3 loops@5 legs

Massive amplitudes on the Coulomb branch of N=4 SYM

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FIAN, CFAR VNIIA

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hep-th 2605.17057
hep-th 2605.17056

N=4 SYM theory

- **N=4 SYM - one may hope that this theory is exactly solvable.**
- **Physical content - resembles perturbative part of QCD (massless QED without running of the coupling). Tree amplitudes identical to QCD.**
- **Toy model for weakly coupled gauge theories.**
- **The correlation functions in this theory can be studied in the weak and strong regimes (via AdS/CFT).**
- **The computation of anomalous dimensions of local operators in N=4 SYM in planar limit can be reduced to the problem of solving some integrable system.**
- **There are numerous results for perturbative expansions of amplitudes (S-matrix) and form factors/cor.functions with some results valid in all orders of PT (BDS ansatz for 4,5 points, collinear OPE).**
- **Results which can be written in "simple" analytical manner are still rare.**

N=4 SYM theory the “harmonic oscillator of 21 century” …

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N=4 SYM

We already know that!

- **N=4 SYM - one may hope that this theory is exactly solvable (in some way or another….)**
- **Let us put good properties of N=4 SYM to good use then!**
- **Let us investigate IR physics of gauge theories with off-shell/ massive particles in controlled regime**
- **&**
- **Let us try to obtain some exact results in D=4 gauge theories along the way or at least make some conjectures how all loop behavior may look like**

IR properties of Gauge theories

(in oversimplified manner)

- **The folklore says that in <u>any</u> gauge theory on can expect that (** here $M_n = (all\ loops)_n/(tree)_n$, color ordering is implied**):**

$$a = \frac{g_{\rm YM}^2 N_c}{8\pi^2}$$

$$M_n \sim \prod_{i=1}^{n} F_2\left(s_{ii+1}/\mu_{\rm IR}, g\right) \times h_n\left(\{s\}, g\right) + \dots$$

Here F_2 is so called Sudakov form factor.

In different theories and kinematical regimes it can have different form but factorization relation is expected to be universal. $\mu_{\rm IR}$ can be anything

$$\Gamma_2 = \langle 0|\mathcal{O}|p_1,p_2\rangle \qquad F_2 = \frac{\Gamma_2}{\Gamma_2^{tree}}$$

Van Neerven 85
Bork, Kazakov et.all 10, Henn et.all 12, Henn et.all 13, Belitsky, Bork, Smirnov, Pikelner 23, etc.

$$\log F_2\left(\frac{\mu^2}{Q^2}, a, \varepsilon\right) = -\sum_{l=0}^{\infty} \frac{a^l}{2}\left(\frac{\Gamma^{(l)}_{cusp}}{(l\varepsilon)^2} + \dots\right)\left(\frac{\mu}{Q^2}\right)^{2\varepsilon}$$

Planar N=4 SYM massless case

$$\log F_2\left(\frac{M^2}{Q^2}, a\right) = -\frac{\Gamma_{cusp}(a)}{4}\log^2\left(\frac{M^2}{Q^2}\right) + \dots$$

Planar N=4 SYM Coulomb branch locust
all external states massless, $1/(k^2 - M^2)$ propagators

$$\log F_2\left(\frac{m^2}{Q^2}, a\right) = -\frac{\Gamma_{oct}(a)}{2}\log^2\left(\frac{m^2}{Q^2}\right) + \dots$$

Planar N=4 SYM different Coulomb branch locust
all external states massive (m^2), $1/k^2$ propagators

IR properties of N=4 SYM on the Coulomb branch

- **Let us concentrate on the planar N=4 SYM case where massive particle are present. But only in external states. Mimics off-shell kinematics.**
- **Based on pervious computations one can expect that:**

$$\log M_n = \sum_{i=1}^{n} \log F_2\left(\frac{m^2}{s_{ii+1}}, a\right) + f_n(\{s\}, a) + \frac{n}{2} D(a)$$

3 loop Sudakov FF.,
2 loop n=5 ampl.,
4loop n=4 ampl.,
2loop n-point FF …

Bork et al. 22,
Belitsky et al. 23,
Huot ,Coronado 21

with:

$$\log F_2\left(\frac{m^2}{Q^2}, a\right) = -\frac{\Gamma_{oct}(a)}{2} \log^2\left(\frac{m^2}{Q^2}\right) - D(a)$$

Note the absence of collinear G(a).

where:

$$\Gamma_{oct}(a) = \frac{2}{\pi^2} \log\left(\cosh\left(\pi\sqrt{2a}\right)\right) = 2a - 4\zeta_2 a^2 + 32\zeta_4 a^3 + \dots,$$

$$D(a) = \frac{1}{4} \log\left(\frac{\sinh(2\pi\sqrt{2a})}{2\pi\sqrt{2a}}\right) = 2\zeta_2 a - 8\zeta_4 a^2 - \frac{128\zeta_6}{3} a^3 + \dots.$$

Belitsky, Korchemsky 19

All loop results!!

IR properties of N=4 SYM on the Coulomb branch

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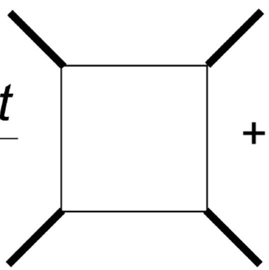
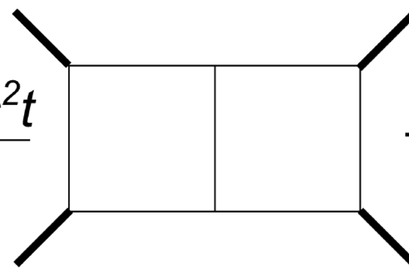
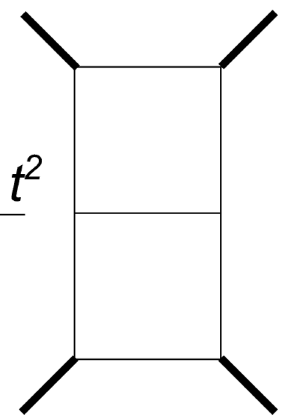
3 loop Sudakov FF.,
2 loop n=5 ampl.,
4loop n=4 ampl.,
2loop n-point FF …

Bork et al. 22,
Belitsky et al. 23,
Huot ,Coronado 21

No all loop solid proof.

n=4 example:

$M_4 = 1 + \frac{a\, st}{2}$  $+ \frac{a^2 s^2 t}{4}$  $+ \frac{a^2 s\, t^2}{4}$  $+ \dots$

Note that integrals above are Ussyukina-Davydychev box functions with good (dual)conformal properties…

IR properties of N=4 SYM Coulomb branch amplitudes

$$\log M_n = \sum_{i=1}^{n} \log F_2\left(\frac{m^2}{s_{ii+1}}, a\right) + f_n\left(\{s\}, a\right) + \frac{n}{2} D(a)$$

- **Can we test this conjecture even further ? More loop&legs.**
- **What about finite part** f_n **?**

For n=4 example one can obtain:

$$\log M_4 = -\frac{\Gamma_{oct}(a)}{2}\left(\log^2\left(\frac{m^2}{s}\right) + \log^2\left(\frac{m^2}{t}\right)\right) + \tilde{\mathcal{F}}_4 + O(m^2)$$

Bork et al. 22,
Huot ,Coronado 21

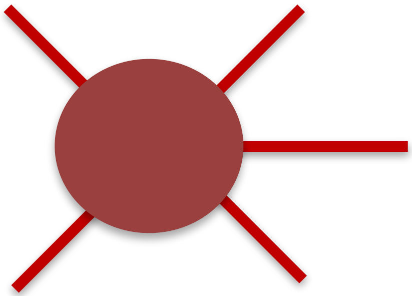
$$\tilde{\mathcal{F}}_4 = \frac{\Gamma_{oct}(a)}{4}\log^2\left(\frac{s}{t}\right) - \frac{D(a)}{2}$$

4loop explicit verification, expected to hold for all loops

IR properties of N=4 SYM Coulomb branch amplitudes

$$\log M_n = \sum_{i=1}^{n} \log F_2\left(\frac{m^2}{s_{ii+1}}, a\right) + f_n\left(\{s\}, a\right) + \frac{n}{2} D(a)$$

- **Can we test this conjecture even further ? More loop&legs.**
- **What about finite part f_n ?**
  - **n=5 amplitude is natural candidate to investigate!**

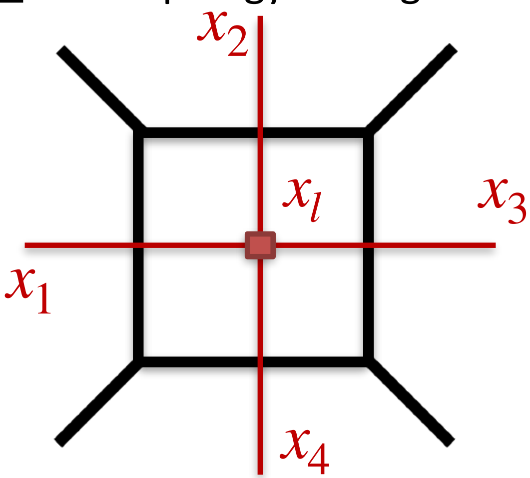


Coulomb branch masses are extra dimensional coordinates

Integrand of 4 and 5 point amplitudes are essentially identical between D=4,6,8 10 SYM theories

Kazakov, Bork et al. 15 and reference therein

We can **choose extra dimensional coordinates to make internal or external lines in diagrams massive**. The topology of diagrams is identical in all D (for n=4,5,…). So for example we can consider:

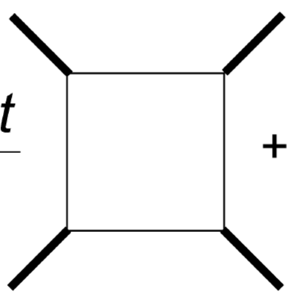
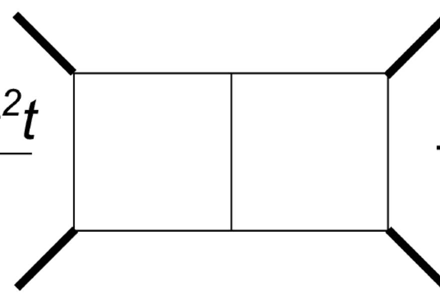
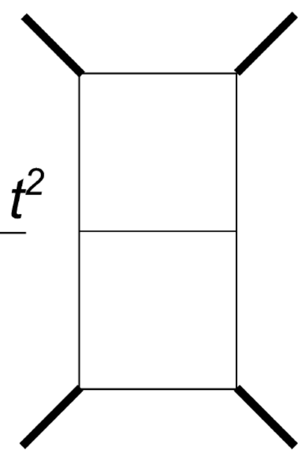


Let us split D demential coordinates as $x_{li}^2 = x_{li}^{2,D=4} + y_{il}^2$

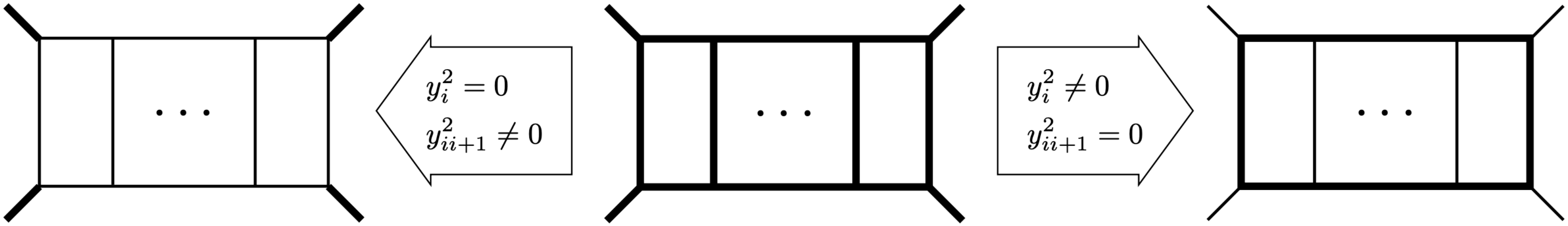
And impose additional constraints: $\delta^{D-4}(y_l)$ and $y_i^2 = 0$

Alday, Henn et al. 09,
Huot ,Coronado 21,
Elvang et al. 11,
Plefka et al. 14

4p amplitude example:

$M_4 = 1 + \frac{a\,st}{2}$  $+ \frac{a^2 s^2 t}{4}$  $+ \frac{a^2 s\, t^2}{4}$  $+ \dots$

Descending on the Coulomb branch from N=(1,1) D=6 MSYM



- **Let us consider on shell momentum superspace for N=(1,1) MSYM**
- **All creation/annihilation operators for on-shell can be combined into:**

$$|\Omega\rangle = \left(\phi + \chi^a\eta_a + \bar{\chi}_{\dot{a}}\bar{\eta}^{\dot{a}} + \eta^2\phi' + \bar{\eta}^2\phi'' + g^a{}_{\dot{a}}\eta_a\bar{\eta}^{\dot{a}} + \psi^a\eta_a\bar{\eta}^2 + \bar{\psi}_{\dot{a}}\bar{\eta}^{\dot{a}}\eta^2 + \eta^2\bar{\eta}^2\phi'''\right)|0\rangle$$

Note that this is essentially N=4 on-shell supermultiplet in disguise. g is gluons et.c.

Siegel et al. 10, Brandhuber et al. 11, Bern et al. 11

See also Ivanov 06, Buchbinder, Ivanov et al. 07

gluons: $g^a{}_{\dot{a}}$,

scalars: $\phi, \phi', \phi'', \phi'''$,

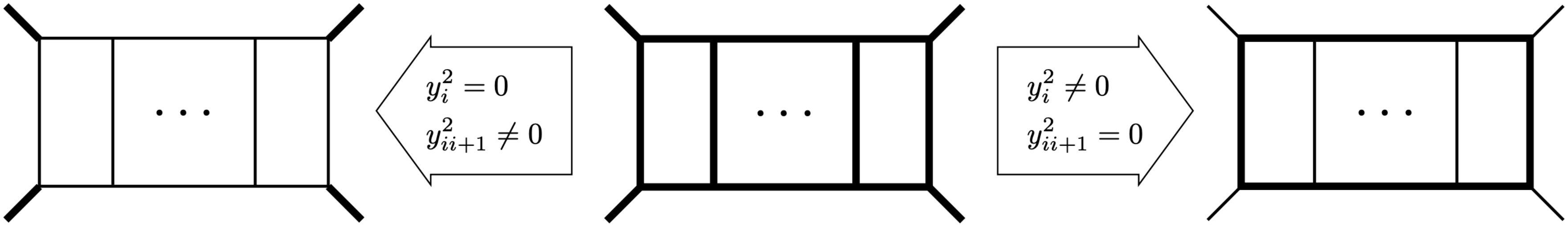
gluinos: $\chi^a, \bar{\chi}_{\dot{a}}, \psi^a, \bar{\psi}_{\dot{a}}$

$a, \dot{a}$ are D=6 helicity little group indices SU(2)XSU(2)

Cheung, O'Connell 09

Superamplitudes in N=(1,1) 0n-shell momentum superspace are defied as: $A_n = \langle 0|S|\Omega_n \ldots \Omega_1\rangle$

Descending on the Coulomb branch from N=(1,1) D=6 MSYM



- **Let us consider on shell momentum superspace for N=(1,1) MSYM**

In on-shell momentum superspace we can define supercharges:

$$Q_i = \langle i^a | \eta_{i,a}\,, \qquad \bar{Q}_i = [i_{\dot{a}} | \bar{\eta}_i^{\dot{a}} \qquad\qquad P_i = |i^a\rangle\langle i_a|\,, \qquad \bar{P}_i = |i_{\dot{a}}][i^{\dot{a}}|$$

$$\mathcal{A}_n = \widehat{\mathcal{A}}_n\, \delta^{(4)}\left(\sum_{i=1}^{5} Q_i\right) \delta^{(4)}\left(\sum_{i=1}^{5} \bar{Q}_i\right)$$

Cheung, O'Connell 09

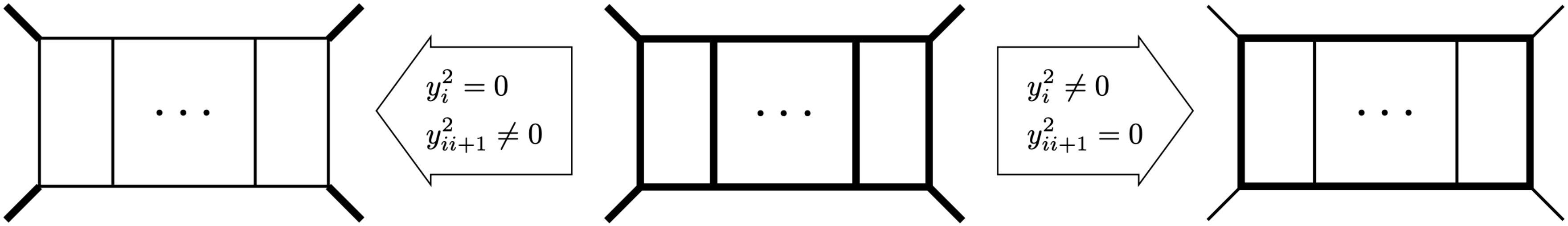
n=4,5 tree level amplitudes are given by: Cheung, O'Connell 09, Plefka et al. 14, Belitsky 23

$$\widehat{\mathcal{A}}_4^{(0)} = \frac{1}{S_{12}S_{23}}\,, \qquad \widehat{\mathcal{A}}_5^{(0)} = \frac{-\Omega}{S_{12}S_{23}S_{34}S_{45}S_{51}}$$

Ω is some (not)complicated function of $|i^a\rangle, \ldots, \eta^a, \bar{\eta}^{\dot{a}}$, similar to the D=4 R_{rst} NMHV functions.

$$S_{ij} = (P_i + P_j)^2 |_{D=6}$$

Descending on the Coulomb branch from N=(1,1) D=6 MSYM



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$$\mathcal{A}_n = \widehat{\mathcal{A}}_n \, \delta^{(4)}\left(\sum_{i=1}^{5} Q_i\right) \delta^{(4)}\left(\sum_{i=1}^{5} \bar{Q}_i\right)$$

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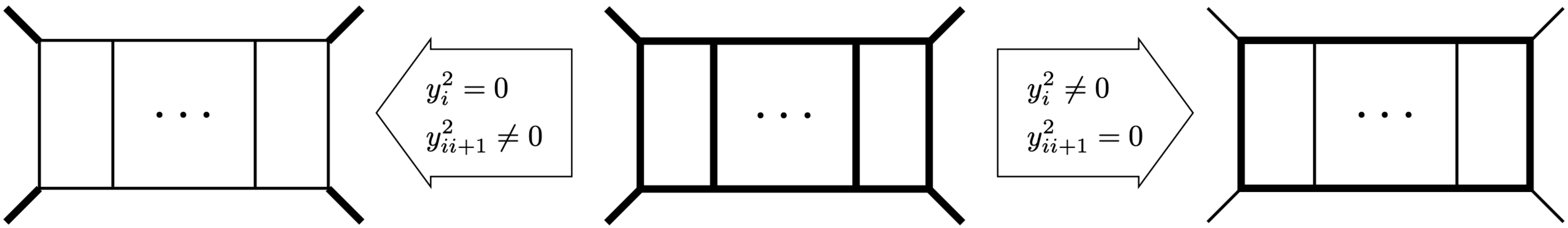
Trees in N=(1,1) are dimensional uplifts from N=4 SYM

Plefka et al. 14

$$S_{ij} = (P_i + P_j)^2 |_{D=6}$$

Descending on the Coulomb branch from N=(1,1) D=6 MSYM



- **Note that such choice of “masses” will imply that all integrals are well defined in d=4.**
- **Moreover they are D=4 dual conformal invariant! This imply that:**

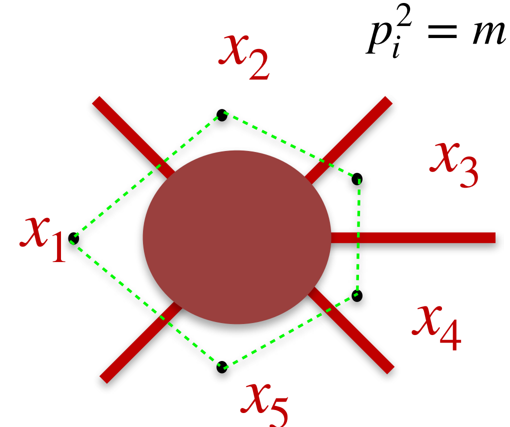
$$\log M_5 = \mathbb{D}(u_1, \ldots, u_5; a)$$

$$p_i^2 = m^2 \ll 1$$

Where u_i are dual conformal cross ratios

$$u_i = \frac{x_{i+1i+1}^2 x_{i-2i-1}^2}{x_{i+2i-1}^2 x_{i+1i-2}^2} = \frac{m^4}{s_{i+1}s_{i+2}}$$

$$s_i \equiv s_{ii+1} = (p_i + p_{i+1})^2$$

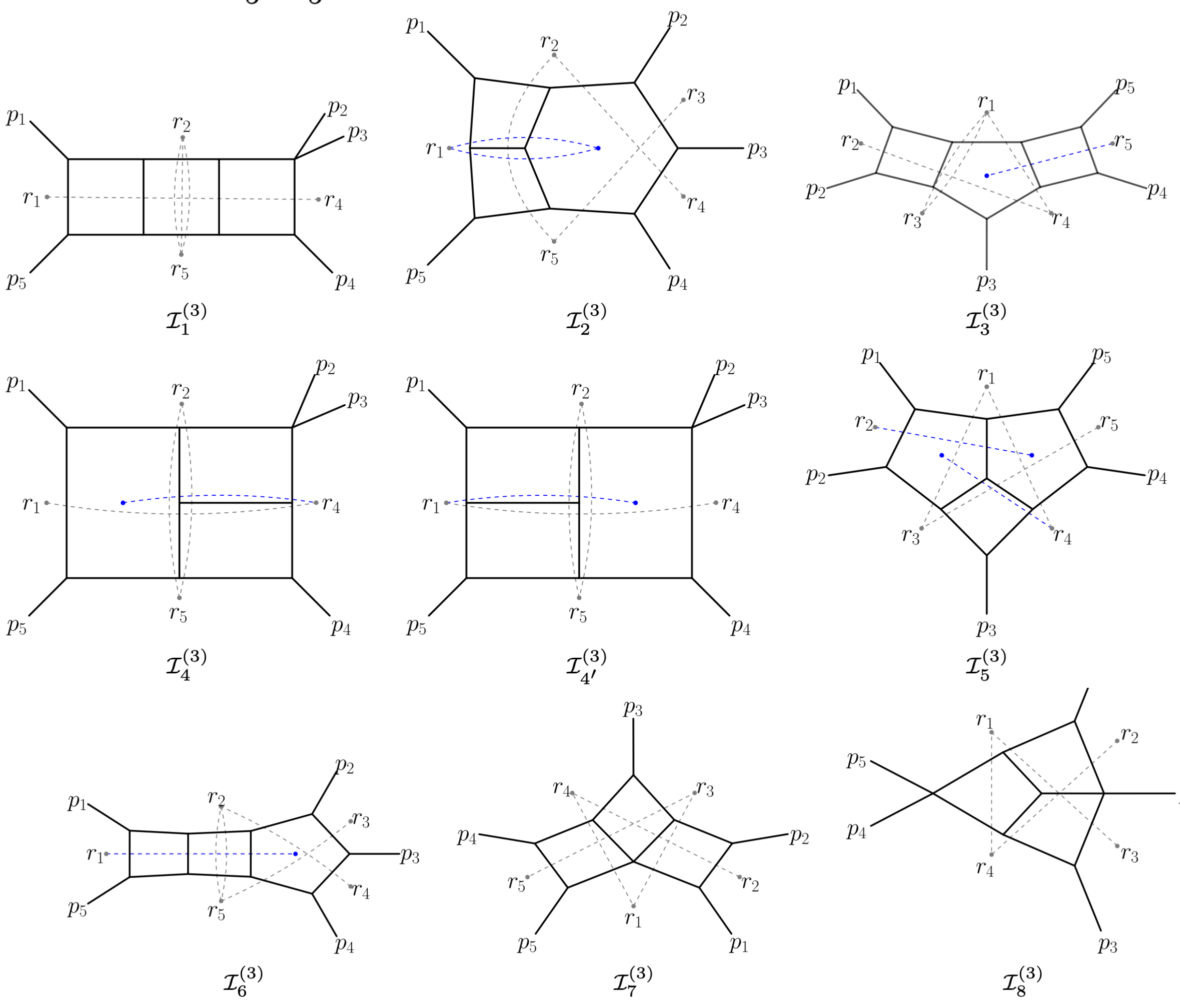


3 loop@5legs integrand

$$M_5^{(3)} = \sum_{\sigma_5 \cup \bar{\sigma}_5} \left[-\frac{1}{2} I_1^{(3)} - \frac{1}{4} I_2^{(3)} - \frac{1}{2} I_3^{(3)} - \frac{1}{2} I_4^{(3)} - \frac{1}{2} I_{4'}^{(3)} - \frac{1}{2} I_5^{(3)} - \frac{1}{4} I_6^{(3)} + \frac{1}{4} I_7^{(3)} + \frac{1}{4} I_8^{(3)} \right]$$

Spradlin et al. 08

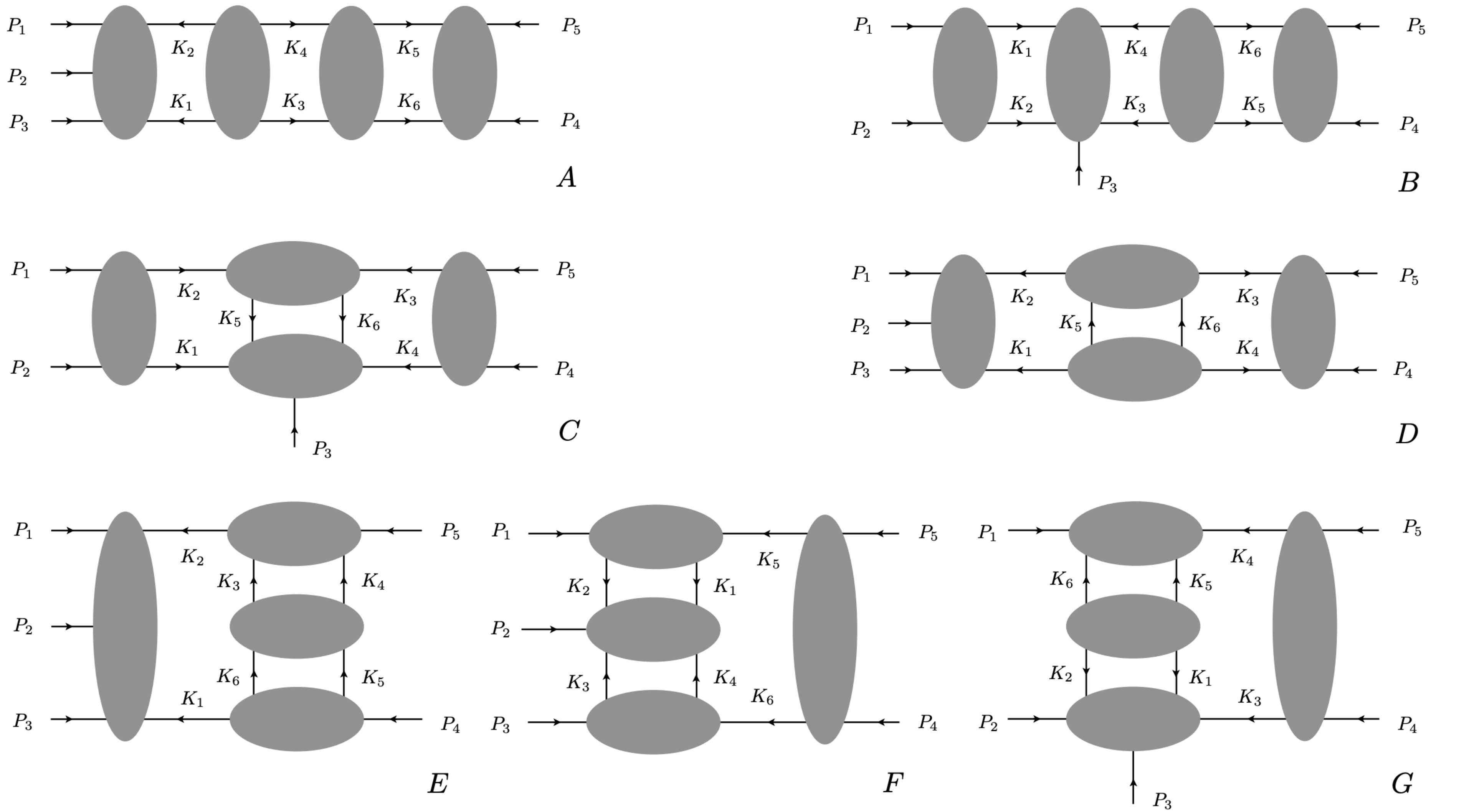
$$\sigma_5 = \{(12345), (23451), (34512), (45123), (51234)\},$$
$$\bar{\sigma}_5 = \{(54321), (43215), (32154), (21543), (15432)\}.$$



- **2 loop results are previously known**
- **The form of $M_5^{(3)}$ can be conjectured form the amplitude/correlator duality** Eden et al. 13
- **But can we really compute it from “first principles” ?**

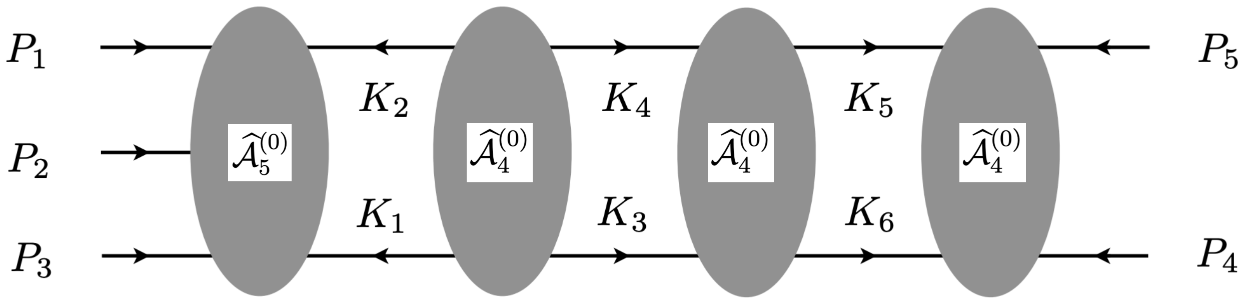
D=6 N=(1,1) MSYM unitarity cuts

- **Yes, we can! D=6 Unitarity:**



D=6 N=(1,1) MSYM unitarity cuts

- **Yes, we can! D=6 Unitarity (3loop example):**

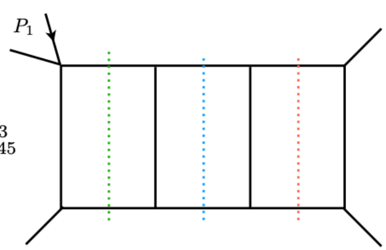


$$(\phi_1''', \phi_2''', g_3, \phi_4, \phi_5)$$

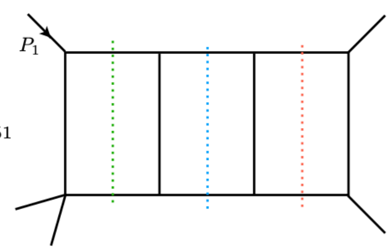
A

$$-\left\langle \mathcal{A}_5^{(3)}|_{\text{cut}-\text{A}} \right\rangle / \left\langle \mathcal{A}_5^{(0)} \right\rangle = \frac{\frac{1}{2} S_{45} \langle\!\langle \Omega_{\text{A}} \rangle\!\rangle}{S_{23} S_{1,K_2} S_{3,K_1} S_{K_2 K_4} S_{K_4,-K_5} S_{5,K_5}}$$

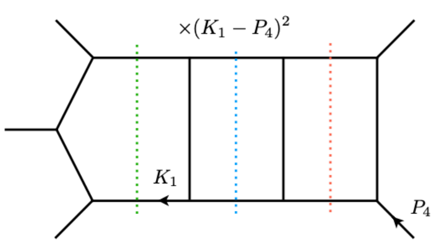
$= \frac{1}{2} S_{34} S_{45}^3$



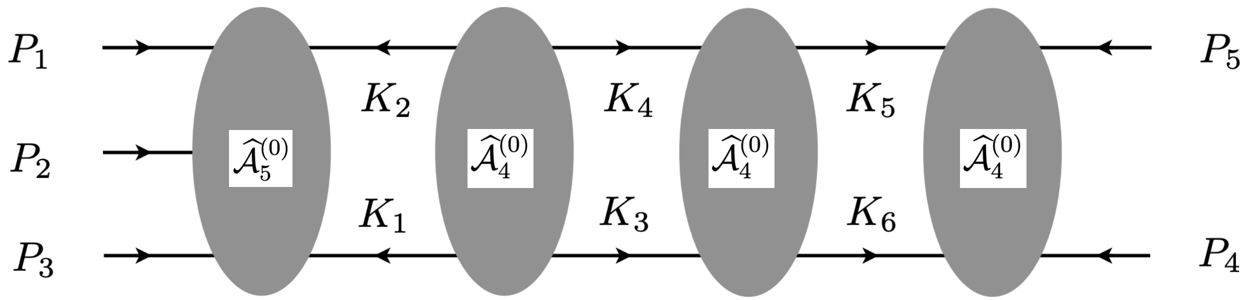
$+\frac{1}{2} S_{45}^3 S_{51}$



$+\frac{1}{2} S_{12} S_{23} S_{45}^2$



D=6 N=(1,1) MSYM unitarity cuts



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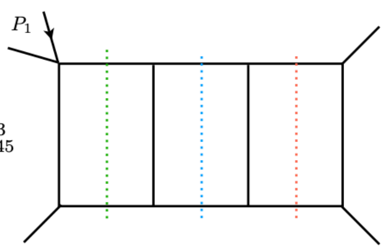

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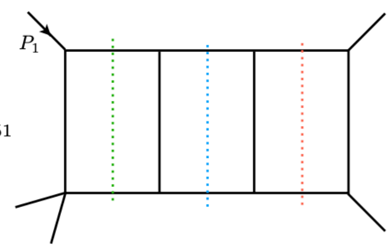
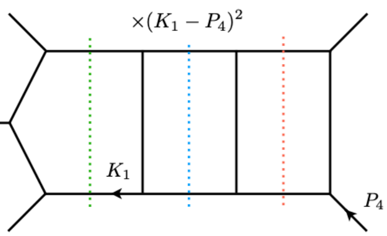


$+\frac{1}{2} S_{12} S_{23} S_{45}^2$



$$\frac{S_{12}}{S_{45}} \langle\!\langle \Omega_A \rangle\!\rangle = \tfrac{1}{2} \operatorname{tr}_4[\bar{P}_3 P_4 \bar{P}_5 P_1 \bar{P}_2 P_3 \bar{P}_2 P_1 \bar{K}_2 K_1] - \tfrac{1}{2} \operatorname{tr}_4[\bar{P}_3 P_4 \bar{P}_5 P_1 \bar{P}_2 P_3 \bar{K}_1 K_2 \bar{P}_1 P_2]$$

$$= S_{12} S_{23} S_{34} S_{45} S_{1,K_2} + S_{12} S_{23} S_{45} S_{51} S_{3,K_1} + S_{12}^2 S_{23}^2 S_{4,-K_1} \,, \qquad (3.18)$$

D=6 N=(1,1) MSYM unitarity cuts and D=4 Dual conformal invariance

- **All integrals such as are Dual conformal Integrals**

$$M_5^{(3)} = \sum_{\sigma_5 \cup \bar{\sigma}_5} \left[-\frac{1}{2} I_1^{(3)} - \frac{1}{4} I_2^{(3)} - \frac{1}{2} I_3^{(3)} - \frac{1}{2} I_4^{(3)} - \frac{1}{2} I_{4'}^{(3)} - \frac{1}{2} I_5^{(3)} - \frac{1}{4} I_6^{(3)} + \frac{1}{4} I_7^{(3)} + \frac{1}{4} I_8^{(3)} \right]$$

as well as at 1 and 2 loop level

- **This imply that:**

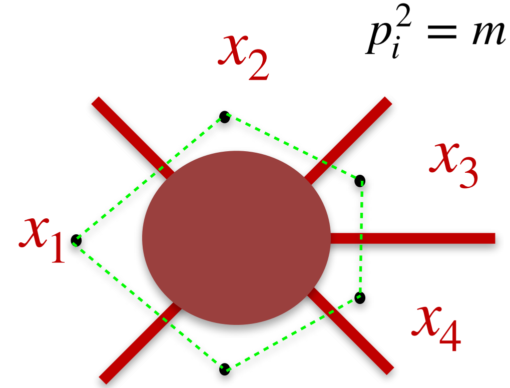
$$\log M_5 = \mathbb{D}(u_1, \ldots, u_5; a)$$

Where u_i are dual conformal cross ratios

$$u_i = \frac{x^2_{i+1\,i+1} x^2_{i-2\,i-1}}{x^2_{i+2\,i-1} x^2_{i+1\,i-2}} = \frac{m^4}{s_{i+1} s_{i+2}}$$

$$s_i \equiv s_{i\,i+1} = (p_i + p_{i+1})^2$$

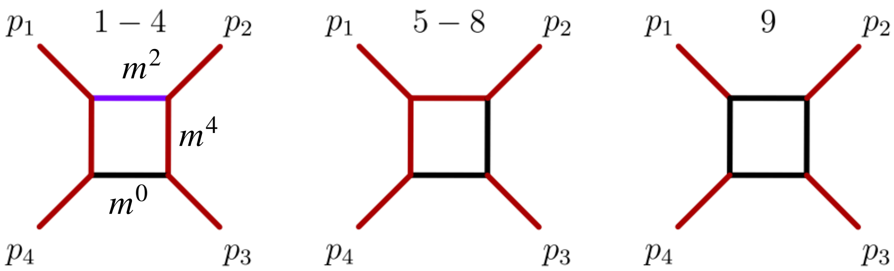
$p_i^2 = m^2 \ll 1$



- **We have explicitly verified this at 3 loop level. However this is to be expected…**

So how to compute such integrals ?
DCI MofR

- **Integrals are well defined in D=4 but when we cut initial integral into regions regularization is required. Usually one promotes $d^4 l \mapsto d^{4-2\epsilon} l$ but this brakes DCI of each region.** Lee, Onishchenko, Bork 25
- **To compensate this we also introduce another regulator α_i for each propagator $1/D_i \mapsto 1/D_i^{\alpha_i}$**
- **This allows us to radically simplify parametric integrals for each region. *Here is 1loop example:***



$$u_1 = \frac{p_1^2 p_3^2}{st}, \quad u_2 = \frac{p_2^2 p_4^2}{st}$$

$$B = \int \frac{d^4 y_5}{\pi^2} \frac{y_{13}^2 y_{24}^2}{y_{15}^2 y_{25}^2 y_{35}^2 y_{45}^2},$$

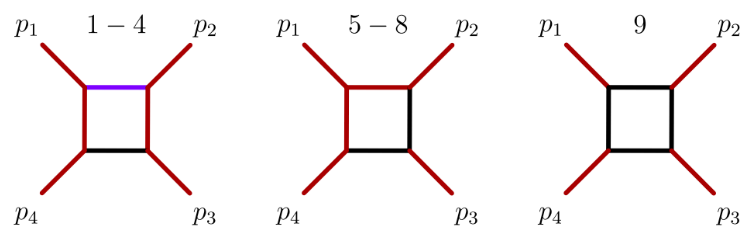
where we use the notation $y_{ij} = y_i - y_j$. We define kinematic invariants

$$p_1^2 = y_{14}^2, \quad p_2^2 = y_{21}^2, \quad p_3^2 = y_{32}^2, \quad p_4^2 = y_{43}^2,$$
$$s = (p_2 + p_1)^2 = y_{24}^2, \quad t = (p_3 + p_2)^2 = y_{13}^2.$$

So how to compute such integrals …

- **Integrals are well defined in D=4 but when we cut initial integral into regions regularization is required. Usually one promotes $d^4 l \mapsto d^{4-2\epsilon} l$ but this brakes DCI of each region.**
- **To compensate this we also introduce another regulator α_i for each propagator $1/D_i \mapsto 1/D_i^{\alpha_i}$**
- **1loop example:**

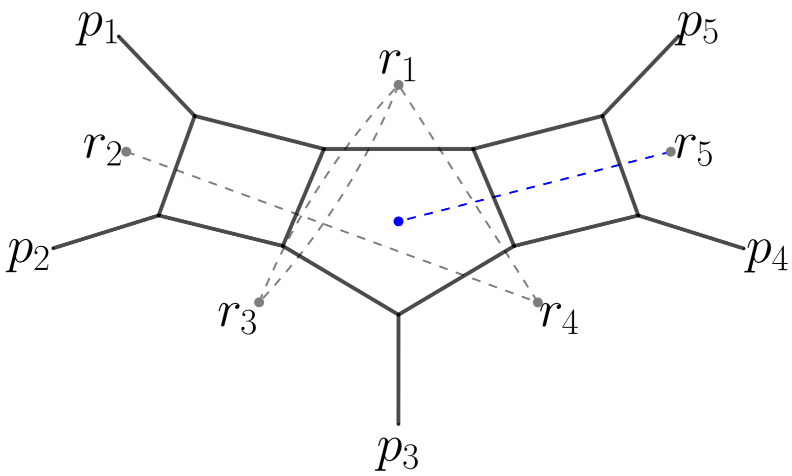
k	B_k^{dim}	B_k^{dci}
1	$s^\epsilon \Gamma(1-\epsilon)\Gamma(\epsilon)^2 \left(p_1^2 p_2^2\right)^{-\epsilon}$	$\frac{\Gamma(1-\alpha_1-\epsilon,\alpha_1+\alpha_2+\epsilon,\alpha_1+\alpha_4+\epsilon)}{\Gamma(1+\alpha_1,1+\alpha_2,1+\alpha_4)} u_1^{\alpha_2+\alpha_3+\epsilon}$
2	$t^\epsilon \Gamma(1-\epsilon)\Gamma(\epsilon)^2 \left(p_2^2 p_3^2\right)^{-\epsilon}$	$\frac{\Gamma(1-\alpha_2-\epsilon,\alpha_1+\alpha_2+\epsilon,\alpha_2+\alpha_3+\epsilon)}{\Gamma(1+\alpha_1,1+\alpha_2,1+\alpha_3)}$
3	$s^\epsilon \Gamma(1-\epsilon)\Gamma(\epsilon)^2 \left(p_3^2 p_4^2\right)^{-\epsilon}$	$\frac{\Gamma(1-\alpha_3-\epsilon,\alpha_3+\alpha_4+\epsilon,\alpha_2+\alpha_3+\epsilon)}{\Gamma(1+\alpha_2,1+\alpha_3,1+\alpha_4)} u_2^{\alpha_1+\alpha_2+\epsilon}$
4	$t^\epsilon \Gamma(1-\epsilon)\Gamma(\epsilon)^2 \left(p_4^2 p_1^2\right)^{-\epsilon}$	$\frac{\Gamma(1-\alpha_4-\epsilon,\alpha_3+\alpha_4+\epsilon,\alpha_1+\alpha_4+\epsilon)}{\Gamma(1+\alpha_1,1+\alpha_3,1+\alpha_4)} u_1^{\alpha_2+\alpha_3+\epsilon} u_2^{\alpha_1+\alpha_2+\epsilon}$
5	$\Gamma(-\epsilon)^2\Gamma(\epsilon) \left(p_1^2\right)^{-\epsilon}/\Gamma(-2\epsilon)$	0
6	$\Gamma(-\epsilon)^2\Gamma(\epsilon) \left(p_2^2\right)^{-\epsilon}/\Gamma(-2\epsilon)$	0
7	$\Gamma(-\epsilon)^2\Gamma(\epsilon) \left(p_3^2\right)^{-\epsilon}/\Gamma(-2\epsilon)$	0
8	$\Gamma(-\epsilon)^2\Gamma(\epsilon) \left(p_4^2\right)^{-\epsilon}/\Gamma(-2\epsilon)$	0
9	Massless box (hard region)	0



$$u_1 = \frac{p_1^2 p_3^2}{st}, \quad u_2 = \frac{p_2^2 p_4^2}{st}$$

$$B = \log u_1 \log u_2 + 2\zeta_2 + O(u_1, u_2)$$

This allows us to compute small m expansions of:



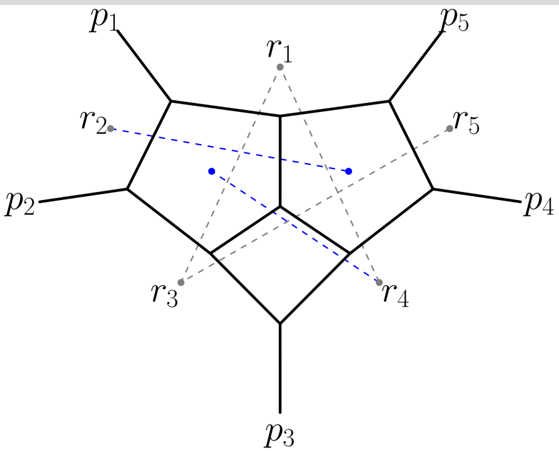
$$p_i^2 = r_{i,i+1}^2 = m_i^2 \text{ and } (p_i + p_{i-1})^2 = r_{i-1,i+1}^2 = s_i$$

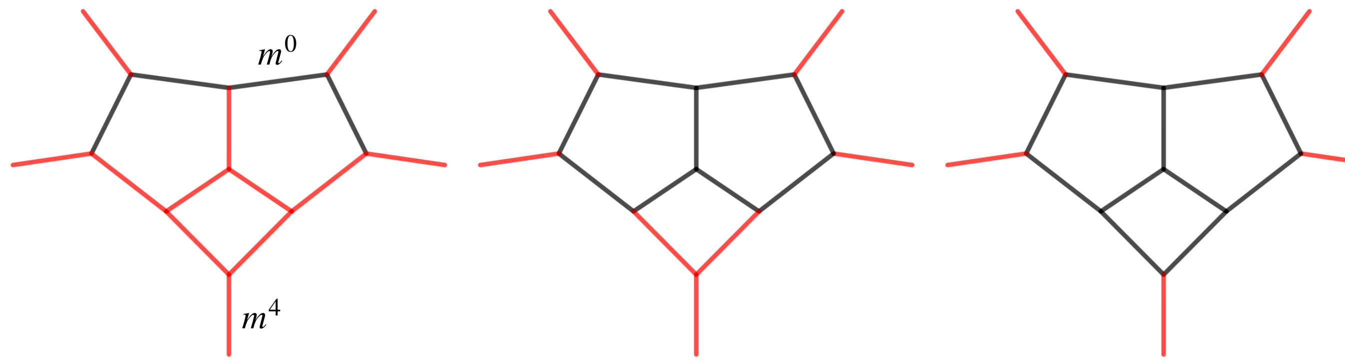
$$v_i = \frac{r_{i-1,i+1}^2 r_{i-2,i+2}^2}{r_{i-2,i+1}^2 r_{i+2,i-1}^2} = \frac{s_i m_{i+2}^2}{s_{i+2} s_{i-2}}$$

$$L_i = \log v_i$$

$$\begin{aligned}
\mathcal{I}_3^{(3)} = {}& \frac{1}{12}\left(2L_2L_1^3 + 3L_2L_3L_1^2 + 3L_2L_4L_1^2 + 6L_2L_3L_4L_1 + L_3L_4^3 + 3L_2L_3L_4^2\right) L_5^2 \\
&+\frac{1}{6}\big(2L_2L_1^3 - 2L_5L_1^3 + 3L_5^2L_1^2 + 3L_2L_3L_1^2 + 3L_2L_4L_1^2 + 6L_2L_5L_1^2 - 3L_3L_5L_1^2 - 3L_4L_5L_1^2 + 3L_2L_5^2L_1 \\
&+6L_4L_5^2L_1 + 6L_2L_3L_4L_1 + 12L_2L_3L_5L_1 - 6L_3L_4L_5L_1 + L_3L_4^3 + 3L_2L_3L_4^2 + 3L_4^2L_5^2 + 3L_2L_3L_5^2 \\
&+ 3L_2L_4L_5^2 + 3L_3L_4L_5^2 - L_4^3L_5 - 3L_2L_4^2L_5 + 12L_2L_3L_4L_5\big)\zeta_2 - \frac{1}{6}\big(2L_1^3 + 3L_3L_1^2 + 3L_4L_1^2 \\
&+ 6L_3L_4L_1 - L_4^3 - 3L_2L_4^2 - 3L_2L_5^2 + 6L_4L_5^2 - 6L_2L_4L_5 - 6L_3L_4L_5\big)\zeta_3 + \frac{1}{4}\big(10L_4^2 + 20L_1L_4 \\
&-11L_2L_4 + L_3L_4 + 20L_5L_4 + 10L_5^2 + 10L_1L_2 - 20L_1L_3 + 42L_2L_3 + 30L_1L_5 + 6L_2L_5 + 4L_3L_5\big)\zeta_4 \\
&+ 2(L_2 + L_3)\zeta_5 - \big(3L_5 + L_1 - 3L_2 - L_3 + 4L_4\big)\zeta_2\zeta_3 - 3\zeta_3^2 + 28\zeta_6 + O(v_i)\ .
\end{aligned}$$

Not all sunshine and rainbows :(



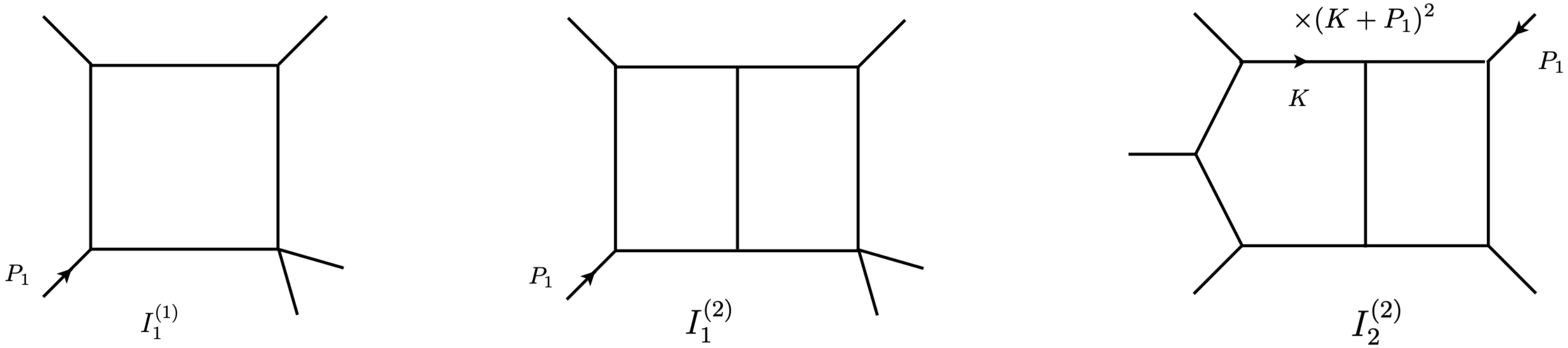


$$
\begin{aligned}
\mathcal{I}_5^{(3)} = \frac{1}{12}\big(&L_5^2L_1^3 + L_2L_5L_1^3 + 3L_4L_5^2L_1^2 + 3L_2L_4L_5L_1^2 + 6L_3L_4L_5^2L_1 + 3L_2L_3^2L_4L_1 + L_2L_3^3L_4 \\
&+6L_2L_3L_4L_5L_1\big)L_2 + \frac{1}{6}\big(-L_2L_3^3 + L_4L_3^3 - 3L_1L_2L_3^2 + 3L_1L_4L_3^2 + 6L_2L_4L_3^2 + 6L_1L_2^2L_3 - 3L_1L_5^2L_3 \\
&-3L_2L_5^2L_3 + 3L_4L_5^2L_3 + 3L_2^2L_4L_3 + 6L_1L_2L_4L_3 - 3L_2^2L_5L_3 + 12L_2L_4L_5L_3 + 3L_1^2L_2^2 + 3L_1L_2L_5^2 \\
&+6L_1L_4L_5^2 + 3L_1L_2^2L_4 + 3L_1L_2^2L_5 + 6L_1^2L_2L_5 + 12L_1L_2L_4L_5\big)\zeta_2 + \frac{1}{6}\big(6L_5^2L_3 - L_3^3 - 3L_1L_3^2 + 3L_2^2L_3 \\
&+6L_1L_2L_3 - 6L_1L_4L_3 - 12L_2L_4L_3 + 6L_1L_5L_3 - 6L_4L_5L_3 - 6L_1L_2^2 + 3L_1L_5^2 + 3L_2L_5^2 - 3L_4L_5^2 \\
&+6L_2^2L_4 + 6L_1L_2L_4 + 3L_2^2L_5\big)\zeta_3 + \frac{1}{4}\big(3L_2^2 + 41L_1L_2 + 20L_3L_2 + L_4L_2 + 28L_5L_2 - 10L_3^2 - 4L_5^2 \\
&+12L_1L_3 + 9L_1L_4 + 9L_3L_4 + 9L_1L_5 - 22L_3L_5 + 31L_4L_5\big)\zeta_4 + \big(7L_3 - 4L_1 - 7L_2 + 2L_4 - 4L_5\big)\zeta_2\zeta_3 \\
&- 2\big(3L_1 + L_2 + 2L_3 - 2L_5\big)\zeta_5 + c_1\zeta_6 + c_2\zeta_3^2 + O(v_i)\,.
\end{aligned}
$$

2 loop results.

$$p_i^2 = r_{i,i+1}^2 = m_i^2 \text{ and } (p_i + p_{i-1})^2 = r_{i-1,i+1}^2 = s_i \qquad v_i = \frac{r_{i-1,i+1}^2 r_{i-2,i+2}^2}{r_{i-2,i+1}^2 r_{i+2,i-1}^2} = \frac{s_i m^2}{s_{i+2} s_{i-2}}$$

$$L_i = \log v_i$$



$$I_1^{(1)} = L_3 L_4 + L_4 L_5 + 2\zeta_2 \,,$$

$$I_1^{(2)} = \tfrac{1}{4} L_4^2 (L_3 + L_5)^2 + \tfrac{1}{2}(L_3^2 + 4L_3L_4 + L_4^2 + 2L_3L_5 + 4L_4L_5 + L_5^2)\zeta_2 + \tfrac{21}{2}\zeta_4 \,,$$

$$\begin{aligned} I_2^{(2)} = &\tfrac{1}{2} L_1 (L_2^2 L_3 + 2L_2L_3L_4 + 2L_3L_4L_5 + L_4L_5^2) \\ &+ \tfrac{1}{2}(4L_1L_2 - L_2^2 + 2L_1L_3 + 2L_1L_4 - 2L_2L_4 + 4L_3L_4 + 4L_1L_5 - 2L_3L_5 - L_5^2)\zeta_2 \\ &+ (L_3 + L_4 - 2L_1)\zeta_3 + 5\zeta_4 \,, \end{aligned}$$

Combining all integrals together

- **We can define the following combination of log's:**

$$\mathbb{L}_0^2(w) = \sum_{i=1}^{5} \log^2 w_i \,, \qquad \mathbb{L}_1^2(w) = \sum_{i=1}^{5} \log w_i \log w_{i+1} \,, \qquad \mathbb{L}_2^2(w) = \sum_{i=1}^{5} \log w_i \log w_{i+2}$$

- **Then up to 3loops we have for $\log M_5$:**

$$g^2 = \frac{g_{\mathrm{YM}}^2 N_c}{16\pi^2}$$

$$\log M_5 = \left(\zeta_2 g^4 - 16\zeta_4 g^6\right) \mathbb{L}_0^2(v) + \left(-g^2 + 2\zeta_2 g^4 - \tfrac{59}{2}\zeta_4 g^6\right) \mathbb{L}_1^2(v) + \left(\zeta_2 g^4 - \tfrac{37}{2}\zeta_4 g^6\right) \mathbb{L}_2^2(v) + d(g)$$

- **Or in different basis of (dual)conformal cross ratios :** $u_i = \dfrac{m^4}{s_{i+1}s_{i+2}}$

$$\log M_5 = \left(-\tfrac{1}{4}g^2 + \zeta_2 g^4 - \tfrac{27}{2}\zeta_4 g^6\right) \mathbb{L}_0^2(u) + \left(-\tfrac{1}{2}g^2 + \zeta_2 g^4 - \tfrac{27}{2}\zeta_4 g^6\right) \mathbb{L}_1^2(u) + \left(\tfrac{1}{2}g^2 - \zeta_2 g^4 + 11\zeta_4 g^6\right) \mathbb{L}_2^2(u) + d(g)$$

- **We see that up to three loops finite parts as well as log's m exponentiate!**

Combining all integrals together

- **We can define the following combination of log's:**

$$\mathbb{L}_0^2(w) = \sum_{i=1}^{5} \log^2 w_i \,, \qquad \mathbb{L}_1^2(w) = \sum_{i=1}^{5} \log w_i \log w_{i+1} \,, \qquad \mathbb{L}_2^2(w) = \sum_{i=1}^{5} \log w_i \log w_{i+2}$$

- **Then up to 3loops we have for $\log M_5$:**

$$g^2 = \frac{g_{\rm YM}^2 N_c}{16\pi^2}$$

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- **Or in differen basis of (dual)conformal cross ratios :** $u_i = \dfrac{m^4}{s_{i+1}s_{i+2}}$

$$\log M_5 = \left(-\tfrac{1}{4}g^2 + \zeta_2 g^4 - \tfrac{27}{2}\zeta_4 g^6\right) \mathbb{L}_0^2(u) + \left(-\tfrac{1}{2}g^2 + \zeta_2 g^4 - \tfrac{27}{2}\zeta_4 g^6\right) \mathbb{L}_1^2(u) + \left(\tfrac{1}{2}g^2 - \zeta_2 g^4 + 11\zeta_4 g^6\right) \mathbb{L}_2^2(u) + d(g)$$

- **No $\zeta(2k+1)$ up to 3loops. No $\zeta(2k+1)$ Zetas on this locust of coulomb branch ?**
- **Maximal trancendentality holds brilliantly!**

IR structure up to 3 loops

- **Both representations of the amplitude**

$$v_i = \frac{s_i m^2}{s_{i+2} s_{i-2}}$$

$$\log M_5 = \left(\zeta_2 g^4 - 16\zeta_4 g^6\right)\mathbb{L}_0^2(v) + \left(-g^2 + 2\zeta_2 g^4 - \tfrac{59}{2}\zeta_4 g^6\right)\mathbb{L}_1^2(v) + \left(\zeta_2 g^4 - \tfrac{37}{2}\zeta_4 g^6\right)\mathbb{L}_2^2(v) + d(g)$$

$$\log M_5 = \left(-\tfrac{1}{4}g^2 + \zeta_2 g^4 - \tfrac{27}{2}\zeta_4 g^6\right)\mathbb{L}_0^2(u) + \left(-\tfrac{1}{2}g^2 + \zeta_2 g^4 - \tfrac{27}{2}\zeta_4 g^6\right)\mathbb{L}_1^2(u) + \left(\tfrac{1}{2}g^2 - \zeta_2 g^4 + 11\zeta_4 g^6\right)\mathbb{L}_2^2(u) + d(g)$$

- **satisfy, as expected:**

$$u_i = \frac{m^4}{s_{i+1} s_{i+2}}$$

$$\log M_5 = -\sum_{i=1}^{5} \frac{\Gamma_{oct}(g)}{2} \log^2\left(\frac{m^2}{s_i}\right) + f_5\left(\{s_i/s_j\}, g\right) + \frac{n}{2} D(g) + O(m^2)$$

IR factorization into product of Sudakov form factors holds !

All loop constraints and conjectures

- **Based on three loop result:**

Looks suspiciously like Γ_{cusp}

$$\log M_5 = \left(-\tfrac{1}{4}g^2 + \zeta_2 g^4 - \tfrac{27}{2}\zeta_4 g^6\right)\mathbb{L}_0^2(u) + \left(-\tfrac{1}{2}g^2 + \zeta_2 g^4 - \tfrac{27}{2}\zeta_4 g^6\right)\mathbb{L}_1^2(u) + \left(\tfrac{1}{2}g^2 - \zeta_2 g^4 + 11\zeta_4 g^6\right)\mathbb{L}_2^2(u) + d(g)$$

- **It is natural to assume that to all loops:**

$$\log M_5 = \gamma_0(g)\,\mathbb{L}_0^2(u) + \gamma_1(g)\,\mathbb{L}_1^2(u) + \gamma_2(g)\,\mathbb{L}_2^2(u) + d(g)$$

- **Where $\gamma_i(g)$ must satisfy:**

$$\gamma_0(g) + \gamma_1(g) + \gamma_2(g) = -\frac{1}{16}\Gamma_{\text{oct}}(g)$$

Up to three loops indeed this is the case!

- **However we where not able to find <u>unique</u> all loop conjectures for all** $\gamma_i(g)$

All loop constraints and conjectures

$$\mathbb{L}_0^2(w) = \sum_{i=1}^{5} \log^2 w_i \,, \qquad \mathbb{L}_1^2(w) = \sum_{i=1}^{5} \log w_i \log w_{i+1} \,, \qquad \mathbb{L}_2^2(w) = \sum_{i=1}^{5} \log w_i \log w_{i+2}$$

$$\log M_5 = \gamma_0(g)\, \mathbb{L}_0^2(u) + \gamma_1(g)\, \mathbb{L}_1^2(u) + \gamma_2(g)\, \mathbb{L}_2^2(u) + d(g)$$

- **Why we think that there must be “simple” solution for $\gamma_i(g)$ to all loops ?**
- **Compere formula above with:**

$$\mathcal{E}(u_i) = \lim_{\epsilon\to 0} \frac{\mathcal{A}_6(s_{ij},\epsilon)}{\mathcal{A}_6^{\text{BDS-like}}(s_{ij},\epsilon)} \qquad u_1 = \frac{s_{12}s_{45}}{s_{123}s_{345}}\,,\ u_2 = \frac{s_{23}s_{56}}{s_{234}s_{123}}\,,\ u_3 = \frac{s_{34}s_{61}}{s_{345}s_{234}}$$

$$\ln\mathcal{E} = -\frac{\Gamma_{\text{oct}}}{24}\ln^2(u_1u_2u_3) - \frac{\Gamma_{\text{hex}}}{24}\sum_{i=1}^{3}\ln^2\left(\frac{u_i}{u_{i+1}}\right) + C_0 + \mathcal{O}(u_i),$$

- **Result above is for the massless case, “origin” of Coulomb branch, but still looks suspiciously similar!**

All loop constraints and conjectures

$$\mathbb{L}_0^2(w) = \sum_{i=1}^{5} \log^2 w_i\,, \qquad \mathbb{L}_1^2(w) = \sum_{i=1}^{5} \log w_i \log w_{i+1}\,, \qquad \mathbb{L}_2^2(w) = \sum_{i=1}^{5} \log w_i \log w_{i+2}$$

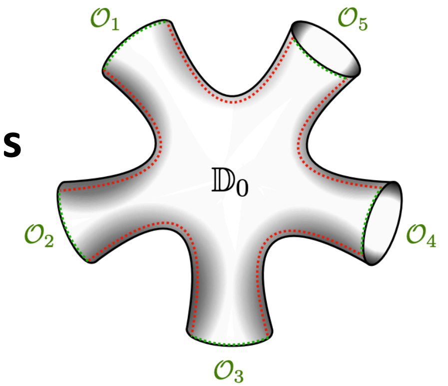
$$\log M_5 = \gamma_0(g)\, \mathbb{L}_0^2(u) + \gamma_1(g)\, \mathbb{L}_1^2(u) + \gamma_2(g)\, \mathbb{L}_2^2(u) + d(g)$$

- **Why we think that there must be "simple" solution for $\gamma_i(g)$ to all loops ?**

We expect that this amplitude is dual to the correlators of $[\mathrm{tr}(\beta_i\, \Phi_i)]^K$ operators for K>>1?

$$G_5 = \frac{\partial^{2K}}{\partial^K \beta \partial^K \gamma} \langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_5) \rangle \Big|_{\beta=\gamma=0} = \frac{\mathbb{D}_0(u_i, a)^2}{(x_{12}^2 \ldots x_{51}^2)^K}$$

$$\boxed{\mathbb{D}_0 = M_5 \quad ?}$$



- **Such correlators are expected to be related to integrable systems**
- **BMN spin chains, 2d worldsheet dynamics**

Komatsu et al. 15, Basso et al 15, Coronado 15,18 ,Gonçalves et al. 22, 24

Conclusions and open questions

- **More loops & legs! n=6 2loop, n=5 4loop**
- **Integrability análisis of n=5 case in m<<1 limit.**
- **Mixed case, where both massless and massive particles are present**
- **Real life case- QCD computation of off-shell sudakov form factor**

Different Gammas:

$$\Gamma_{cusp}(a) = a\left[\mathbb{Q}\frac{1}{1+\mathbb{K}}\right]_{11},$$

where the elements $(\mathbb{K})_{nm}$ and $(\mathbb{Q})_{nm}$ are given by:

$$(\mathbb{K})_{nm} = 2m(-1)^{m(n+1)}\int_0^\infty \frac{dt}{t}\frac{J_n(\sqrt{2a}t)J_m(\sqrt{2a}t)}{e^t-1}, \quad (\mathbb{Q})_{nm} = \delta_{nm}\, n\,(-1)^{n+1}.$$

This definition is especially useful in weak coupling limit. For example one can easily get:

```
GammaCuspPhys + O[g] ^ (L + 2)
```

$$1 - \frac{\pi^2 g^2}{3} + \frac{11\,\pi^4 g^4}{45} + \left(-\frac{73\,\pi^6}{315} + 8\,\text{Zeta}[3]^2\right) g^6 + \left(\frac{3548\,\pi^8}{14\,175} - \frac{16}{3}\pi^2\,\text{Zeta}[3]^2 - 160\,\text{Zeta}[3]\,\text{Zeta}[5]\right) g^8 +$$
$$\left(-\frac{136\,883\,\pi^{10}}{467\,775} + \frac{64}{15}\pi^4\,\text{Zeta}[3]^2 + \frac{320}{3}\pi^2\,\text{Zeta}[3]\,\text{Zeta}[5] + 816\,\text{Zeta}[5]^2 + 1680\,\text{Zeta}[3]\,\text{Zeta}[7]\right) g^{10} +$$
$$\left(\frac{15\,360\,178\,\pi^{12}}{42\,567\,525} - \frac{752}{189}\pi^6\,\text{Zeta}[3]^2 + 64\,\text{Zeta}[3]^4 - \frac{1312}{15}\pi^4\,\text{Zeta}[3]\,\text{Zeta}[5] - 544\,\pi^2\,\text{Zeta}[5]^2 -\right.$$
$$\left. 1120\,\pi^2\,\text{Zeta}[3]\,\text{Zeta}[7] - 17\,472\,\text{Zeta}[5]\,\text{Zeta}[7] - 18\,816\,\text{Zeta}[3]\,\text{Zeta}[9]\right) g^{12} + O[g]^{14}$$

$$g^2 \sim a$$

Different Gammas:

arXiv: 2001.05460

$$\Gamma(a|\alpha) = a\left[\frac{1}{1+\mathbb{K}(\alpha)}\right]_{11},$$

even x even etc.

where $\mathbb{K}(\alpha)$ is now given by

$$\mathbb{K}(\alpha) = 2\cos(\alpha)\begin{pmatrix}\cos(\alpha)\mathbb{K}_{\circ\circ} & \sin(\alpha)\mathbb{K}_{\circ\bullet}\\ \sin(\alpha)\mathbb{K}_{\bullet\circ} & \cos(\alpha)\mathbb{K}_{\bullet\bullet}\end{pmatrix}$$

with $\alpha = 0, \pi/4$ and $\pi/3$ for $\Gamma_{\rm oct}, \Gamma_{\rm cusp}$ and $\Gamma_{\rm hex}$:

$$\frac{\Gamma(g^2|\alpha)}{4g^2} = 1 - 4c^2\zeta_2 g^2 + 8c^2(3+5c^2)\zeta_4 g^4 - 8c^2\left[(25+42c^2+35c^4)\zeta_6 + 4s^2\zeta_3^2\right]g^6 + \ldots$$

$c = \cos(\alpha)$
$s = \sin(\alpha)$

$g^2 \sim a$

Different perspective …

Web of dualities in planar N=4 SYM

1 **$A_n \leftrightarrow W_n$: MHV-амплитуды ↔ светоподобные Wilson loops.**

Основание: dual x_i, dual conformal symmetry, AdS/CFT.
Проверки: strong coupling, 1–2 loops, много дальнейших тестов.

2 **$A_n \leftrightarrow$ super / twistor Wilson loop: гипотеза для полного super-S-matrix.**

Проверки: деревья, 1-loop integrands, 2-loop MHV.

3 **G_n и $G_n(x, \theta, \bar{\theta})$ в light-like limit ↔ Wilson loops / superamplitudes.**

Проверки: много примеров; 4-point correlator/amplitude известен до 12 loops.

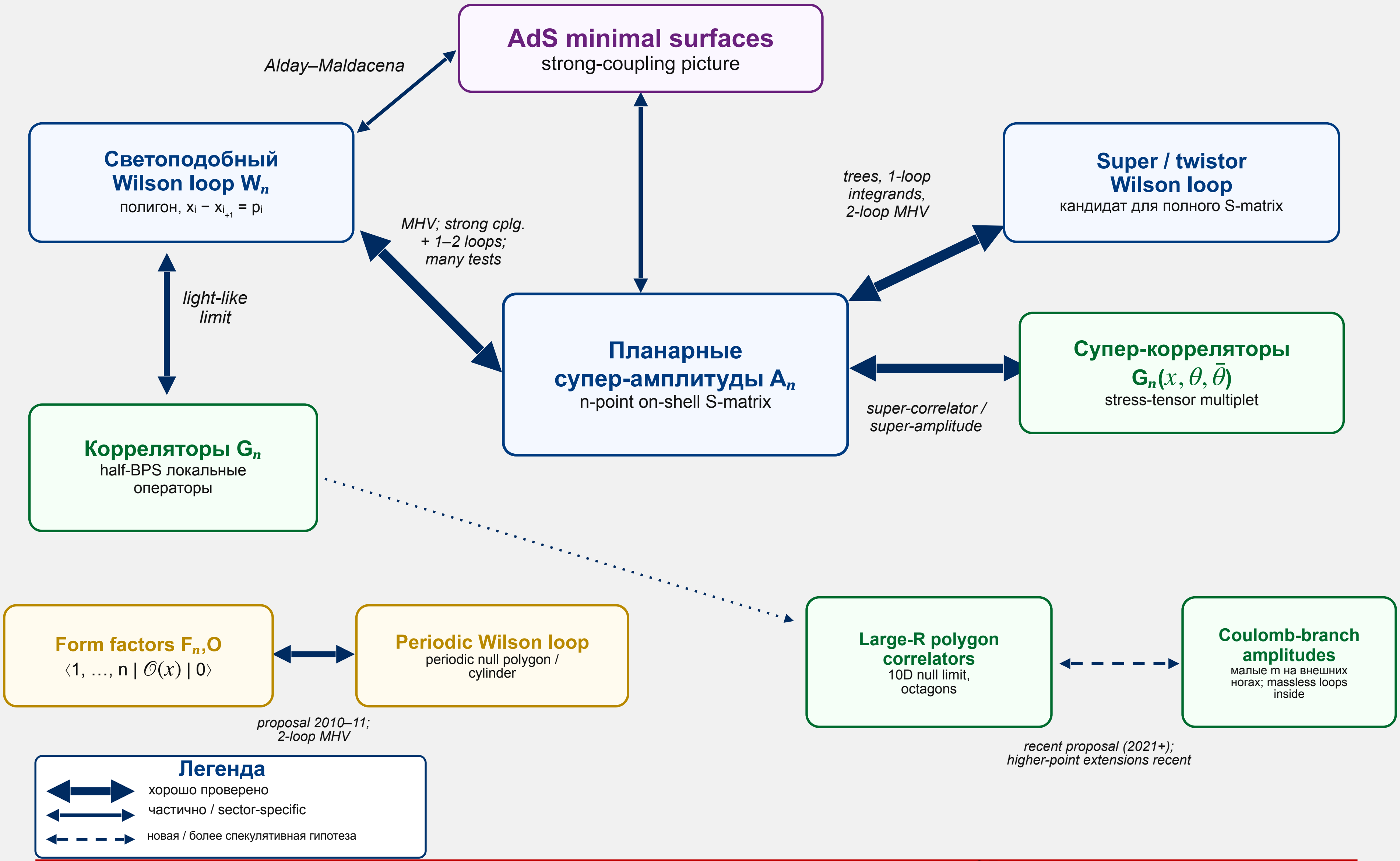
4 **F_n,O ↔ periodic Wilson loop.**

Предложено в 2010–2011; explicit 2-loop MHV checks есть.

5 **Large-R polygon correlators ↔ Coulomb-branch amplitudes.**

Недавняя 10D / null-limit гипотеза; higher-point extensions появились недавно;
evidence пока ограничен.

Web of dualities in planar N=4 SYM



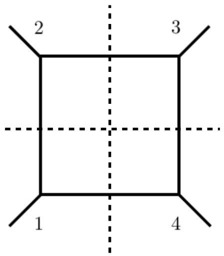
Alday–Maldacena (2007); Drummond et al. (2007–2010); Mason–Skinner (2010); Eden et al. (2011); Caron–Huot et al. (2021); recent higher-point large-R work (2025).

Amplitudes/Willson loops/Correlation function **duality**

Amplitudes

n=4 illustration

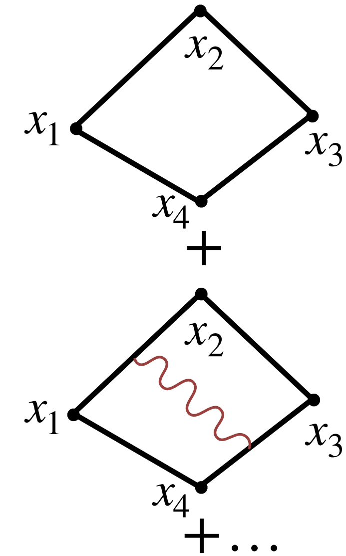
$$\frac{A_n}{A_n^{tree}} \quad 1+g^2\left(st \right.$$



$$\left. \right) + \ldots$$

Wilson loops

$$\langle W(C_n) \rangle$$



$$x_{ij} = x_i - x_j = \sum_{k=1}^{j-1} p_k$$

$$x_{ii+1}^2 = p_i^2 = 0 \qquad D = 4 - 2\varepsilon$$

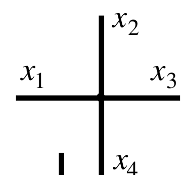
correlation functions

$$\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \rangle$$

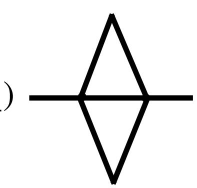
$$\mathcal{O} = \mathrm{tr}(\phi_{AB}^2)$$

n=4 illustration

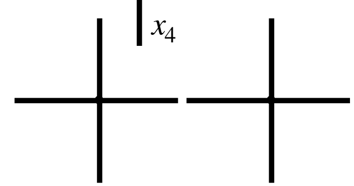
$1 + \ a \ 2x_{13}^2 x_{24}^2$



$+ \ 2a^2 \ (x_{13}^2 x_{24}^4 + x_{13}^4 x_{24}^2)$



$+a^2 \ x_{13}^4 x_{24}^4$



$+\ldots$

n=4 illustration

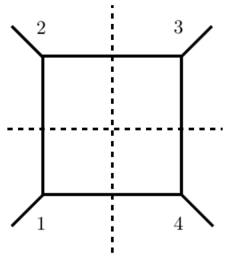
see arXiv: 0807.1889 1007.3246 and reference therein

Amplitudes/Correlation function duality

n=4 illustration

Amplitudes

$$\frac{A_n^{MHV}}{A_n^{MHV,tree}} = M_n$$

$$1 + g^2 \left(st \right. $$  $$\left. \right) + \dots$$

$$x_{ij} = x_i - x_j = \sum_{k=1}^{j-1} p_k$$

$$x_{ii+1}^2 = p_i^2 = 0 \qquad D = 4 - 2\varepsilon$$

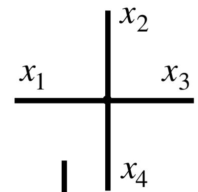
Duality "on the **integrand** level"

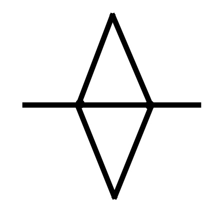
$$\lim_{x_{ii+1}^2 \to 0} \frac{G_n}{G_n^{tree}} = (M_n)^2$$

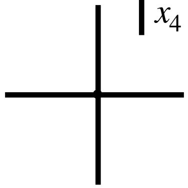
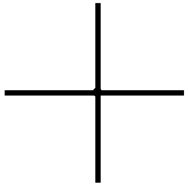
correlation functions

$$G_n = \langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \rangle$$

$$\mathcal{O} = \mathrm{tr}(\phi_{AB}^2)$$

$1 + \ a \ 2x_{13}^2 x_{24}^2$ 

$+ \ 2a^2 \ (x_{13}^2 x_{24}^4 + x_{13}^4 x_{24}^2)$ 

$+a^2 \ x_{13}^4 x_{24}^4$   $+ \dots$

n=4 illustration

see arXiv: 0807.1889 1007.3246 and reference therein

5p tree

$$\mathcal{A}_n = \widehat{\mathcal{A}}_n \, \delta^{(4)} \left(\sum_{i=1}^{5} Q_i \right) \delta^{(4)} \left(\sum_{i=1}^{5} \bar{Q}_i \right) \qquad \widehat{\mathcal{A}}_5^{(0)} = \frac{-\Omega}{S_{12} S_{23} S_{34} S_{45} S_{51}}$$

$$\Omega = \tfrac{1}{2} \langle B | \bar{B}] + \mathrm{cc}$$

$$\langle B| = -S_{34} \langle Q_5| + \langle Q_{34}| \bar{P}_{34} P_5 \,, \quad |\bar{B}] = -S_{51} |\bar{Q}_4] + \bar{P}_4 P_{51} |\bar{Q}_{51}]$$