Causal perturbative QFT and space-time geometry

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Based on the doctoral dissertation

Contents

Author's publications including results presented for defense

[A1] 'Causal perturbative QED and white noise", Theoretical and Mathematical Physics **218**, 483-502 (2024).

[A2] "Bogoliubov's Causal perturbative QED and white noise. Interacting fields", Theoretical and Mathematical Physics **211**, 775-816 (2022). Errata, 212, 1312-1314 (2022).

[A3] "Causal perturbative QFT and white noise". Infinite Dimensional Analysis, Quantum Probability and Related Topics **26**, 2350012 (54 pages) (2024).

[A4] "Bogoliubov's Causal perturbative QED with Hida operators", International Journal of Modern Physics A, **37**, No. 20n21, 2243024 (2022).

[A5] "Positivity of the invariant kernel underlying quantum theory of the Coulomb field", Acta Phys. Polon **B 53**, 215 (2022).

Author's publications including results presented for defense (continued)

[A6] "A new theorem on the representation structure of the SL(2,C) group acting in the Hilbert space of the quantum Coulomb field", Acta Phys. Polon. **B 49**, 171 (2018).

[A7] "A commentary on single-photon wave function advocated by Białynicki-Birula", Acta Phys. Polon. **B 47**, 2163 (2016).

[A8] "Index Theory and Adiabatic Limit in QFT", Int. J. Theor. Phys. **52**, 2910 (2014).

[A9] " Feynman Integral in QFT and White Noise on a Compactified Version of a Space-Time With the Lie Group Structure", Theoretical and Mathematical Physics **223**, 166 (2025).

[A10] (with Wawrzycki, T.) "Representation Structure of the $SL(2,\mathbb{C})$ Acting in the Hilbert Space of the Quantum Coulomb Field", in press: Acta Phys. Polon. **B 56** (No 6).

[A11] "Causal Perturbative QFT and Space-time Geometry". arXiv: math-ph/180206719.

The main results

For causal perturbative QFT on the Minkowski space-time $\mathcal{M}_0$, with the Hida operators as the creation-annihilation operators and (first order) interaction Lagrangian $\mathcal{L}$, we have proved

R4 $\Big[$ The higher order contributions to interacting fields in the adiabatic limit $g \to 1$ exist as sums of generalized integral kernel operators in the sense of Obata $\Big] \Longleftarrow \Big[$ $\mathcal{L}$ is of first order **at most** in mass-less fields, and for the "'on mass-shell" normalization $\Big]$.

R5 $\Big[$ $\mathcal{L}$ is of order **greater than one** in mass-less fields $\Big] \Longrightarrow \Big[$ the higher order contributions to interacting fields in the adiabatic limit $g \to 1$ does not exist, even as sums of generalized integral kernel operators, and for each choice of normalization $\Big]$.

R6 We have given a rigorous construction of the Feynman integral averaging for the computation of the complete Green functions for QFT on the compactified EU using white noise calculus, giving a solution to the problem posed by Bogoliubov and Shirkov.

The main results (continued)

R7

- The infrared and ultraviolet asymptotic of the interacting fields in perturbative QED with Hida operators, involving several necessary massive charged fields coupled to the potential, exists for each (asymptotic) homogeneity degree whose real part is -1.

- The homogeneity component of degree -1 contains the component which coincides with the quantum theory of the infrared electric type fields due to Staruszkiewicz.

- We have proved that in the Staruszkiewicz theory the integer part of $\sqrt{\frac{\pi}{\mathfrak{e}^2}}$ distinguishes two regimes of the value of the total charge with substantially different classes of representations of $SL(2,\mathbb{C})$ acting in the eigenvalue $n\mathfrak{e}$ spaces of the total charge. For $n > \sqrt{\frac{\pi}{\mathfrak{e}^2}}$ they are all equivalent, and become nonequivalent for any two n-s less than $\sqrt{\frac{\pi}{\mathfrak{e}^2}}$.

This asymptotic is naturally and uniquely determined by the decomposition of the representation of $SL(2,\mathbb{C})$ acting in the Fock space of free fields.

The main results (continued)

Theorems R4-R5 should be compared with the following results, which we have obtained on EU, using white noise calculus and Hida operators, but which could have also been obtained with more traditional methods, and in case of QED and the scalar φ^4-theory, was obtained Segal and Zhou for QED and φ^4-theory on EU:

EU1 For the causal perturbative spinor QED on the Einstein Universe the interacting fields become ordinary operator valued distributions with the Noether integrals for interacting fields being ordinary self adjoint operators if and only if the charged field coupled to the potential is massive. Renormalization is in this case uniquely determined by the condition of preservation of the singularity degree: retarded and advanced parts of a causal distribution have the same singularity degree as the causal distribution itself.

EU2 $\Big[$For the causal perturbative QFT on the Einstein Universe the interacting fields become ordinary operator valued distributions with the Noether integrals for interacting fields being ordinary self adjoint operators$\Big]$ $\Longleftrightarrow$ $\Big[$ $\mathcal{L}$ is at most of first order in massless fields.$\Big]$ Renormalization is in this case uniquely determined by the condition of preservation of the singularity degree.

A technical key point

In our approach the free fields and higher order contributions to interacting fields

$$\mathbb{A}(x) = \int \kappa_{0,1}(\boldsymbol{p};x)\partial_{\mathbf{p}}\, \mathrm{d}^3\boldsymbol{p} + \int \kappa_{1,0}(\boldsymbol{p};x)\partial_{\mathbf{p}}^{+}\, \mathrm{d}^3\boldsymbol{p},$$

$$\sum_{\ell,m} \Xi\left((\kappa_{\ell m}(x)\right) =$$

$$\sum_{\ell,m} \int \kappa_{\ell m}\big(\mathbf{p}_1,\ldots,\mathbf{p}_\ell,\mathbf{k}_1,\ldots,\mathbf{k}_m;x\big)\ \partial_{\mathbf{p}_1}^{+}\ldots\partial_{\mathbf{p}_\ell}^{+}\partial_{\mathbf{k}_1}\ldots\partial_{\mathbf{k}_m} d\mathbf{p}_1\ldots d\mathbf{p}_\ell d\mathbf{k}_1\ldots d\mathbf{k}_m$$

are finite sums of the generalized white noise integral kernel operators.

a) **Whitman approach**. $\kappa_{\ell m}(\phi)$ should represent a normalizable $\ell+m$ particle state as the spin-momenta function, rapidly decreasing in these variables.

b) **Berezin white noise approach**. It is sufficient that $\kappa_{\ell m}(\phi)$ represents (not necessary normalizable) $\ell+m$ particle generalized state, in the spin-momenta variables, *i.e.* it is sufficient that $\kappa_{\ell m}(\phi)$ is a distribution in $E^{*\otimes(\ell+m)}$, continuously depending on ϕ.

Experimental sigificance (example)

From theorems R4, R5, applied to the Weinberg-Salam model, it follows that the neutrino has a nonzero mass. Otherwise, the Lagrangian of interaction would contain monomials

$$\overline{\nu_e}(x)\gamma^\mu(1-\gamma_5)\nu_e(x)Z_\mu(x)$$

which are of degree two in the mass-less field ν_e, which would result in the nonexistence of the adiabatic limit for interacting fields, according to R4, R5.

Equivalence principle in QFT

(EP1) With each free field there is canonically associated linear wave equation. It is extended on general causal space-time by covariance requirement, so that locally and for the Minkowski metric reduces to the Minkowski wave equation. Single particle Hilbert space is constructed of **all** those solutions which can be smoothly extended over the whole space-time (constraint which in practice some of the continuous quantum numbers reduces to a discrete subset).

In particular the singularity order ω of a causal distribution (Green function) **should coincide** with the singularity order of the corresponding Minkowski distribution.

(EP2) The high energy limit of the scattering phenomena, experienced by any (approximately) freely falling observer, should be the same regardless of the location and orientation of the experimental equipment. In other words: the Standard Model, in the high-energy limit, is the same regardless of the location in space-time.

Question: **Are the conditions (EP1) and (EP2) consistent ?**
Answer: **No!**. **(EP1) need to be modified if we assume (EP2)**.

Singularity order ω at $x = 0$ is a local property

Causal $d \in \mathcal{S}^*$ has singularity order ω at $x = 0$ if $d_0 \in \mathcal{S}^*$, defined by the scaling $S_\lambda \phi(x) := \phi(\lambda x)$:

$$\lim_{\lambda \to +\infty} \left(\tfrac{1}{\lambda}\right)^\omega \langle d, S_\lambda \phi \rangle = \lim_{\lambda \to +\infty} \left(\tfrac{1}{\lambda}\right)^\omega \left(\tfrac{1}{\lambda}\right)^n \langle \widetilde{d}, S_{1/\lambda} \widetilde{\phi} \rangle = \langle d_0, \phi \rangle$$

exists and is nonzero.

In this case splitting of d into retarded and advanced parts

$$\langle \widetilde{\mathrm{ret}\, d}, \widetilde{\phi} \rangle = \langle \mathrm{ret}\, d, \phi \rangle := \langle d, \theta \Omega \phi \rangle = \langle \widetilde{d}, \widetilde{\theta} * \widetilde{\Omega \phi} \rangle$$

is well-defined, up to a finite dimensional subspace of functionals on $\ker \Omega$, where Ω – the idempotent operator projecting test space onto the subspace of elements, with first ω Taylor coefficients at zero, equal zero.

We can restrict ϕ to the closed subspace of elements supported in a compact closure $\overline{\mathcal{O}}$ of a relatively compact neighborhood $\mathcal{O}$ of $x = 0$.

Equivalence principle in QFT (corrected)

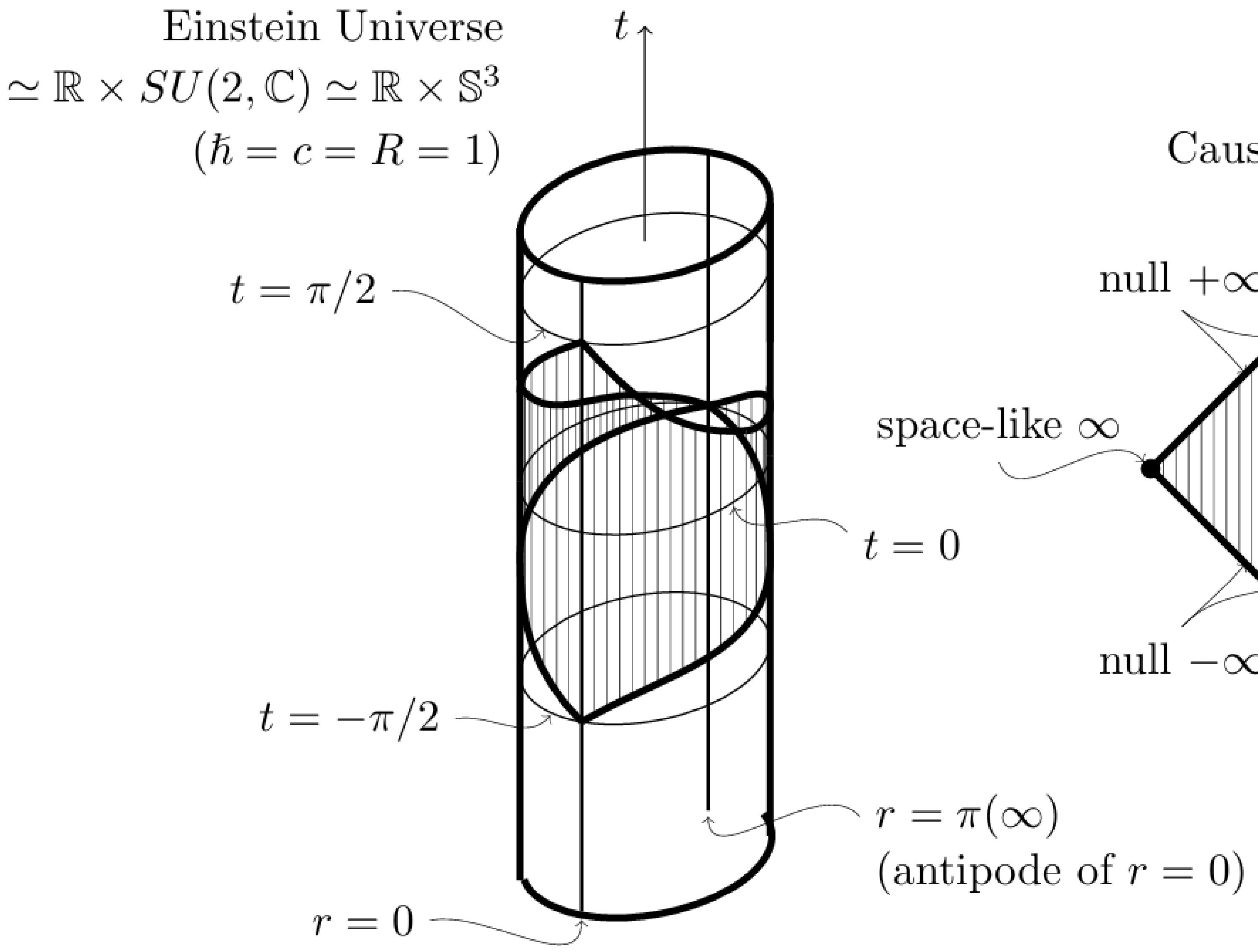
(EP1) (Corrected version). With each free field there is canonically associated linear wave equation. It is extended on general causal space-time by covariance requirement, so that locally and for the Minkowski metric reduces to the Minkowski wave equation. Single particle Hilbert space is constructed of **not all** those solutions which can be smoothly extended over the whole space-time. Some of the solutions must be excluded not only by the topology, but also by the condition (EP2), so that the allowed spectra of quantum numbers is further reduced.

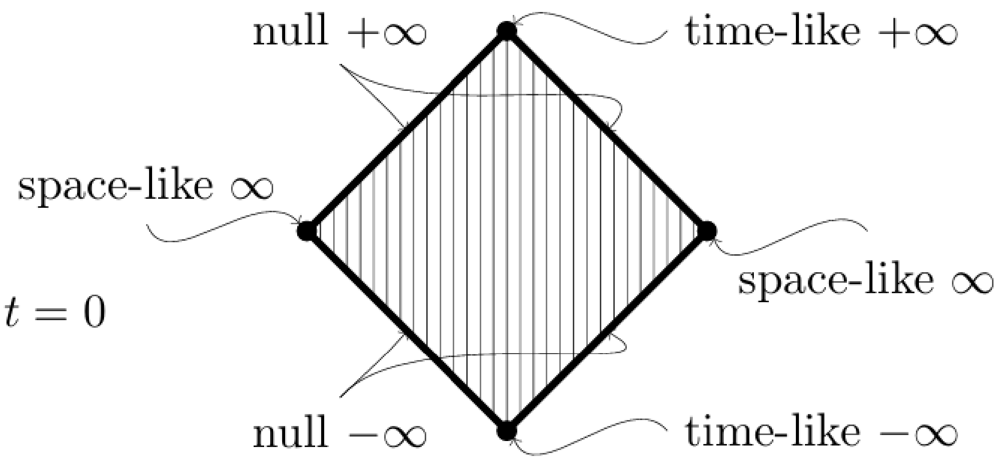
In general the singularity order ω of a causal distribution (Green function) **does not coincide** with the singularity order of the corresponding Minkowski distribution.

(EP2) The high energy limit of the scattering phenomena, experienced by any (approximately) freely falling observer, should be the same regardless of the location and orientation of the experimental equipment. In other words: the Standard Model, in the high-energy limit, is the same regardless of the location in space-time.

(EP2) intervenes through a restriction of causal character, put everywhere, *i.e.* globally, on the allowed *In* and *Out* plane wave states. We explain it using the example of the Einstein Universe (EU).

Equivalence principle in QFT (continued)





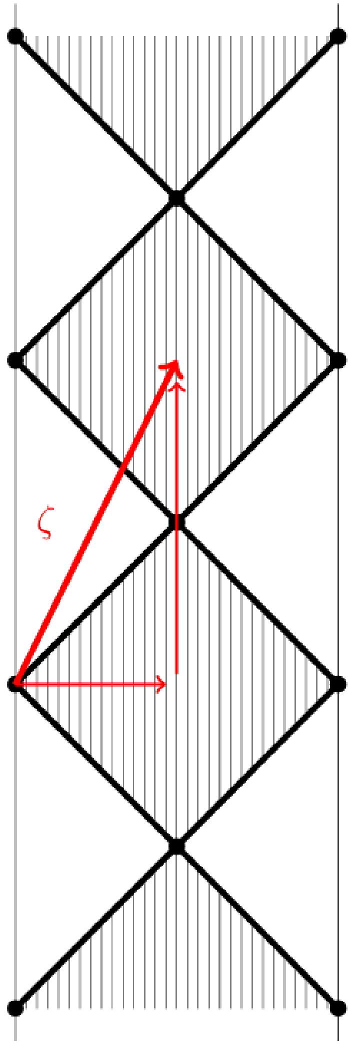
$\mathcal{M}_0$ causally periodically embedded into EU

Equivalence principle in QFT (continued)

$(\hbar = c = R = 1)$

$\zeta : (t, \boldsymbol{u}) \to (t + \pi, A\boldsymbol{u})$

ζ commutes with all conformal automorphisms of EU



Each free EU wave is locally a Minkowski mass packet, with $\mathcal{M}_0$ regarded as embedded into EU

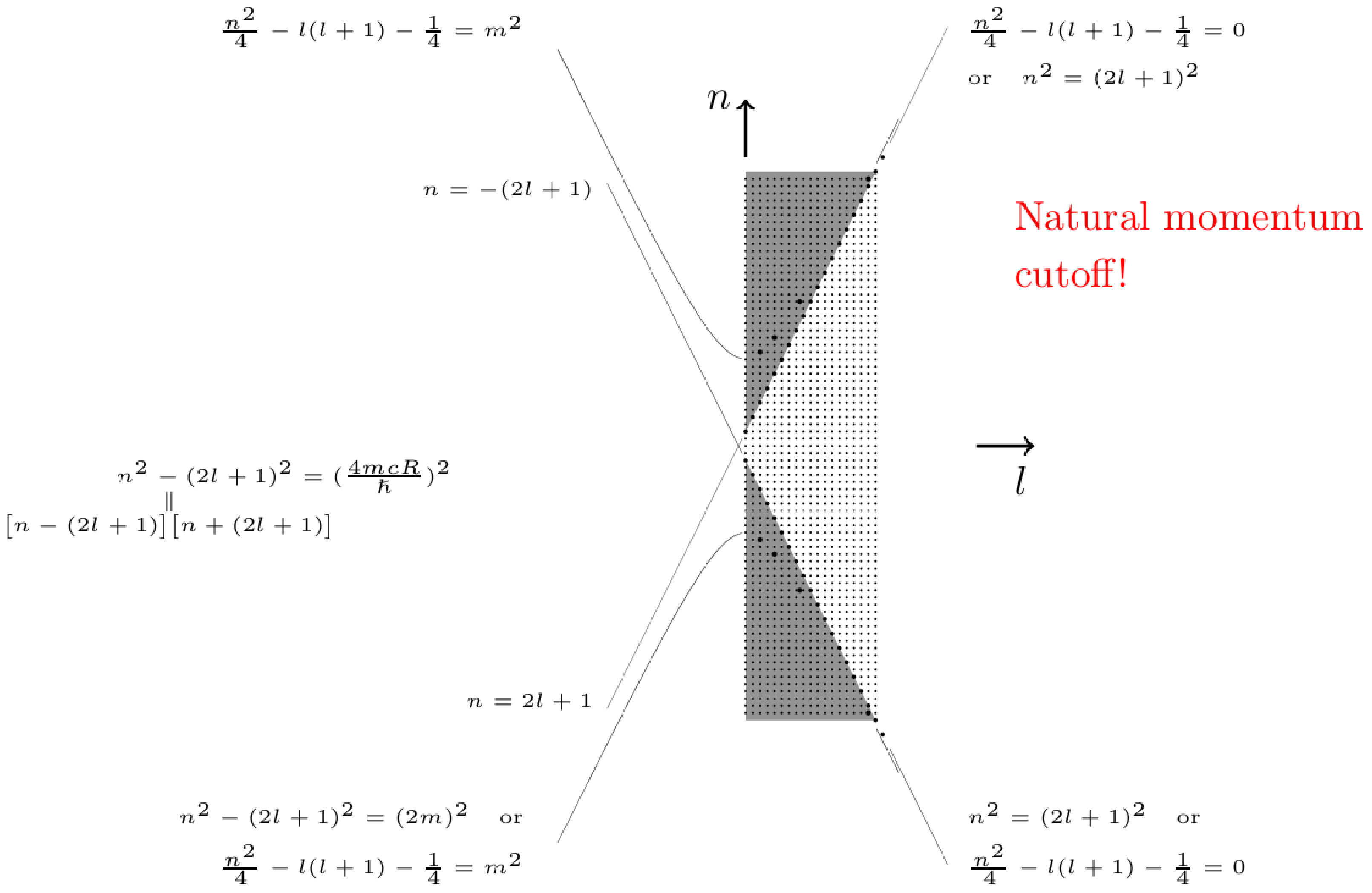
A Minkowski mass packet,
with $\mathcal{M}_0$ regarded as embedded into EU
which is extendible over whole EU,
regarded as a causal manifold,
is invariant either under ζ or under ζ^4
A physical free EU wave on EU – agrees with (EP2)

A Minkowski mass packet,
which cannot be extended over the whole EU,
is pathological, with the support of the FT,
regarded over $\mathcal{M}_0$ embedded into EU
lying outside the cone $p^2 > 0$
Nonphysical free EU wave on EU – disagrees with (EP2), as we do not observe neither tachyonic particles nor excess of degeneracy of Minkowski quantum numbers in scattering processes

[Paneitz, Segal, 1982]

Free fields on EU

finite support of Fourier transform
of massive waves on EU



Importance of finite support of FT of single particle massive wave functions on EU

$$\begin{aligned}\boldsymbol{\psi}^a(x) = \boldsymbol{\psi}^{(-)\,a}(x) + \boldsymbol{\psi}^{(+)\,a}(x) &= \Xi\big(\kappa_{01}(a,x)\big) + \Xi\big(\kappa_{10}(a,x)\big) \\ &= \sum_{s\in\{1,\dots,(2l+1)^2 2\},\widehat{n}\cdot\widehat{l}\in\mathscr{O}_+} \kappa_{01}(s,\widehat{n}\cdot\widehat{l};a,x) b_{{}_s}(\widehat{n}\cdot\widehat{l}) \\ &\quad + \sum_{s\in\{1,\dots,(2l+1)^2 2\},\widehat{n}\cdot\widehat{l}\in\mathscr{O}_-} \kappa_{10}(s,\widehat{n}\cdot\widehat{l};a,x) d_{{}_s}(\widehat{n}\cdot\widehat{l})^+,\end{aligned}$$

$$(s,\widehat{n}\cdot\widehat{l}) \mapsto \kappa_{0,1}(s,\widehat{n}\cdot\widehat{l};a,x) = \sum_{i,j\in\{-l,\dots,l\}} \sqrt{2l+1}\, u^a_{s\,{}_{ji}}(\widehat{n}\cdot\widehat{l})\widehat{n}(t)\widehat{l}_{{}_{ij}}(\boldsymbol{w}),$$

$$(s,\widehat{n}\cdot\widehat{l}) \mapsto \kappa_{1,0}(s,\widehat{n}\cdot\widehat{l};a,x) = \sum_{i,j\in\{-l,\dots,l\}} \sqrt{2l+1}\, v^a_{s\,{}_{ji}}(\widehat{n}\cdot\widehat{l})\overline{\widehat{n}(t)\widehat{l}_{{}_{ij}}(\boldsymbol{w})},$$

$$\text{here } x = t\times\boldsymbol{w}\in\mathbb{R}\times SU(2,\mathbb{C}).$$

Plane wave character $e_{p_0}(t)e_{\mathbf{p}}(\mathbf{x}) = e^{ip_0t - i\mathbf{p}\cdot\mathbf{x}}$ is replaced with the character $\widehat{n}(t)\widehat{l}(\boldsymbol{w}) = e^{int}\widehat{\boldsymbol{l}}(\boldsymbol{w})$ of the group $\mathbb{R}\times SU(2,\mathbb{C})$, where $\widehat{l}$ are the standard unitary irreducibles of the $SU(2,\mathbb{C})$ of weight l.

Importance of finite support of FT of single particle massive wave functions on EU (continued)

$$T\left[\mathcal{L}(x_1)\ldots\mathcal{L}(x_i)\ldots\mathcal{L}(x_j)\ldots\mathcal{L}(x_n)A(x)\right]$$

- Products of pairings of free massive fields have negative singularity order ω, with the pairings being finite linear combinations of analytic functions, and the retarded and abvanced parts of the pairing can be computed by ordinary multiplication by the step θ-function. This is true because of the finite suppport of FT of single particle massive wave functions!

- If $\mathcal{L}$ is of **first order at most in massless fields**, then we have **at most one pairing of massles fields**, joinning the same space-time pair (x_i, x_j). Such pairings have **negative** singularity order ω. Chronological product is uniquely defined

- If $\mathcal{L}$ is of **second, or higher, order in massless fields**, then we encounter products of **two, or more, pairings of massles fields**, joinning the same space-time pair (x_i, x_j). Such products have **nonnegative** singularity order ω, with non unique splitting, determined up to a free number of arbitrary constants depending on ω (*i.e.* with nontrivial renormalization needed)

Importance of finite support of FT of single particle massive wave functions on EU (spinor QED)

First order contributions to interacting $\boldsymbol{\psi}(x)$ and $A^{\mu}(x)$:

$$\boldsymbol{\psi}^{a\,(1)}(g=1;x) = -e \int\limits_{\mathbb{R}\times SU(2,\mathbb{C})} S^{aa_1}_{\text{ret}}\left(xx_1^{-1}\right)\gamma^{\nu_1\,a_1a_2}\boldsymbol{\psi}^{a_2}(x_1)A_{\nu_1}(x_1)\,\mathrm{d}^4x_1,$$

$$A^{\mu\,(1)}(g=1;x) = -e \int\limits_{\mathbb{R}\times SU(2,\mathbb{C})} D_0^{\text{av}\,\mu\nu}\left(x_1x^{-1}\right) :\boldsymbol{\psi}^{\sharp\,a_1}(x_1)\gamma_{\nu}^{a_1a_2}\boldsymbol{\psi}^{a_2}(x_1):\mathrm{d}^4x_1,$$

Examples of one-loop 2$^{\text{nd}}$-ord. contr. to int. $\boldsymbol{\psi}(x)$ and $A^{\mu}(x)$:

$$e^2 \int\limits_{\left[\mathbb{R}\times SU(2,\mathbb{C})\right]^{\times 2}} S^{(+)}_{\text{ret}}(xx_1^{-1})\Sigma_{\text{ret}}(x_1x_2^{-1})\boldsymbol{\psi}(x_2)\,\mathrm{d}^4x_1\mathrm{d}^4x_2,$$

$$e^2 \int\limits_{\left[\mathbb{R}\times SU(2,\mathbb{C})\right]^{\times 2}} D_0^{(-)\,\text{av}}\left(x_1x^{-1}\right)\Pi^{\text{av}\,\mu\nu}\left(x_2x_1^{-1}\right)A_{\nu}(x_2)\,\mathrm{d}^4x_1\,\mathrm{d}^4x_2,$$

$$\theta(x) = \theta(t\times\boldsymbol{w}) = \theta(t), \quad \Sigma_{\text{ret}}(x) \;:=\; \theta(x)D_0^{(-)}\left(x\right)\gamma^{\nu}S^{(-)}(x)\gamma_{\nu},$$

$$\Pi^{\text{av}\,\mu\nu}(x_1x_2^{-1}) \;:=\; i\theta\left(x_2x_1^{-1}\right)\text{tr}\left[\gamma^{\mu}S^{(-)}\left(x_1x_2^{-1}\right)\gamma^{\nu}S^{(+)}\left(x_2x_1^{-1}\right)\right],$$

A paradox resolved

- The results EU1, EU2 are consequences of the equivalence principle (EP1) and (EP2). EU1, EU2 , imply, e.g. that neutrinto should be massive, or that elementary electrically charged particle should be massive.
- Question: Why we have no such theoretical prediction in perturbative QFT on the Minkowski space-time $\mathcal{M}_0$?
- Answer: Because we regard the higher order contributions to S and to interacting fields, as operator distributions in Wightman sense. In the Berezein approach, with the creation-annihilation operators equal to Hida operators, we obtain results R4 and R5 which are in agreement with the results EU1 and EU2.

Constraints.
Example of QED with Lorentz gauge

Choice of spacetime and Lorentz condition are constraints.

- Choice of space-time $(\mathcal{M}, g)$
- Quantization (operator counterpart) only of the dynamical part of matter equations
$$\frac{\delta S_{matt}}{\delta \psi} = 0 \text{ e.g. } \Box A_{\text{int}} = j_{\text{int}}$$
- Constraints are counted only at the very end by putting the condition on the allowed states in which the average
$$\left\langle \text{constraint part of } \frac{\delta S_{matt}}{\delta \psi} \right\rangle = 0 \text{ e.g. } \langle \partial A_{\text{int}} \rangle = 0$$
of the constraint part is fulfilled, but constraint has no operator realization.

Gravity coupled to matter

$$\mathcal{L} = \mathcal{L}_{grav} + \alpha_{matt}\mathcal{L}_{matter}$$

$$S = \int \left[\mathcal{L}_{grav} + \alpha_{matt}\mathcal{L}_{matt}\right] d^4x = S_{grav} + \alpha_{matt} S_{matt}$$

$$\frac{\delta S_{matt}}{\delta \psi} = 0$$

$$G_{\mu\nu} = -\kappa \mathbb{T}_{\mu\nu},$$

$$\mathbb{T}_{\mu\nu} = \frac{\alpha_{matt}}{\kappa} \frac{1}{\sqrt{-g}} \frac{\delta S_{matt}}{\delta g_{\mu\nu}},$$

Gravity as a constraint

- Choice of space-time $(\mathcal{M}, g)$
- Quantization of the dynamical part of matter equations

$$\frac{\delta S_{matt}}{\delta \psi} = 0$$

- Constraints put on the states in which the average

$$\left\langle \text{constraint part of } \frac{\delta S_{matt}}{\delta \psi} \right\rangle = 0$$

of constraint part is fulfilled and

$$-\kappa \Delta \mathbb{T}_{\mu\nu}(\phi) \leq G_{\mu\nu}(\phi) - \big\langle -\kappa \mathbb{T}_{\mu\nu}(\phi) \big\rangle \leq \kappa \Delta \mathbb{T}_{\mu\nu}(\phi),$$

where

$$G(\phi) \stackrel{\text{df}}{=} \int G(x)\phi(x)\, d^4x$$

with the fluctuaction

$$\Delta \mathbb{T}_{\mu\nu}(\phi) = \sqrt{\langle \mathbb{T}_{\mu\nu}(\phi)^2 \rangle - \langle \mathbb{T}_{\mu\nu}(\phi) \rangle^2}$$

Conclusion

- Presented principles provide a consistent unification of gravity and quantum field theory without any need for quantizing gravity.
- Geometry of space-time cannot be considered independently of QFT. Space-time geometry not only intervenes through the causality condition (e.g. in the perturbative construction of the scattering operator), but is restricted by the consistency coming from the requirements put on the operators of interacting fields to be well-defined as well as by the condition put on the energy-momentum operator to be well-defined. In particular the Minkowski space-time is excluded by the principles of the last slide, as nonphysical.
- QFT on the Minkowski space-time $\mathcal{M}_0$ has application restricted only to high energy scattering processes, and with restricted meaning, with the interacting field operators and the energy-momentum tensor being generalized operators, which even after smearing with test function, are not operators acting within the Fock space.