Nonrenormalizable theories and finite formulation of QFT

In collaboration with P. Petrov and M. Shaposhnikov

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Dubna

1 Is it possible to proceed to all calculations in QFT without any divergences?

- In standard approaches in QFT $\rightarrow$ meet some **divergences** during calculations of loops...
- One of the goals of QFT is to compute **n-point Green's functions** $\rightarrow$ related to physical observables like particle lifetimes and cross-sections.
- Example: review the standard approach to the renormalization of these functions in the following theory:

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\mu \phi_0 \partial^\mu \phi_0 - \frac{m_0^2}{2}\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4.$$

The metric signature is $(-+++)$.

- Consider the one-particle-irreducible (OPI) two- and four-point Green functions in dimensional regularization. In standard approach they are given by (up to one loop)

$$\Gamma^{(2)}(k) = i(k^2 + m_0^2) + \frac{\lambda_0 \mu^{4-d}}{2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + m_0^2},$$

$$\Gamma^{(4)}(\kappa_i) = -i\lambda_0 \mu^{4-d} + \sum_{3 \text{ opt}} \frac{\lambda_0^2 \mu^{8-2d}}{2} \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 + m_0^2)} \frac{1}{(l + \kappa_i)^2 + m_0^2}.$$

- Since Green functions are directly related to observables, **they must be finite**.
- In the standard approach $\rightarrow$ regularise the infinities.
- Having regularised the UV divergent integrals $\rightarrow$ move to **renormalisation** $\rightarrow$ add counterterms to the Lagrangian and subtracts the divergences.

1 Simple example in standard approach: ϕ^4 theory

- After renormalization we arrive to finite answers:

$$\bar{\Gamma}^{(2)} = i(k^2 + m_1^2) + O(\lambda^2),$$

$$\bar{\Gamma}^{(4)} = -i\lambda_1 + \sum_{3 \text{ opt}} \frac{i\lambda_1^2}{32\pi^2} \int_0^1 dx \cdot \ln\Big(\frac{m_1^2}{x(1-x)\kappa_i^2 + m_1^2}\Big) + O(\lambda^3),$$

where λ_1 and m_1 are physical and finite parameters now.

1 Is it possible to proceed to all calculations in QFT without any divergences?

- In standard approach, although it uses the UV divergent integrals, but at the end of the day, this is just a **mapping** between the well-defined set of **finite parameters**, that characterise the theory and the set of **experimental observables**.
- Thus, from this "mapping argument" point of view it is quite **natural to require** the existence of the formulation of **QFT without infinities** at all.
- We would like to explore such a procedure, which provides no divergent expressions at any stage of the computation in QFT.

1 Is it possible to proceed to all calculations in QFT without any divergences? Yes!

More fundamental motivation:

- **Hierarchy problem** (e.g. why the Higgs mass is so much smaller than the Planck scale?).
- Consider the bare mass m_0 for Higgs field. The quantum correction **shifts** m_0^2 by a **huge quadratically cutoff** dependent amount:

$$\delta m_0 \sim f\Lambda^2,$$

  where Λ is some characteristic mass scale (say, Planck mass) and where f denotes some dimensionless coupling.
- To have physical mass

$$m_P^2 = m_0^2 + \delta m_0,$$

  of order ~ 125 GeV $\rightarrow$ we require an extremely fine-tuned and highly unnatural cancellation between m_0^2 and δm_0.

2 Divergence-free methods

- Such divergence-free methods have been invented already in the past.
- We stick to the scheme which is based on **Callan-Symanzik** (differential) equations. C. G. Callan'1970; A. S. Blaer, K. Young'1974 $\rightarrow$ was designed to prove the validity of the standard multiplicative renormalization program.
- But the solution of these differential equations with boundary conditions $\rightarrow$ renormalized n-point OPI Green functions!

We have learned a great deal. First, we have shown that the multiplicative renormalization scheme actually produces renormalized Green functions which have a finite $\Lambda \to \infty$ limit. Second, we have shown that the renormalized Green functions satisfy a set of "renormalization group" equations,

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} + n\gamma(\lambda)\right] \Gamma^{(n)}(p;\lambda,\mu) = -i\mu^2 \alpha(\lambda) \Gamma_\theta^{(n)}(0;p;\lambda,\mu)\,,$$

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} + n\gamma(\lambda) + \gamma_\theta(\lambda)\right] \Gamma_\theta^{(n)}(q;p;\lambda,\mu) = -i\mu^2 \alpha(\lambda) \Gamma_{\theta\theta}^{(n)}(0,q;p;\lambda,\mu)\,, \qquad (2.13)$$

which, together with the normalization conditions, allow one to systematically compute in a unique fashion (indeed, in a way which never encounters a divergent Feynman integral) the perturbation expansion of the renormalization parts. Thus, any two renormalization schemes which yield Green functions satisfying these equations will necessarily yield identical Green functions. The re-

C. G. Callan'1970

- In order to obtain CS equations, which only include renormalised (= finite) quantities, we firstly turn to the bare Lagrangian

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{m_0^2}{2}\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4.$$

- The CS approach is all based on the observation that differentiating the (bare) scalar field propagator with respect to m_0^2 yields (minus i times) two propagators:

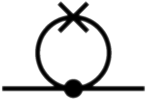
$$\frac{d}{dm_0^2}\left[\frac{-i}{k^2+m_0^2}\right] = -i\left(\frac{-i}{k^2+m_0^2}\right)^2.$$

- Obviously, **adding an extra propagator** to the diagram **reduces its degree** of divergence by two.

- Taking this derivative (and multiplying by $-i$) is now denoted as acting with a θ-**operation** on a propagator.
- **The algebraic** representation of the θ-operation is

$$\Gamma_\theta^{(n)}(k^2) \equiv -i \times \frac{d}{dm_0^2}\Gamma^{(n)}(k^2).$$

- Since this operation splits every propagator, one by one, in two parts $\rightarrow$ it equals to **inserting a new kind of "cross" vertex**, which comes with Feynman rule (-1):



2 Obtain CS equations

- In order to obtain equations → rewrite both sides of

$$\Gamma_\theta^{(n)}(k^2) \equiv -i \times \frac{d}{dm_0^2}\Gamma^{(n)}(k^2),$$

in terms of renormalized quantities.

- Need to know the relation between bare and renormalized correlation functions. Recall that the renormalized field and bare one are connected as

$$\phi_{ph} = \frac{\phi_0}{\sqrt{Z}}, \quad \Gamma^{(n)}(\lambda_0, m_0) = Z^{n/2}\bar{\Gamma}^{(n)}(\lambda, m).$$

- Also introduce

$$\Gamma_\theta^{(n)}(\lambda_0, m_0) = Z^{n/2} Z_\theta \bar{\Gamma}_\theta^{(n)}(\lambda, m).$$

- Use the following decomposition of the "bare" total derivative in terms of "physical" partial derivatives

$$\frac{d}{dm_0^2} = \frac{\partial m^2}{\partial m_0^2}\frac{\partial}{\partial m^2} + \frac{\partial \lambda}{\partial m_0^2}\frac{\partial}{\partial \lambda}.$$

- Then, the **first Callan-Symanzik equation** reads

$$2im^2 G\bar{\Gamma}_\theta^{(n)} = \left[n\gamma + \left(2m^2 \frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial \lambda}\right)\right]\bar{\Gamma}^{(n)},$$

where

$$G \equiv \left[\frac{\partial m^2}{\partial m_0^2}\right]^{-1} Z_\theta,$$

$$\beta \equiv 2m^2 \left[\frac{\partial m^2}{\partial m_0^2}\right]^{-1} \frac{\partial \lambda}{\partial m_0^2},$$

$$\gamma \equiv m^2 \left[\frac{\partial m^2}{\partial m_0^2}\right]^{-1} \frac{\partial \ln Z}{\partial m_0^2}.$$

- CS equation contains only finite quantities!

2 Obtain CS equations

- Now, recall that for bare $\Gamma^{(2)}$ up to one loop in ϕ^4-theory we have

$$\Gamma^{(2)}(k) = i(k^2 + m_0^2) + \frac{\lambda_0 \mu^{4-d}}{2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + m_0^2},$$

  and here we need to apply two θ-operations in order to obtain finite value $\to$ we need one more CS equation!

$$2im^2 G \bar{\Gamma}^{(n)}_{\theta\theta} = \left[\left(2m^2 \frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial \lambda} \right) + n\gamma + \gamma_\theta \right] \bar{\Gamma}^{(n)}_{\theta}, \qquad (1)$$

  introducing

$$\gamma_\theta \equiv 2m^2 \left[\frac{\partial m^2}{\partial m_0^2} \right]^{-1} \frac{\partial \ln Z_\theta}{\partial m_0^2},$$

  and using the definition of

$$\Gamma^{(n)}_{\theta\theta}(k^2) \equiv -i \times \frac{d}{dm_0^2} \Gamma^{(n)}_{\theta}(k^2),$$

- Later on, for non-renormalizable theory we will need **even more CS equations** ...

2 Obtain CS equations

- One can derive the most general form of CS equation:

$$2m^2 iG\bar{\Gamma}^{(n)}_{k\theta} =$$
$$= \left[\left(2m^2 \frac{\partial}{\partial m^2} + \sum \beta \frac{\partial}{\partial \lambda} \right) + n\gamma + (k-1)\gamma_\theta \right] \bar{\Gamma}^{(n)}_{(k-1)\theta},$$

and another definition was used

$$\Gamma^{(n)}_{k\theta} \equiv -i \times \frac{d}{dm_0^2} \Gamma^{(n)}_{(k-1)\theta},$$

where $\Gamma^{(n)}_{1\theta} \equiv \Gamma^{(n)}_{\theta}$, $\Gamma^{(n)}_{2\theta} \equiv \Gamma^{(n)}_{\theta\theta}$ and etc.

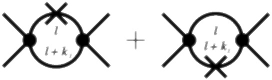
- Recall the bare 4-point function

$$\Gamma^{(4)} = -i\lambda_0 + \sum_{3\text{ opt}} \frac{\lambda_0^2}{2} \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 + m_0^2)} \frac{1}{(l+\kappa_i)^2 + m_0^2}.$$

- Here it is enough to only consider the one CS equation

$$2im^2(1+\gamma)\bar{\Gamma}_\theta^{(4)} = \left[4\gamma + \left(2m^2 \frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial \lambda}\right)\right]\bar{\Gamma}^{(4)},$$

so firstly one needs to find $\bar{\Gamma}_\theta^{(4)}$. It is shown in figure below:



2 Use CS equations: derive 4-point correlation function in ϕ^4 theory

- These diagrams are given by

$$[\bar{\Gamma}_\theta^{(4)}]_{\lambda^2} = -\sum_{3 \text{ opt}} \lambda^2 \int_0^1 dx \frac{1}{32\pi^2 \left(m^2 + \kappa_i^2 x(1-x)\right)}.$$

- Next, use $\bar{\Gamma}^{(4)}(0) = -i\lambda$ at $\kappa_i^2 = 0$, one can find

$$-\frac{3i\lambda^2}{16\pi^2} = -i\left([\beta]_{\lambda^2} + 4\lambda[\gamma]_\lambda\right).$$

- And turning back to CS equation:

$$\frac{\partial}{\partial m^2}\left[\bar{\Gamma}^{(4)}\right]_{\lambda^2} = -\frac{i\lambda^2}{32\pi^2}\sum_{3 \text{ opt}} \int_0^1 dx \frac{1}{x(1-x)\kappa_i^2 + m^2} + \frac{3i\lambda^2}{32\pi^2} \cdot \frac{1}{m^2},$$

one finds

$$\bar{\Gamma}^{(4)} = -i\lambda + \frac{i\lambda^2}{32\pi^2}\sum_{3 \text{ opt}} \int_0^1 dx \ln \frac{m^2}{x(1-x)\kappa_i^2 + m^2} + \mathcal{O}\left(\lambda^3\right).$$

The result:

This answer coincides with the result from standard approach, but now it was obtained in a **manifestly finite way**!

- CS for ϕ^4 (for theory with two fields, and etc) $\rightarrow$ see **S. Mooij, M. Shaposhnikov, Nucl. Phys. B 2023**
- Next orders, more loops? $\rightarrow$ CS method is recursive, so order by order, one can recover the usual results (up to all orders) for n-point functions! $\rightarrow$ see **S. Mooij, M. Shaposhnikov, Nucl. Phys. B 2023**

- Let us introduce the CS equations for effective action! To this end, recall that

$$\Gamma_{\text{eff}} = \sum_n \frac{1}{n!} \int d^4x_1 \dots d^4x_n \bar{\Gamma}^{(n)}(x_1 \dots x_n) \phi_0(x_1) \dots \phi_0(x_n),$$

$$\Gamma_{\text{eff},\theta} = \sum_n \frac{1}{n!} \int d^4x_1 \dots d^4x_n \bar{\Gamma}^{(n)}_\theta(x_1 \dots x_n) \phi_0(x_1) \dots \phi_0(x_n),$$

- ϕ_0 denotes the classical background field here and below!
- Then the generalization is straightforward

$$2im^2 G \Gamma_{\text{eff},\theta\theta} = \left[\left(2m^2 \frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial \lambda} \right) + \gamma \phi_0 \frac{\delta}{\delta \phi_0} + \gamma_\theta \right] \Gamma_{\text{eff},\theta},$$

$$2im^2 G \Gamma_{\text{eff},\theta} = \left[\left(2m^2 \frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial \lambda} \right) + \gamma \phi_0 \frac{\delta}{\delta \phi_0} \right] \Gamma_{\text{eff}},$$

- $\delta/\delta\phi_0$ is the functional derivative with respect to ϕ_0.

- To generalize we introduce a functional:

$$\Gamma_{\text{eff}}(\theta, \phi_0) = \sum_{n=0}^{\infty} \Gamma_{\text{eff},n\theta}(\phi_0) \frac{\theta^n}{n!},$$

where we use the shorthand notations $\Gamma_{\text{eff},1\theta} \equiv \Gamma_{\text{eff},\theta}$, $\Gamma_{\text{eff},2\theta} \equiv \Gamma_{\text{eff},\theta\theta}$, etc.

- The introduction of this functional immediately allows us to write an equation

$$\left[2m^2\left(\frac{\partial}{\partial m^2} - iG\frac{\partial}{\partial\theta}\right) + \gamma_\theta\theta\frac{\partial}{\partial\theta} + \gamma\phi_0\frac{\delta}{\delta\phi_0} + \sum_i \beta_i\frac{\partial}{\partial\lambda_i}\right]\Gamma_{\text{eff}}(\theta, \phi_0) = 0,$$

which unifies all possible CS equations for effective action and manifests itself as a general CS equation we are looking for.

- The key point is that non-renormalizable theories may include different operators, each of a different dimension. Such operators produce diagrams with an arbitrarily high degree of UV divergence. However, this is not the problem for the CS method since one can apply as many theta operations as needed to make the relevant Feynman graphs convergent.
- Operators are always included in the Lagrangian together with corresponding coupling constants. This means that the generalization of the CS equation will contain new beta functions related to these new coupling constants.

- In the case of $\lambda\phi^4$ plus some higher dimension operator $g\phi^6$ we have

$$\sum_i \beta_i \frac{\partial}{\partial\lambda_i} \to \beta \frac{\partial}{\partial\lambda} + \Omega_g \frac{\partial}{\partial g},$$

where Ω_g is a beta-function for g coupling constant.

- So, in the framework of CS method we also can study the non-renormalizable theories! Can compare our results with other methods...
  D. I. Kazakov, Phys. Part. Nucl. 2020;
  D. I. Kazakov, 2311.01109;
  D. I. Kazakov, D. M. Tolkachev and R. M. Yahibbaev, Theor. Math. Phys 2023, JCAP 2023...;
  A. V. Manohar, J. Pages and J. Roosmale Nepve, JHEP 2024...

- As was shown in **S. Mooij and M. Shaposhnikov Nucl. Phys. B 2024**, the Callan-Symanzik equations can be written for *effective potential*, using:

$$\Gamma_{\text{eff}} = -i \int d^4x [\Gamma(\phi_0) + K(\phi_0) + \ldots],$$

where $\Gamma(\phi_0)$ is an effective potential and $K(\phi_0) \propto (\partial_\mu \phi_0)^2$ is an effective kinetic term, so

$$2im^2 G\Gamma_{\theta\theta} = \Big[\Big(2m^2 \frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial \lambda} \Big) + \gamma \phi_0 \frac{\partial}{\partial \phi_0} + \gamma_\theta \Big] \Gamma_\theta ,$$

$$2im^2 G\Gamma_{\theta} = \Big[\Big(2m^2 \frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial \lambda} \Big) + \gamma \phi_0 \frac{\partial}{\partial \phi_0} \Big] \Gamma .$$

1. Main idea and motivaion
2. Divergence-free QFT: generalities
3. Calculations in non-renormalizable case
4. Results and outlook

- Consider now

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + \frac{\xi}{M^2}\phi(\Box^2\phi) + \frac{g}{6!M^2}\phi^6 + \frac{f}{3!M^2}\phi^3\Box\phi.$$

- However, let us make use of reparametrization freedom. Indeed, considering the following field redefinition:

$$\phi \to \phi + C_1\frac{\phi^3}{M^2} + C_2\frac{\Box\phi}{M^2} + C_3\frac{m^2\phi}{M^2},$$

- This helps to get rid of some terms and arrive to

$$\mathcal{L}_{\mathrm{New}} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\tilde{m}^2\phi^2 - \frac{\tilde{\lambda}}{4!}\phi^4 - \frac{\tilde{g}}{6!M^2}\phi^6.$$

- Introduce the expansion

$$\Gamma = \Gamma_0 + \hbar \cdot \phi_0^4 \cdot \Gamma_1(\phi_0^2/m^2) + \mathcal{O}(\hbar^2),$$

where Γ_0 is the classical potential, which reads

$$\Gamma_0 = \frac{m^2\phi_0^2}{2} + \frac{\lambda\phi_0^4}{4!} + \frac{g\phi_0^6}{6!M^2},$$

and Γ_1 is the one-loop correction which we are after.

3 One loop correction to effective potential

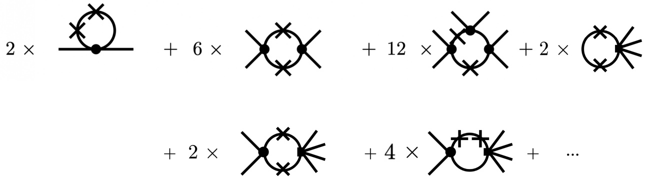


Figure: One loop contributions to $\Gamma_{\theta\theta,1}$. Square vertex corresponds to $g\phi_0^4$ term.

- So, we need to introduce

$$\Gamma_\theta = \Gamma_{\theta,0} + \hbar \cdot \Gamma_{\theta,1} + \mathcal{O}(\hbar^2),$$
$$\Gamma_{\theta\theta} = \Gamma_{\theta\theta,0} + \hbar \cdot \Gamma_{\theta\theta,1} + \mathcal{O}(\hbar^2).$$

- We denote the corrections to all γ, γ_θ, β and Ω_g as:

$$
\begin{aligned}
&G = G_0 + \hbar \cdot G_1, \\
&\gamma = \gamma_0 + \hbar \cdot \gamma_1, \\
&\gamma_\theta = \gamma_{\theta,0} + \hbar \cdot \gamma_{\theta,1}, \\
&\beta = \beta_0 + \hbar \cdot \beta_1, \\
&\Omega_g = \Omega_{g,0} + \hbar \cdot \Omega_{g,1}.
\end{aligned}
$$

- Zero-order parameters can be found as follows: consider the effective kinetic term:

$$K = \frac{1}{2}(\partial_\mu \phi_0)^2 \Big\{ K_0 + \hbar \cdot K_1\big(\phi_0^2/m^2, \lambda, g/M^2\big) + \mathcal{O}(\hbar^2) \Big\},$$

and the corresponding CS equation reads:

$$2m^2 iGK_\theta = \left[\left(2m^2 \frac{\partial}{\partial m^2} + \sum_i \beta_i \frac{\partial}{\partial \lambda_i} \right) + \gamma \phi_0 \frac{\partial}{\partial \phi_0} \right] K,$$

where

$$K_\theta = \frac{1}{2}(\partial_\mu \phi_0)^2 \cdot \Big\{ K_{\theta,0} + \hbar \cdot K_{\theta,1} + \mathcal{O}(\hbar^2) \Big\}.$$

- At the tree level, we have

$$K_0 = 1, \quad K_{\theta,0} = 0.$$

3 One loop correction to effective potential: $\hbar^0$ order

- We also will use the equations for effective potential

$$2im^2 G\Gamma_{\theta\theta} = \left[\left(2m^2 \frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial \lambda} \right) + \gamma \phi_0 \frac{\partial}{\partial \phi_0} + \gamma_\theta \right] \Gamma_\theta ,$$

$$2im^2 G\Gamma_{\theta} = \left[\left(2m^2 \frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial \lambda} \right) + \gamma \phi_0 \frac{\partial}{\partial \phi_0} \right] \Gamma ,$$

and at tree level we have

$$\Gamma_{\theta\theta,0} = 0,$$

$$\Gamma_{\theta,0} = -i \frac{\phi_0^2}{2},$$

since we have

$$\Gamma_0 = \frac{m^2 \phi_0^2}{2} + \frac{\lambda \phi_0^4}{4!} + \frac{g \phi_0^6}{6! M^2}.$$

- So, using CS equations for kinetic term and potential, we arrive to:

$$\gamma_0 = 0,$$
$$G_0 = 1,$$
$$\beta_0 = 0,$$
$$\Omega_{g,0} = 0,$$
$$\gamma_{\theta,0} = 0,$$

so we have found $\hbar^0$-order values!

3 One loop correction to effective potential: $\hbar^1$ order

- Next order contribution we obtain in the same manner! Take CS equation for kinetic term at $\hbar^1$ order:

$$\gamma_1 - im^2 K_{\theta,1} - x\frac{\partial K_1}{\partial x} = 0, \quad x \equiv \frac{\phi_0^2}{m^2}.$$

- Here $K_{\theta,1}$ is finite and can be found using the background field method with adiabatic expansion of effective action. Finally, arrive to

$$K_{1,\theta} = \frac{i}{(4\pi)^2}\frac{(\partial_\mu \phi_0)^2}{2}\left(\frac{\lambda}{2m^2}\right)^2 \frac{\phi_0^2}{(1+\frac{\lambda\phi_0^2}{2m^2})^2}.$$

- So, the second and the third terms in CS equation above are both proportional to ϕ_0^2 (in the leading order by λ), and we conclude that

$$\gamma_1 = 0.$$

- Now, turn to effective potential. Firstly, recall, that all $\hbar$ contributions to $\Gamma_{\theta\theta}$ are finite!
- The corresponding formula for $\Gamma_{\theta\theta}$ in $\hbar$ order reads

$$\Gamma_{\theta\theta,1} = -\frac{1}{32\pi^2}\ln\left[1 + \frac{\lambda\phi_0^2}{2m^2} + \frac{g}{4!M^2}\frac{\phi_0^4}{m^2}\right],$$

which is obtained after the summation of all one-loop contributions.

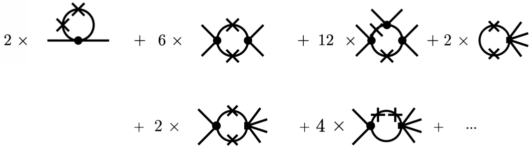


Figure: One loop contributions to $\Gamma_{\theta\theta,1}$. Square vertex corresponds to $g\phi_0^4$ term.

- Again, begin with equation

$$2im^2 G\Gamma_{\theta\theta} = \Big[\Big(2m^2 \frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial \lambda} \Big) + \gamma \phi_0 \frac{\partial}{\partial \phi_0} + \gamma_\theta \Big] \Gamma_\theta ,$$

and find

$$\frac{\Gamma_{\theta,1}}{m^2} = i \Big\{ \frac{xG_1}{2} + x^3 \Gamma_1'(x) - \frac{x^2 \beta_1}{48} - \frac{x^3 m^2 \Omega_{g,1}}{1440 M^2} \Big\},$$

recall that $x \equiv \frac{\phi_0^2}{m^2}$.

- Having the latter, we turn to

$$2im^2 G\Gamma_\theta = \left[\left(2m^2 \frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial \lambda} \right) + \gamma \phi_0 \frac{\partial}{\partial \phi_0} \right] \Gamma \,,$$

and find

$$\Gamma_1 = \frac{1}{64\pi^2} \left\{ \frac{\ln[1 + \frac{x}{24}(12\lambda + g \cdot x \cdot m^2/M^2)]}{x^2} \right.$$
$$\left. + \ln[24 + x(12\lambda + g \cdot x \cdot m^2/M^2)] \left(\frac{\lambda}{x} + \frac{\lambda^2}{4} + \frac{gm^2}{12M^2} \left(1 + \frac{\lambda x}{2} \right) \right) \right\}$$
$$+ \ln[x] \cdot \left[\frac{1}{4x} \left(\gamma_{\theta,1} - \frac{\lambda}{16\pi^2} \right) + \frac{1}{48} \left(\beta_1 - \frac{3\lambda^2}{16\pi^2} \right) \right.$$
$$\left. - \frac{g}{768M^2} \left(\frac{m^2}{\pi^2} + \frac{\lambda x m^2}{2\pi^2} \right) + \frac{x m^2 \Omega_{g,1}}{1440M^2} \right]$$
$$+ \frac{1}{4x} \left(\gamma_{\theta,1} - \frac{\lambda}{32\pi^2} - \frac{g \cdot x \cdot m^2}{384\pi^2 M^2} + c_1 \right) + c_2 .$$

- Need to know $\gamma_{\theta,1}$, β_1 and $\Omega_{g,1}$!
- To this end, we require that our result satisfies the analyticity requirement. In other words, we impose that the solution for Γ_1 is regular with $\phi_0 = 0$ (what guarantees that Green functions exist perturbatively).

- That is why from expression

$$\Gamma_1 = \frac{1}{64\pi^2}\left\{\frac{\ln[1+\frac{x}{24}(12\lambda + g\cdot x\cdot m^2/M^2)]}{x^2}\right.$$
$$\left. + \ln[24 + x(12\lambda + g\cdot x\cdot m^2/M^2)]\left(\frac{\lambda}{x} + \frac{\lambda^2}{4} + \frac{gm^2}{12M^2}\left(1+\frac{\lambda x}{2}\right)\right)\right\}$$
$$+ \ln[x]\cdot\left[\frac{1}{4x}\left(\gamma_{\theta,1} - \frac{\lambda}{16\pi^2}\right) + \frac{1}{48}\left(\beta_1 - \frac{3\lambda^2}{16\pi^2}\right)\right.$$
$$\left. - \frac{g}{768M^2}\left(\frac{m^2}{\pi^2} + \frac{\lambda x m^2}{2\pi^2}\right) + \frac{xm^2\Omega_{g,1}}{1440M^2}\right]$$
$$+ \frac{1}{4x}\left(\gamma_{\theta,1} - \frac{\lambda}{32\pi^2} - \frac{g\cdot x\cdot m^2}{384\pi^2 M^2} + c_1\right) + c_2,$$

we find: $\gamma_{\theta,1} = \frac{\lambda}{16\pi^2}$, $\beta_1 = \frac{3\lambda^2}{16\pi^2} + \frac{gm^2}{16\pi^2 M^2}$, $\Omega_{g,1} = \frac{15g\lambda}{16\pi^2}$.

- Above results coincide with, for instance, with **A. V. Manohar, J. Pages and J. Roosmale Nepve, JHEP 2024**!

- The integration constants c_1 and c_2 can be found by imposing appropriate boundary conditions at some convenient field value.
- The choice of c_1 and c_2, or in other words, the choice of boundary conditions, actually defines the physical parameters m, λ, and g.

$$\Gamma_1 = \frac{1}{64\pi^2}\Bigg\{\frac{\ln[1+\frac{x}{24}(12\lambda + g\cdot x\cdot m^2/M^2)]}{x^2} + \ln[24 + x(12\lambda + g\cdot x\cdot m^2/M^2)]\left(\frac{\lambda}{x} + \frac{\lambda^2}{4} + \frac{gm^2}{12M^2}\left(1+\frac{\lambda x}{2}\right)\right)\Bigg\} + \frac{1}{4x}\left(\gamma_{\theta,1} - \frac{\lambda}{32\pi^2} - \frac{g\cdot x\cdot m^2}{384\pi^2 M^2} + c_1\right) + c_2,$$

The CS method for n-point correlation functions contains the following finite ingredients:

- convergent connected diagrams;
- a set of CS equations between n-point functions and their derivatives with respect to the mass parameter and;
- the boundary conditions to fix integration constants or to define parameters from the Lagrangian.

Below, we use all these ingredients to find two-, four- and six-point functions at one loop level.

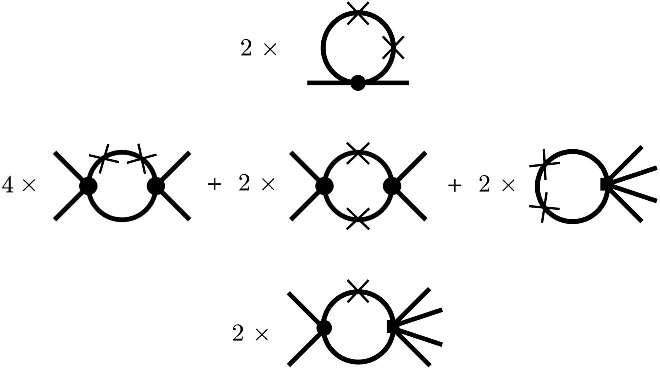


Figure: Graphs for 2-, 4-, and 6-point correlation functions with one (for a six-point function) and two (for two- and four-point functions) θ-operations. The square vertex corresponds to the ϕ^6 term.

- The tree contributions to two-, four-, and six-point correlation functions (which can be found from the Lagrangian) are:

$$\bar{\Gamma}^{(2)} = i(m^2 + k^2)\,,$$
$$\bar{\Gamma}^{(4)} = -i\lambda\,,$$
$$\bar{\Gamma}^{(6)} = -\frac{ig}{M^2},$$

while the expression for $\bar{\Gamma}^{(2,4)}_{\theta\theta}$ and $\bar{\Gamma}^{(6)}_{\theta}$ can be found from graphs.
- They are

$$\bar{\Gamma}^{(2)}_{\theta\theta} = -\frac{i\lambda}{32\pi^2 m^2},$$
$$\bar{\Gamma}^{(4)}_{\theta\theta} = -\frac{i\lambda^2}{2(4\pi)^2}\sum_{3\text{ opt}}\int_0^1 \frac{dx}{\Delta^2} + \frac{ig}{32\pi^2 m^2 M^2},\; \Delta \equiv m^2 + \kappa_i^2(1-x)x,$$
$$\bar{\Gamma}^{(6)}_{\theta} = -\frac{g\lambda}{2(4\pi)^2 M^2}\sum_{n=1}^{15}\int_0^1 \frac{dy}{\Delta_6},\; \Delta_6 \equiv m^2 + s_n^2(1-y)y.$$

- Next, these $\bar{\Gamma}^{(2)}_{\theta\theta}$, $\bar{\Gamma}^{(4)}_{\theta\theta}$, and $\bar{\Gamma}^{(6)}_{\theta}$ are used when solving the following CS equations:

$$2m^2 iG\bar{\Gamma}^{(2)}_{\theta\theta} = \left[\left(2m^2\frac{\partial}{\partial m^2} + \beta\frac{\partial}{\partial\lambda} + \Omega_g\frac{\partial}{\partial g}\right) + 2\gamma + \gamma_\theta\right]\bar{\Gamma}^{(2)}_{\theta},$$

$$2m^2 iG\bar{\Gamma}^{(2)}_{\theta} = \left[\left(2m^2\frac{\partial}{\partial m^2} + \beta\frac{\partial}{\partial\lambda} + \Omega_g\frac{\partial}{\partial g}\right) + 2\gamma\right]\bar{\Gamma}^{(2)},$$

$$2m^2 iG\bar{\Gamma}^{(4)}_{\theta\theta} = \left[\left(2m^2\frac{\partial}{\partial m^2} + \beta\frac{\partial}{\partial\lambda} + \Omega_g\frac{\partial}{\partial g}\right) + 4\gamma + \gamma_\theta\right]\bar{\Gamma}^{(4)}_{\theta},$$

$$2m^2 iG\bar{\Gamma}^{(4)}_{\theta} = \left[\left(2m^2\frac{\partial}{\partial m^2} + \beta\frac{\partial}{\partial\lambda} + \Omega_g\frac{\partial}{\partial g}\right) + 4\gamma\right]\bar{\Gamma}^{(4)},$$

$$2m^2 iG\bar{\Gamma}^{(6)}_{\theta} = \left[\left(2m^2\frac{\partial}{\partial m^2} + \beta\frac{\partial}{\partial\lambda} + \Omega_g\frac{\partial}{\partial g}\right) + 6\gamma\right]\bar{\Gamma}^{(6)}.$$

- Final results are

$$\bar{\Gamma}_1^{(4)} = b_2 + ib_1 m^2 + \frac{igm^2}{32\pi^2 M^2} - \frac{i\lambda^2}{32\pi^2} \sum_{3\text{ opt}} \int_0^1 dx \cdot \ln\frac{\Delta}{m^2},$$

$$\bar{\Gamma}_1^{(2)} = b_4 + im^2(b_3 + G_1),$$

$$\bar{\Gamma}_1^{(6)} = b_5 - \frac{i}{32\pi^2}\frac{g\lambda}{M^2} \sum_{15\text{ opt}} \int_0^1 dy \cdot \ln\frac{\Delta_6}{m^2};$$

- The answers for G, γ, γ_θ, β, and Ω_g coincide with the results from the effective potential consideration;
- The boundary conditions can be used at the final stage of all evaluations to define the integration constants b_i.

4 Results and outlook

- We have considered **non-renormalizable theory** and found corresponding 2-, 4- and 6-point correlation functions as well as correction to effective potential there (1-loop) in a fully finite way!
- The CS method as it stands cannot work for massless particles $\rightarrow$? What about fermions as well $\rightarrow$?
- Does CS method respect SUSY?
- It would be also interesting to see what happens with naturalness in other formulations of finite QFT (e.g. **Gerard 't Hooft, Int.J.Mod.Phys.A 2005**).
- We find more inspiration in:
  K. Nishijima, Phys. Rev. 1960;
  F. V. Tkachov, Sov. J. Part. Nucl., 1994;
  J. W. Moffat, Eur. Phys. J. Plus, 2011;
  A. L. Kataev and K. V. Stepanyantz, Nucl. Phys. B, 2013;
  D. I. Kazakov, Phys. Part. Nucl., 2020;
  I. Y. Arefeva, Teor. Mat. Fiz., 1977;
  and others!!!

THIS IS THE END OF PRESENTATION

Questions: are welcomed

THANK YOU FOR YOUR ATTENTION

- Consider the model with concrete realisations of "UV physics":

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - \frac{1}{2}\left(\partial_\mu \Phi\right)\left(\partial^\mu \Phi\right) - \frac{m^2}{2}\phi^2 - \frac{M^2}{2}\Phi^2 - \frac{\lambda_\phi}{4!}\phi^4 - \frac{\lambda_{\phi\Phi}}{4}\phi^2\Phi^2 - \frac{\lambda_\Phi}{4!}\Phi^4.$$

- Assume $m \ll M$.
- Physics involving the field $\Phi \to$ a toy representation of "new physics" living at large energy scales.
- We see that even after subtracting the formal UV divergences, $\bar{\Gamma}^{(2\phi)}$ still receives large contributions of order M^2:

$$\bar{\Gamma}^{(2\phi)} = i\left(k^2 + m^2\right) - \frac{i\lambda_\phi m^2}{32\pi^2}\left(1 + \ln\frac{\mu^2}{m^2}\right) - \frac{i\lambda_{\phi\Phi} M^2}{32\pi^2}\left(1 + \ln\frac{\mu^2}{M^2}\right).$$

- Therefore, it seems that heavy scale physics of order M^2 has **a dramatic influence** on the physics of order m^2.

4 Background field method and adiabatic expansion: derive effective kinetic term

- Substitute $\phi \to \phi_0 + \delta\phi$ into lagrangian and write the quadratic by $\delta\phi$ part:

$$\mathcal{L}^{(2)} = -\frac{1}{2}(\partial_\mu \delta\phi)^2 - \frac{m^2}{2}(\delta\phi)^2 - \frac{\lambda}{4}\phi_0^2\delta\phi^2 - \frac{g}{2\cdot 4! M^2}\phi_0^4\delta\phi^2.$$

- The equation of motion for $\delta\phi$ is

$$\left(\Box - m^2 - \frac{\lambda\phi_0^2}{2} - \frac{g\phi_0^4}{4!M^2}\right)\delta\phi = 0.$$

- The $\hbar$ correction $\Gamma_{\text{eff}}(\phi_0)$ to the classical action

$$\Gamma_{\text{cl}}(\phi_0) = -i\int d^4x\left[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{m^2\phi_0^2}{2} + \frac{\lambda\phi_0^4}{4!} + \frac{g\phi_0^6}{6!M^2}\right],$$

(where $\mathcal{D}_0 = \Box - m^2 - \frac{\lambda\phi_0^2}{2}, \quad \mathcal{D}_1 = -\frac{g\phi_0^4}{4!M^2}$) is:

$$\Gamma_{\text{eff}}(\phi_0) = \frac{i}{2}\text{Tr}\ln(-\mathcal{D}_0 - \mathcal{D}_1),\ \Gamma_{\text{eff}}(\phi_0) = \frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\ln(-\mathcal{D}_0 - \mathcal{D}_1),$$

4 Background field method and adiabatic expansion: derive effective kinetic term

- We would like to find $K_{\theta,1}$, so we need to consider the application of θ-operation on the quantum effective action $\Gamma_{\text{eff}}(\phi_0)$.
- So, the leading term in $\Gamma_{\text{eff},\theta}$, which is connected to $\mathcal{D}_0$ operator, is:

$$\Gamma_{\text{eff},\theta} = -i\frac{d}{dm_0^2}\Big\{\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\ln[-\mathcal{D}_0]\Big\},$$

- Using $\mathcal{D}_0 = -k^2 - m^2 - \frac{\lambda\phi_0^2}{2}$, we arrive to

$$\frac{d}{dm_0^2}\ln\Big[k^2 + m^2 + \frac{\lambda\phi_0^2}{2}\Big] = \frac{1}{k^2 + m^2 + \frac{\lambda\phi_0^2}{2}}\big(1 + \mathcal{O}(\hbar)\big) \equiv G.$$

- We introduce the factor $1 + \mathcal{O}(\hbar)$ to show that we work in the leading by $\hbar$ order.
- In other words, this factor comes from $\partial m^2/\partial m_0^2$ and $\partial\lambda/\partial m_0^2$.
- Introduce Green function G for $\mathcal{D}_0$ operator and, finally

$$\Gamma_{\text{eff},\theta} = \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}G.$$

- Firstly, for the simplicity we introduce $\Phi = \lambda\phi_0^2/2$ and then expand it with respect to small x_μ:

$$\Phi = \Phi(0) + \partial_\mu\Phi \cdot x^\mu + \frac{1}{2}\partial_\mu\partial_\nu\Phi \cdot x^\mu x^\nu + \ldots.$$

- In the momentum space $x_\mu = i\partial/\partial k^\mu$, and in this representation (up to $x^\mu x^\nu$):

$$\left(k^2 + m^2 + \Phi(0) + \partial_\mu\Phi \cdot i\frac{\partial}{\partial k^\mu} - \partial_\mu\partial_\nu\Phi \cdot \frac{\partial}{\partial k^\mu}\frac{\partial}{\partial k^\nu}\right)G = 1,$$

recalling

$$\frac{1}{k^2 + m^2 + \frac{\lambda\phi_0^2}{2}}\left(1 + \mathcal{O}(\hbar)\right) \equiv G.$$

4 Adiabatic expansion

- Since $x_\mu = i\partial/\partial k^\mu$ is a small parameter, we use the perturbation theory and power-counting with respect to $x_\mu \sim \epsilon$

$$G = G_0 + \epsilon G_1 + \epsilon^2 G_2 + \mathcal{O}(\epsilon^3).$$

- Also, the term $\partial_\mu \Phi \cdot i\frac{\partial}{\partial k^\mu}$ is of ϵ order and the term $\partial_\mu\partial_\nu\Phi \cdot \frac{\partial}{\partial k^\mu}\frac{\partial}{\partial k^\nu}$ is $\sim \epsilon^2$.
- Finally,

$$\begin{aligned} G_0 &= \frac{1}{k^2 + m^2 + \Phi}, \\ G_1 &= -\frac{i}{2}\partial_\mu\Phi \frac{\partial}{\partial k^\mu} G_0^2, \\ G_2 &= -\frac{1}{2} G_0 \frac{\partial}{\partial k^\mu}\frac{\partial}{\partial k^\nu} G_0^2 \cdot \partial_\mu\Phi\partial_\nu\Phi \\ &- \frac{1}{2} G_0 \frac{\partial}{\partial k^\mu}\frac{\partial}{\partial k^\nu} G_0 \cdot \partial_\mu\partial_\nu\Phi. \end{aligned}$$

4 Adiabatic expansion: derive effective kinetic term

- We are after the one-loop correction to $K_{1,\theta} \to$ consider the terms proportional to $(\partial_\mu \phi_0)^2$.
- This is the G_2 expression, so

$$K_{1,\theta} = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} G_2$$
$$= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\Big(-\frac{1}{2}G_0 \frac{\partial}{\partial k^\mu}\frac{\partial}{\partial k^\nu} G_0^2 \cdot \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{2}\partial_\nu \Phi \partial_\mu \Big[G_0 \frac{\partial}{\partial k_\mu}\frac{\partial}{\partial k_\nu} G_0\Big]\Big),$$

or

$$K_{1,\theta} = \frac{i}{(4\pi)^2}\frac{(\partial_\mu \phi_0)^2}{2}\Big(\frac{\lambda}{2m^2}\Big)^2 \frac{\phi_0^2}{(1+\frac{\lambda\phi_0^2}{2m^2})^2}.$$