(n+|\ell|)!}}, \qquad w_m = \frac{2}{\sqrt{|e|B}}$$

Here p_z=const. We have shown that there are also solutions of the Dirac equations describing spatially oscillating Laguerre-Gauss beams with $p_z \neq$ const.

L. Zou, P. Zhang, and A. J. Silenko, General quantum-mechanical solution for twisted electrons in a uniform magnetic field, Phys. Rev. A 103, L010201 (2021).

In vacuum, one uses the paraxial wave equation:

$$\left(\nabla_\perp^2 + 2ik\frac{\partial}{\partial z}\right)\Psi = 0, \qquad \nabla_\perp^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}.$$

Its solution is the paraxial wave function of the Laguerre-Gauss beam:

$$\Psi = \mathbb{A}\,\exp(i\Phi), \qquad \int \Psi^\dagger \Psi\, r\, dr\, d\phi = 1,$$

$$\mathbb{A} = \frac{C_{n\ell}}{w(z)}\left(\frac{\sqrt{2}r}{w(z)}\right)^{|\ell|} L_n^{|\ell|}\left(\frac{2r^2}{w^2(z)}\right)\exp\left(-\frac{r^2}{w^2(z)}\right)\eta,$$

$$\Phi = \ell\phi + \frac{kr^2}{2R(z)} - \Phi_G(z),$$

$$w(z) = w_0\sqrt{1+\frac{z^2}{z_R^2}}, \qquad R(z) = z + \frac{z_R^2}{z}, \qquad z_R = \frac{kw_0^2}{2},$$

$$\Phi_G(z) = N \arctan\left(\frac{z}{z_R}\right), \qquad N = 2n + |\ell| + 1,$$

$$\sigma_z \eta^{\pm} = \pm \eta^{\pm}, \qquad \eta^{+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \eta^{-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The paraxial wave equation in the uniform magnetic field reads

$$\left(\nabla_{\perp}^{2} - ieB\frac{\partial}{\partial\phi} - \frac{e^2B^2r^2}{4} + 2es_zB + 2ik\frac{\partial}{\partial z}\right)\Psi = 0,$$

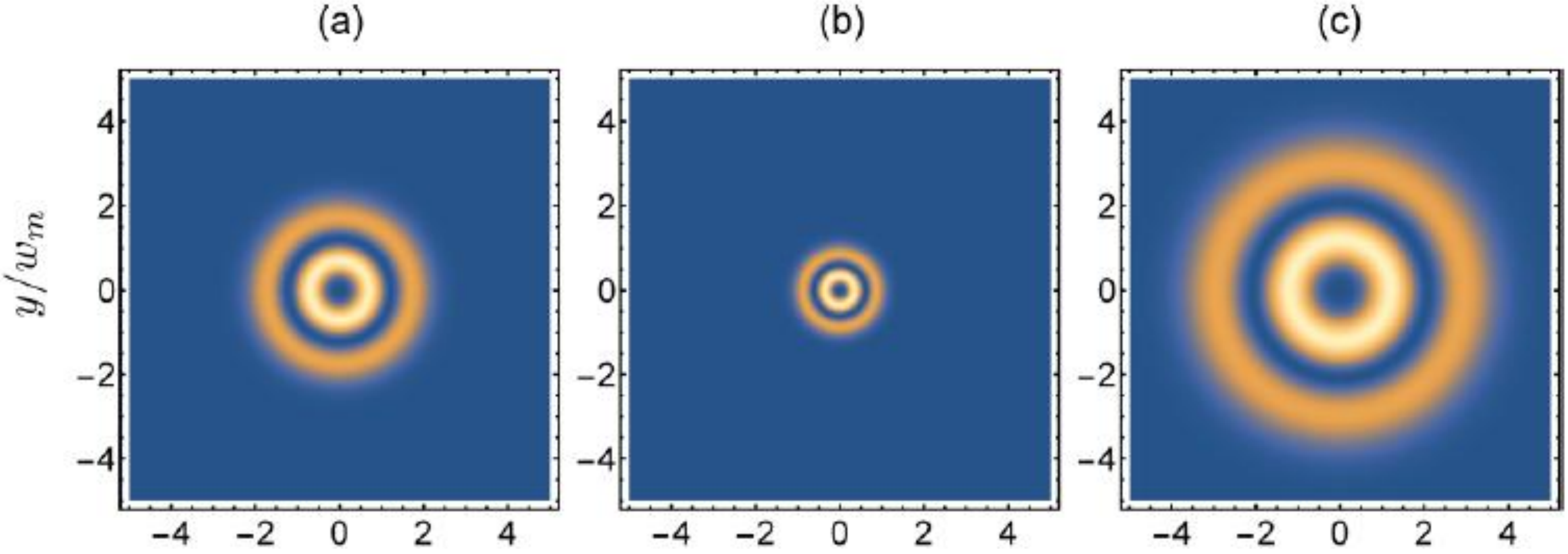
where s_z is the spin projection onto the field direction. The solution obtained is given by

$$w(z) = w_0\sqrt{\frac{1}{2}\left[1 + \frac{w_m^4}{w_0^4} - \left(\frac{w_m^4}{w_0^4} - 1\right)\cos\frac{2z}{z_m}\right]}$$

$$= w_0\sqrt{\cos^2\frac{z}{z_m} + \frac{w_m^4}{w_0^4}\sin^2\frac{z}{z_m}}, \qquad z_m = \frac{kw_m^2}{2},$$

$$R(z) = k w_m^2 \frac{\cos^2 \frac{z}{z_m} + \frac{w_m^4}{w_0^4} \sin^2 \frac{z}{z_m}}{\left(\frac{w_m^4}{w_0^4} - 1\right) \sin \frac{2z}{z_m}},$$

$$\Phi_G(z) = N \arctan\left(\frac{w_m^2}{w_0^2} \tan \frac{z}{z_m}\right) + \frac{(\ell + 2s_z)z}{z_m}.$$

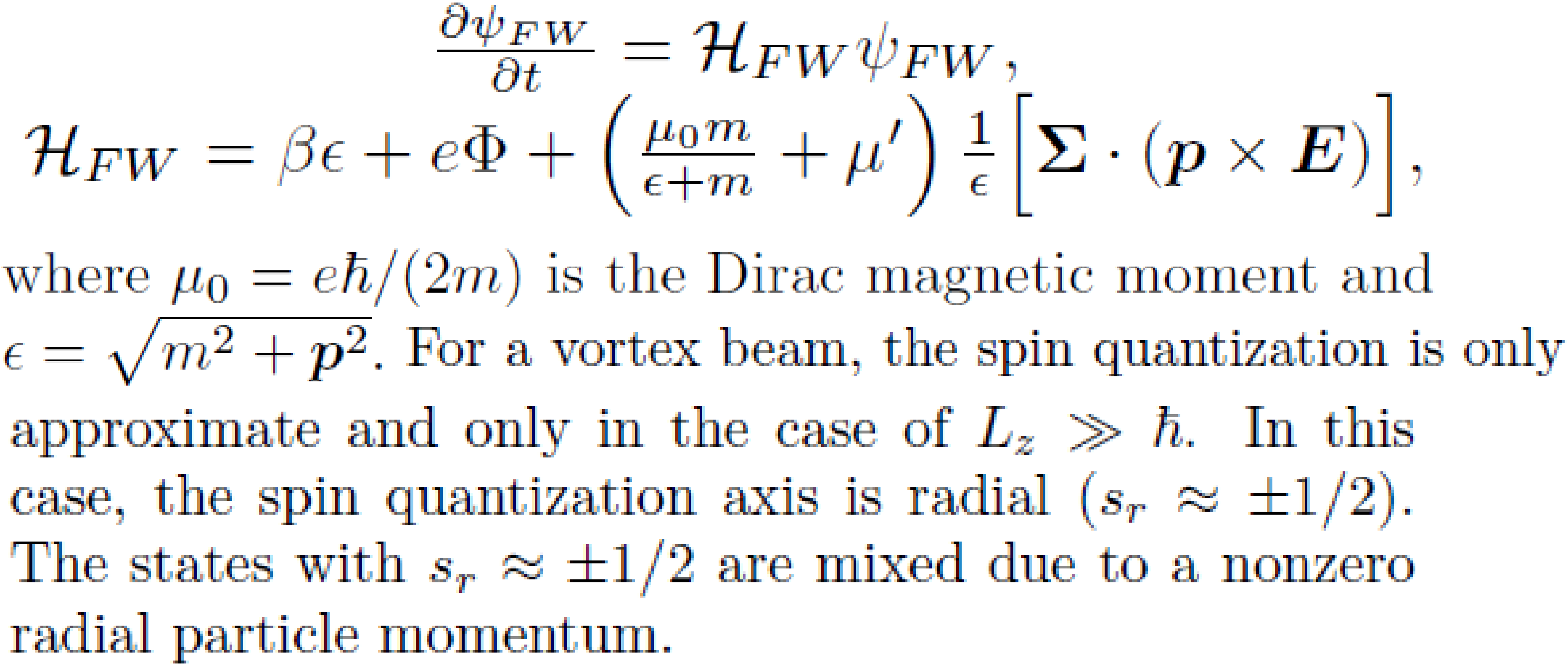


Landau beam **spatially oscillating LG beam**

Solution of quantum-mechanical equation for a twisted beam in a uniform electric field. Suppression of transverse spreading of the beam in an electric field

The relativistic FW transformation for a Dirac particle in a uniform electric field results in

A. J. Silenko, Foldy–Wouthuysen transformation for relativistic particles in external fields, J. Math. Phys. 44, 2952 (2003); **Leading correction to the relativistic Foldy-Wouthuysen Hamiltonian, Phys. Rev. A 111,** 032210 (2025)



$$\frac{\partial\psi_{FW}}{\partial t} = \mathcal{H}_{FW}\psi_{FW},$$

$$\mathcal{H}_{FW} = \beta\epsilon + e\Phi + \left(\frac{\mu_0 m}{\epsilon+m} + \mu'\right)\frac{1}{\epsilon}\Big[\boldsymbol{\Sigma}\cdot(\boldsymbol{p}\times\boldsymbol{E})\Big],$$

where $\mu_0 = e\hbar/(2m)$ is the Dirac magnetic moment and $\epsilon = \sqrt{m^2 + \boldsymbol{p}^2}$. For a vortex beam, the spin quantization is only approximate and only in the case of $L_z \gg \hbar$. In this case, the spin quantization axis is radial ($s_r \approx \pm 1/2$). The states with $s_r \approx \pm 1/2$ are mixed due to a nonzero radial particle momentum.

The transition from the first-order to the second-order relativistic quantum-mechanical equation is given by

$$\left[\left(i\frac{\partial}{\partial t}-\mathfrak{E}\right)^2-\boldsymbol{p}^2-m^2\right]\psi=0,$$

$$\mathfrak{E}=e\Phi+\left(\frac{\mu_0 m}{\epsilon+m}+\mu'\right)\frac{1}{\epsilon}\left[\boldsymbol{\Sigma}\cdot(\boldsymbol{p}\times\boldsymbol{E})\right].$$

For stationary solutions,

$$\left[(\mathbb{E}_0-\mathfrak{E})^2-\mathbf{p}^2-m^2\right]\psi=0,$$

$$\mathbb{E}_0=\epsilon_0+e\Phi_0=\sqrt{m^2+p_0^2}+e\Phi_0,$$

where $\mathbb{E}_0$ is the conserved total energy. The spin-dependent term in the formula for $\mathfrak{E}$ is rather small compared with $e\Phi$ and can be neglected ($\mathfrak{E}=e\Phi$).

$\boldsymbol{p}^2\approx pp_z+\boldsymbol{p}_\perp^2/2$ **is the paraxial approximation**

$$\left[2(\mathbb{E}_0-e\Phi)^2-(2pp_z+\boldsymbol{p}_\perp^2)-2m^2\right]\psi=0.$$

wavenumbers $k \equiv k(z) = p/\hbar$ and $k_0 = p_0/\hbar$

$$k(z) = k_0\sqrt{1 + 2K_1 z + K_2^2 z^2}$$

$$K_1 = \frac{\epsilon_0|eE_z|}{c^2p_0^2}, \qquad K_2 = \frac{|eE_z|}{cp_0}.$$

The substitution $\psi = \exp\left[i\int k(z)dz\right]\Psi$ into the equation

$$\left(2\boldsymbol{p}^2 + 2i\hbar p\frac{\partial}{\partial z} - \boldsymbol{p}_\perp^2\right)\psi = 0$$

results in

$$\left[\boldsymbol{\nabla}_\perp^2 + 2ik(z)\frac{\partial}{\partial z}\right]\Psi = 0.$$

This paraxial equation has the usual form, but *k*=*k(z)*.

The beam parameters satisfy the equations:

$$\frac{k_0}{R(z)} = \frac{k(z)w'(z)}{w(z)},$$

$$\left[\frac{ik_0}{R(z)} - \frac{2}{w(z)^2}\right]^2 + ik(z)\left[\frac{ik_0}{R(z)} - \frac{2}{w(z)^2}\right]' = 0,$$

$$\Phi'_G(z) = \frac{2(2n+|\ell|+1)}{k(z)w(z)^2}.$$

We denote $f(z) = \frac{ik_0}{R(z)} - \frac{2}{w(z)^2}$ **and obtain**

$$f(z)^2 + ik(z)f'(z) = 0$$

$$f(z) = \frac{k_0K_2(ik_0w_0w'_0 - 2)}{k_0K_2w_0^2 + 2(2i + k_0w_0w'_0)A(z)},$$

where $A(z) = \operatorname{arctanh}\left[\frac{K_2}{2K_1 + K_2^2 z}\left(\frac{k(z)}{k_0} - 1\right)\right]$

The final result reads

$$w(z) = w_0\sqrt{\left(1 + \frac{2A(z)w_0'}{K_2 w_0}\right)^2 + \frac{16A(z)^2}{k_0^2 K_2^2 w_0^4}},$$

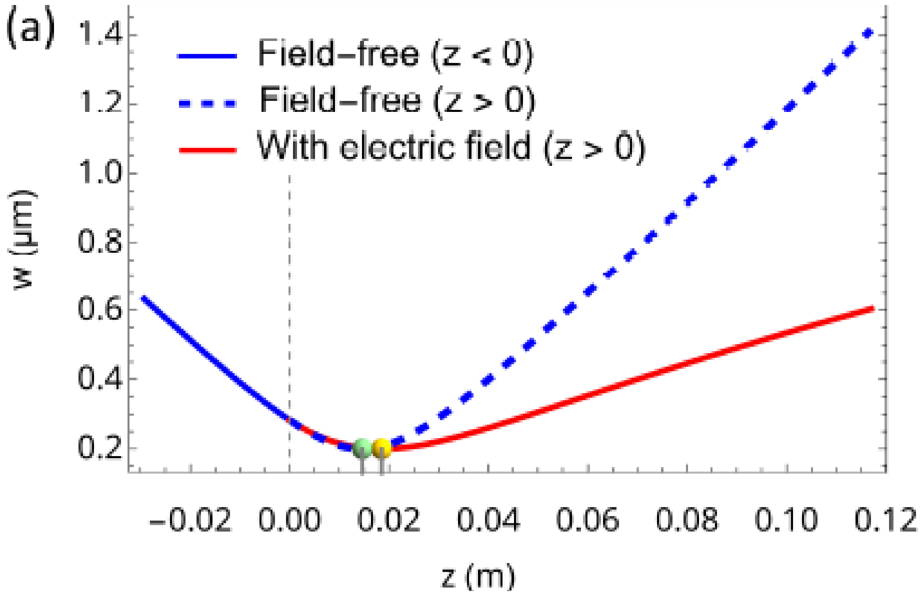
When $w_0' = 0$,

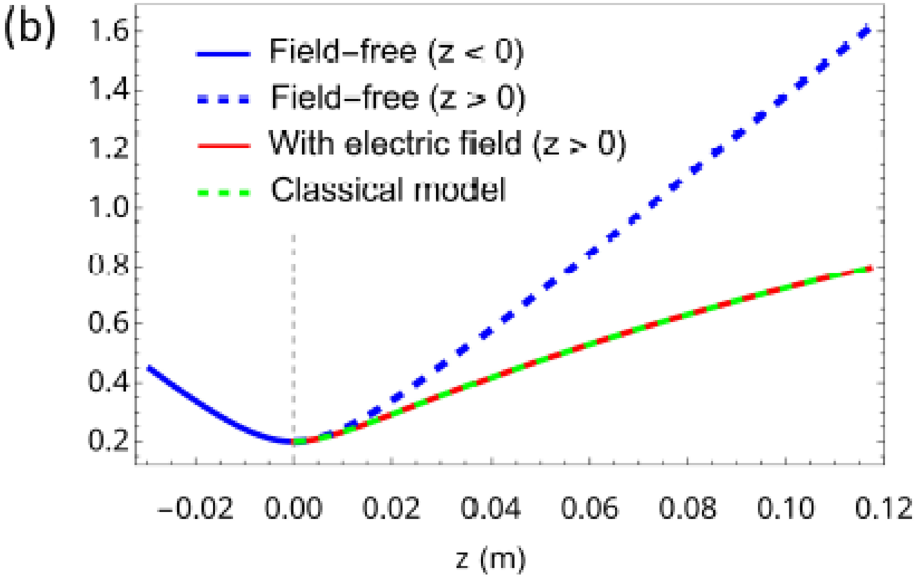
$$w(z) = w_0\sqrt{1 + \frac{4}{k_0^2 w_0^4 K_2^2}\ln^2\left|\frac{K_1 + K_2^2 z + K_2\sqrt{1 + 2K_1 z + K_2^2 z^2}}{K_1 + K_2}\right|}.$$

The Gouy phase is given by

$$\Phi_G(z) = \Phi_G(0) + N \operatorname{arccot}\left[\frac{k_0 w_0 w_0'}{2} + \frac{k_0 K_2 w_0^2}{4A(z)}\right].$$

Suppression of transverse spreading of the beam in an electric field

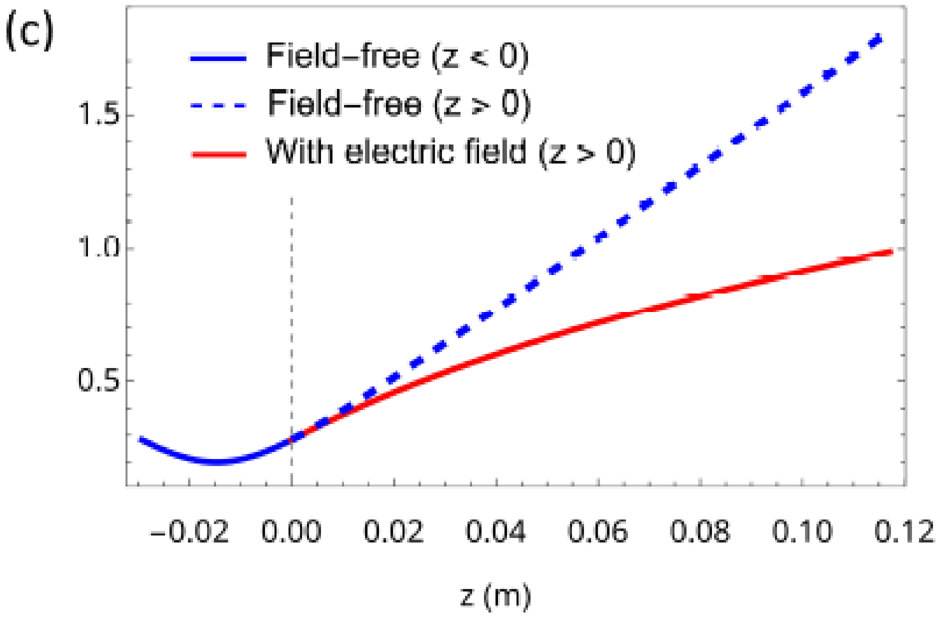




$$(a)\ w_0' = -2/k_0 w_0,$$

$$(b)\ w_0' = 0, \text{ and}$$

$$(c)\ w_0' = 2/k_0 w_0.$$

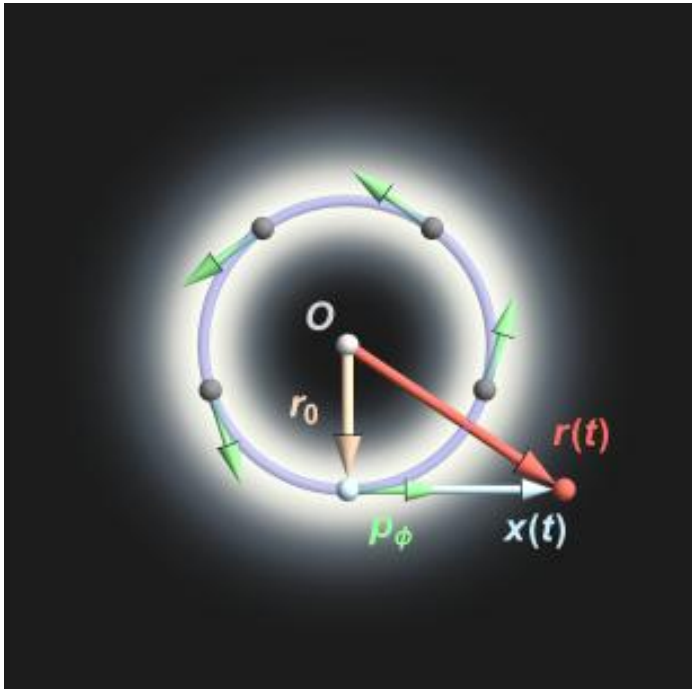


The longitudinal field significantly alters the beam dynamics, extraordinarily suppressing its natural spreading and giving rise to distinct focusing behavior that depends sensitively on the initial conditions.

Suppressing of beam spreading is a kinematic effect.

The transverse momentum $\boldsymbol{p}_\perp = const$ **and** $\boldsymbol{v}_\perp = \boldsymbol{p}_\perp/(m^2+\boldsymbol{p}^2)^{1/2}$.

As a result, $\boldsymbol{v}_\perp$ **decreases with the beam acceleration.**



We can explain the beam evolution with a simple classical model. Let r_0 **be the distance between the particle and the axis of symmetry of the beam. In the transversal plane, any particle moves freely with constant velocity. At the waist,** $v_r = 0$ **and the direction** x **of its movement is orthogonal to** r: $dx = vdt$.

- This direction remains unchanged all the time. After time *t*, the azimuth of the particle position changes, the radial velocity becomes nonzero, and the distance to the axis of symmetry increases:

$$r(t) = \sqrt{r_0^2 + v_\phi^2 t^2}$$

To check the compatibility with wave theory, it is convenient to determine the connection between *r* and *z*. Since $dz = v_z dt \approx v dt$ and $v_\varphi / v = p_\varphi / p$, we obtain

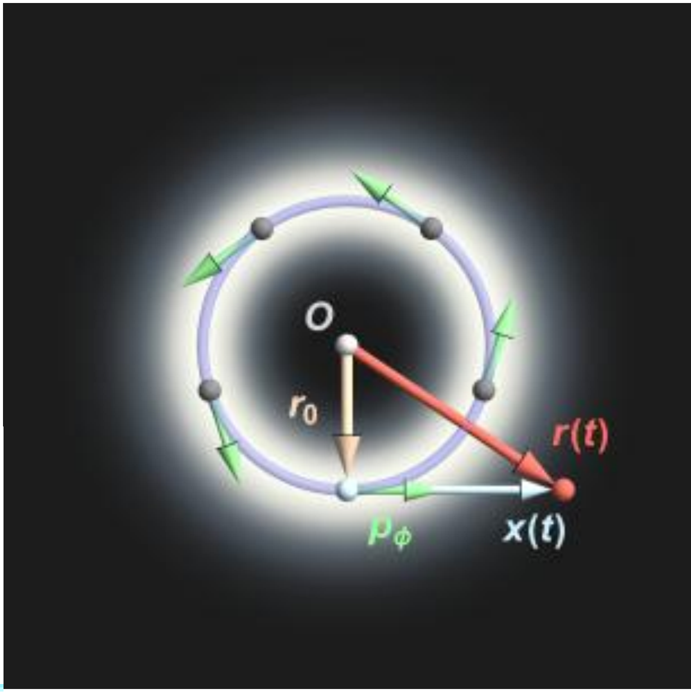
$$r(z) = \sqrt{r_0^2 + \frac{p_\phi^2 z^2}{p^2}}.$$

$$\underline{r_0 = w_0, \ p_\phi = 2\hbar / w_0.}$$

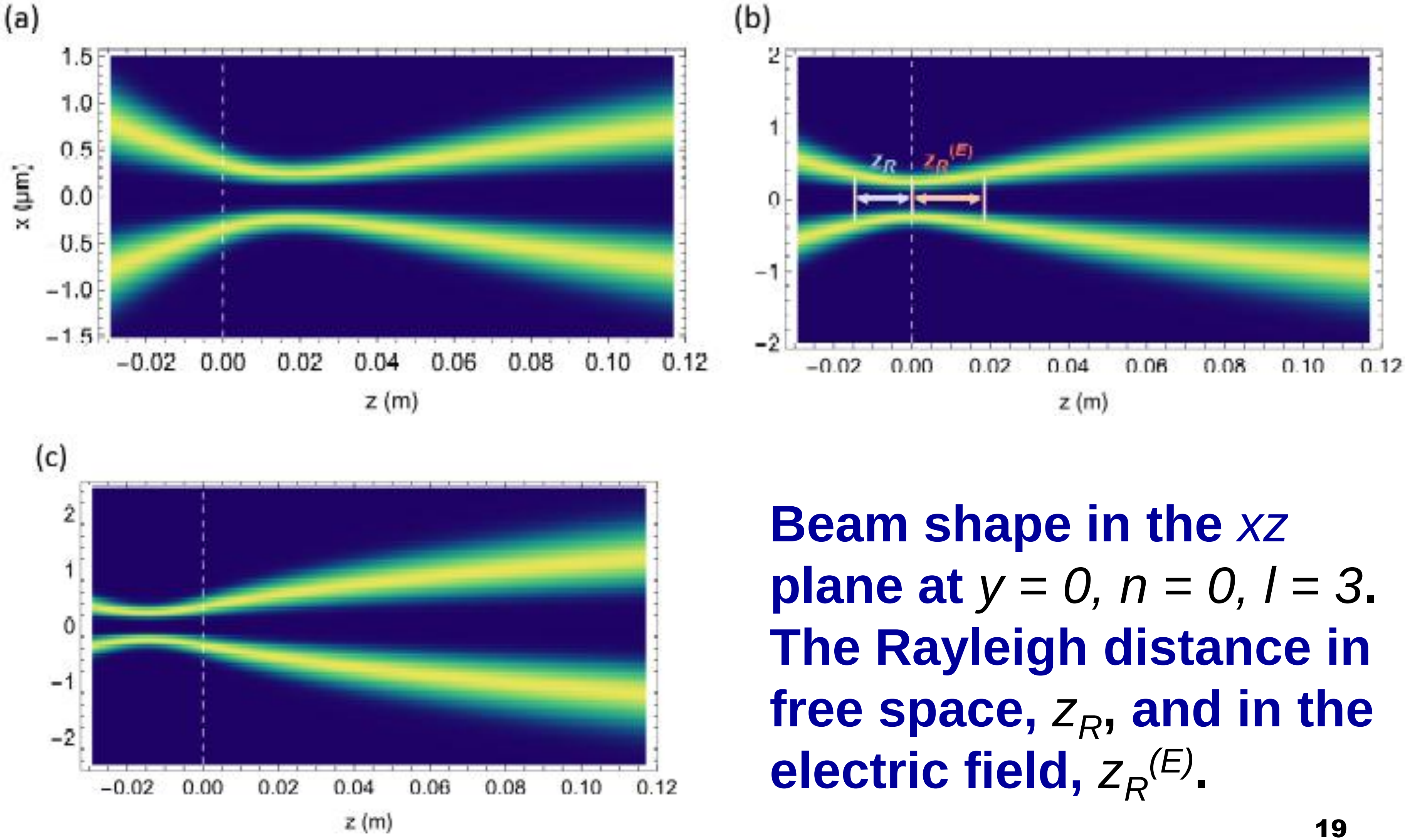
$$dx = [p_\phi/\epsilon(z)]dt = [p_\phi/p(z)]dz.$$

$$x = \frac{p_\phi}{p_0}\int_0^z \frac{dz}{\sqrt{1+2K_1z+K_2^2z^2}}$$

$$= \frac{p_\phi}{p_0 K_2}\ln\left|\frac{K_1+K_2^2z+K_2\sqrt{1+2K_1z+K_2^2z^2}}{K_1+K_2}\right|.$$



This simple classical model leads to the same equation which describes an evolution of the beam width.



Beam shape in the *xz* plane at $y = 0$, $n = 0$, $l = 3$. The Rayleigh distance in free space, z_R, and in the electric field, $z_R^{(E)}$.

Discussion

- Particles can loose OAMs at their helical motion in a magnetic field which is followed by the radiation of vortex photons. Certainly, the decrease of the particle OAM is equal to the total OAM of radiated photons. This decrease is defined by

$$\frac{dL_z}{dt} = (\dot{\boldsymbol{r}} \times \boldsymbol{p} + \boldsymbol{r} \times \dot{\boldsymbol{p}})_z, \qquad \dot{\boldsymbol{r}} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}} = \frac{\boldsymbol{p}}{\sqrt{m^2+\boldsymbol{p}^2}},$$

$$\mathcal{H} = \sqrt{m^2 + \boldsymbol{p}^2} + e\Phi,$$

and is equal to zero. Thus, charged vortex beams accelerated in an electric field do not emit vortex photons. The radiation of non-vortex photons cannot change the particle OAM and destroy the beam coherence, but it can influence the wave function.

Our results can be directly applied to electrostatic linear accelerators (linacs) accelerating electrons by a static and approximately uniform electric field. They also cover RF linacs with a time-varying RF field. When the beam motion is synchronized with the RF oscillations, the beam is governed by an approximately static and uniform electric field.

Summary

- **Stationary states of Dirac electrons in a uniform magnetic field can be spatially oscillating Laguerre-Gauss beams**
- **Charged particle beams keep their coherence and OAMs in a uniform electric field, despite the emission of photons**
- **The beam evolution in the electric field is considered in detail. Extraordinary suppression of transverse spreading of a vortex beam is discovered, analyzed, and explained**
- **Our results can be successfully used for the acceleration of charged vortex beams in linacs**

Thank you for your attention