Anomalous dimension of the heavy-light quark current in HQET up to four loops

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$$j_0^{(5)} = \bar{q}_0^{(5)} \Gamma b_0^{(5)} = Z_j^{(5)}(\alpha_s^{(5)}(\mu)) j^{(5)}(\mu) \qquad \langle 0 | j^{(5)}(\mu) | \bar{B} \rangle$$

Running $j^{(5)}(\mu)$ via $j^{(5)}(m_b)$

Matching $j^{(5)}(m_b)$ via HQET$^{(4)}$ operators

$$j^{(5)}(m_b) = C_\Gamma^{(4)}(m_b) \tilde{\jmath}^{(4)}(m_b) + \mathcal{O}(1/m_b)$$
$$\tilde{\jmath}_0^{(4)} = \bar{q}_0^{(4)} \Gamma \tilde{b}_0^{(4)} = \tilde{Z}_j^{(4)}(\alpha_s^{(4)}(\mu)) \tilde{\jmath}^{(4)}(\mu)$$

Running $\tilde{\jmath}^{(4)}(m_b)$ via $\tilde{\jmath}^{(4)}(m_c)$: $\tilde{\gamma}_j^{(4)}(\alpha_s^{(4)})$

Matching $\tilde{\jmath}^{(4)}(m_c)$ via HQET$^{(3)}$ operators

$$\tilde{\jmath}^{(4)}(m_c) = \tilde{C}^{(3)}(m_c) \tilde{\jmath}^{(3)}(m_c) + \mathcal{O}(1/m_c)$$

Running $\tilde{\jmath}^{(3)}(m_c)$ via $\tilde{\jmath}^{(3)}(\mu)$

$C_\Gamma(m_b)$

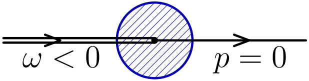
1 loop	Eichten, Hill (1990)
2 loops	Broadhurst, Grozin (1995) Grozin (1998)
3 loops	Bekavac, Grozin, Marquard, Piclum, Seidel, Steinhauser (2010)

$\tilde{C}(m_c)$

2 loops	Grozin (1998)
3 loops	Grozin, A. Smirnov, V. Smirnov (2006)

$\tilde{\gamma}_j$

1 loop	Voloshin, Shifman (1987)
	Politzer, Wise (1988)
2 loops	Ji, Musolf (1991)
	Broadhurst, Grozin (1991)
	Giménez (1992)
3 loops	Chetyrkin, Grozin (2003)
4 loops	Grozin (2023)



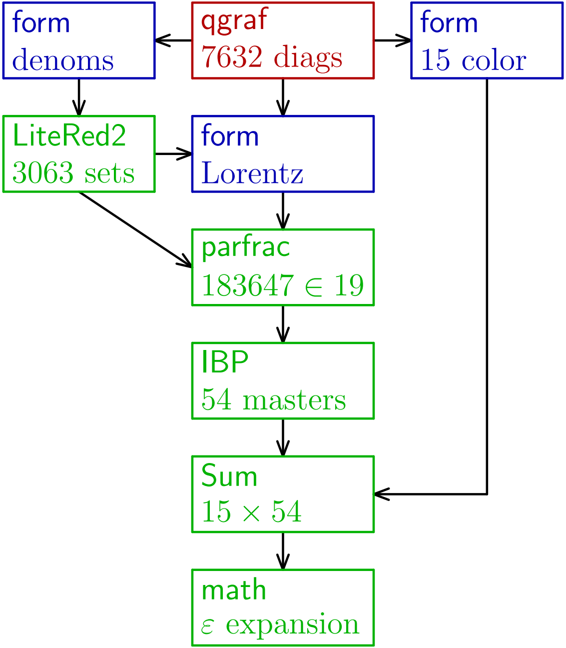
$$= \tilde{\Gamma}(\omega)$$

$$\tilde{\Gamma}(\omega) = \tilde{Z}_\Gamma(\alpha_s(\mu), a(\mu))\tilde{\Gamma}(\omega;\mu)$$

$$\log \tilde{\Gamma}(\omega) = \log \tilde{Z}_\Gamma(\alpha_s(\mu), a(\mu)) + \mathcal{O}(\varepsilon^0)$$

$$\tilde{Z}_j(\alpha_s) = \tilde{Z}_Q^{1/2}(\alpha_s, a) Z_q^{1/2}(\alpha_s, a)\tilde{Z}_\Gamma(\alpha_s, a)$$

$$\tilde{\gamma}_j(\alpha_s) = \tilde{\gamma}_\Gamma(\alpha_s, a) + \frac{1}{2}\left[\tilde{\gamma}_Q(\alpha_s, a) + \gamma_q(\alpha_s, a)\right]$$



No A^4, $t^a_{\mu\nu}$

4-loop: ξ^0, ξ^1

$$
\begin{aligned}
\tilde{\gamma}_j(\alpha_s) &= -3C_F\frac{\alpha_s}{4\pi} \\
&+ C_F\left(\frac{\alpha_s}{4\pi}\right)^2\left[-C_F\left(\frac{8}{3}\pi^2-\frac{5}{2}\right)+\frac{C_A}{3}\left(2\pi^2-\frac{49}{2}\right)+\frac{10}{3}T_Fn_f\right] \\
&+ C_F\left(\frac{\alpha_s}{4\pi}\right)^3\left[-C_F^2\left(36\zeta_3+\frac{8}{9}\pi^4-\frac{32}{3}\pi^2+\frac{37}{2}\right)\right. \\
&+ \frac{C_FC_A}{3}\left(142\zeta_3-\frac{8}{15}\pi^4-\frac{592}{9}\pi^2-\frac{655}{12}\right) \\
&- \frac{C_A^2}{3}\left(22\zeta_3+\frac{4}{5}\pi^4-\frac{130}{9}\pi^2-\frac{1451}{36}\right) \\
&- \frac{2}{3}C_FT_Fn_f\left(88\zeta_3-\frac{112}{9}\pi^2-\frac{235}{3}\right) \\
&+ \left.\frac{8}{3}C_AT_Fn_f\left(19\zeta_3-\frac{7}{9}\pi^2-\frac{64}{9}\right)+\frac{140}{27}(T_Fn_f)^2\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{\alpha_s}{4\pi}\right)^4\Bigg[C_F^4\Bigg(1200\zeta_5-168\zeta_3^2-\frac{896}{3}\pi^2\zeta_3+394\zeta_3\\
&\qquad+\frac{3884}{2835}\pi^6-\frac{4}{15}\pi^4+\frac{136}{3}\pi^2-\frac{691}{8}\Bigg)\\
&-C_F^3C_A\Bigg(\frac{5660}{3}\zeta_5-192\zeta_3^2-\frac{4576}{9}\pi^2\zeta_3+1275\zeta_3\\
&\qquad+\frac{2659}{2835}\pi^6-\frac{119}{45}\pi^4+\frac{2398}{9}\pi^2-\frac{3991}{12}\Bigg)\\
&+C_F^2C_A^2\Bigg(\frac{434}{3}\zeta_5-42\zeta_3^2-\frac{1916}{9}\pi^2\zeta_3+\frac{39047}{27}\zeta_3\\
&\qquad+\frac{2087}{1890}\pi^6-\frac{2663}{90}\pi^4+\frac{41026}{243}\pi^2-\frac{189671}{324}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+ C_F C_A^3 \left(492\zeta_5 + 30\zeta_3^2 + \frac{352}{9}\pi^2\zeta_3 - \frac{14666}{27}\zeta_3 \right. \\
&\qquad \left. - \frac{1439}{8505}\pi^6 + \frac{23}{90}\pi^4 - \frac{7246}{243}\pi^2 + \frac{179089}{648} \right) \\
&+ 8 d_{FA} \left(30\zeta_5 + \frac{106}{3}\pi^2\zeta_3 - 16\zeta_3 - \frac{452}{567}\pi^6 + \frac{29}{9}\pi^4 + \frac{46}{3}\pi^2 - 8 \right) \\
&+ 4 C_F^3 T_F n_f \left(\frac{580}{3}\zeta_5 - \frac{224}{9}\pi^2\zeta_3 - 24\zeta_3 - \frac{29}{45}\pi^4 + \frac{68}{3}\pi^2 - \frac{119}{3} \right) \\
&- \frac{C_F^2 C_A T_F n_f}{3} \left(1096\zeta_5 - \frac{736}{3}\pi^2\zeta_3 + \frac{18980}{9}\zeta_3 \right. \\
&\qquad \left. - \frac{1138}{45}\pi^4 - \frac{9404}{81}\pi^2 - \frac{32093}{27} \right)
\end{aligned}
$$

$$
\begin{aligned}
&- C_F C_A^2 T_F n_f \left(308\zeta_5 + 24\zeta_3^2 + \frac{128}{9}\pi^2\zeta_3 - \frac{20792}{27}\zeta_3 \right.\\
&\qquad \left. - \frac{874}{8505}\pi^6 + \frac{56}{27}\pi^4 + \frac{5240}{243}\pi^2 + \frac{27269}{162} \right)\\
&- 32 d_{FF} n_f \left(15\zeta_5 + \frac{8}{3}\pi^2\zeta_3 - 8\zeta_3 - \frac{437}{2835}\pi^6 + \frac{4}{9}\pi^4 + \frac{20}{3}\pi^2 - 4 \right)\\
&+ \frac{16}{27} C_F^2 (T_F n_f)^2 \left(326\zeta_3 - \frac{11}{5}\pi^4 + \frac{16}{9}\pi^2 - \frac{206}{3} \right)\\
&- \frac{2}{27} C_F C_A (T_F n_f)^2 \left(2272\zeta_3 - \frac{76}{5}\pi^4 + \frac{32}{9}\pi^2 - \frac{761}{3} \right)\\
&- \left. \frac{8}{9} C_F (T_F n_f)^3 \left(16\zeta_3 - \frac{83}{9} \right) \right] + \mathcal{O}(\alpha_s^5)
\end{aligned}
$$

Check: $\tilde{\gamma}_Q$ (4 loops: only ξ^0, ξ^1)

$$\tilde{\gamma}_j = -\frac{\alpha_s}{\pi} - 2.487726\left(\frac{\alpha_s}{\pi}\right)^2 - 6.292698\left(\frac{\alpha_s}{\pi}\right)^3 - 13.878042\left(\frac{\alpha_s}{\pi}\right)^4$$

Large β_0 limit $b = \beta_0\alpha_s/(4\pi) \sim 1$, $1/\beta_0 \ll 1$

$$\tilde{\gamma}_j = -C_F \frac{b}{\beta_0} \frac{\left(1 + \frac{2}{3}b\right)\Gamma(4 + 2b)}{\Gamma^2(2 + b)\Gamma(3 + b)\Gamma(1 - b)} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$$= -\frac{\alpha_s}{\pi} - 1.736111\left(\frac{\alpha_s}{\pi}\right)^2 + 4.219715\left(\frac{\alpha_s}{\pi}\right)^3 + 11.314887\left(\frac{\alpha_s}{\pi}\right)^4$$

$$+ 2.083958\left(\frac{\alpha_s}{\pi}\right)^5 + \cdots$$

$$\langle 0|j^{(5)}|\bar{B}\rangle = m_B f_B \qquad \Gamma = \gamma_5^{\mathrm{AC}} \not{v}$$

$$|\bar{B}\rangle = \sqrt{2m_B}|\bar{B}\rangle_{\mathrm{nr}} \qquad \langle 0|\tilde{\jmath}^{(4)}(\mu)|\bar{B}\rangle_{\mathrm{nr}} = F^{(4)}(\mu)$$

$$f_B = \sqrt{\frac{2}{m_B}} C_{\not{v}}^{(4)}(m_b) F^{(4)}(m_b) \Bigg[1 + \frac{1}{2m_b}\left(C_{\not{v},\Lambda}^{(4)}(m_b)\bar{\Lambda} + G_k^{(4)}(m_b) + C_m^{(4)}(m_b) G_m^{(4)}(m_b) \right) + \mathcal{O}\left(\frac{1}{m_b^2}\right) \Bigg]$$

$$\bar{\Lambda} = m_B - m_b \qquad \langle 0|O_{j,k}^{(4)}(\mu)|\bar{B}\rangle_{\mathrm{nr}} = F^{(4)}(\mu) G_k^{(4)}(\mu)$$

$$O_{j,k0}^{(4)} = \int dx\, T\{\tilde{\jmath}_0^{(4)}(0), O_{k0}^{(4)}(x)\}$$

$G_m^{(4)}(\mu)$ similar

$$L = \bar{\tilde{Q}}_0 iD \cdot v \tilde{Q}_0 + \frac{O_{k0} + C_{m0} O_{m0}}{2m_Q} + \mathcal{O}\left(\frac{1}{m_Q^2}\right)$$

$$F^{(n_f)}(\mu) = \hat{F}^{(n_f)}\left(\frac{\alpha_s^{(n_f)}(\mu)}{4\pi}\right)^{\tilde{\gamma}_{j0}/(2\beta_0^{(n_f)})} K^{(n_f)}(\alpha_s^{(n_f)}(\mu))$$

$$K^{(n_f)}(\alpha_s) = \exp\int_0^{\alpha_s}\frac{d\alpha_s}{\alpha_s}\left(\frac{\tilde{\gamma}_j^{(n_f)}(\alpha_s)}{2\beta^{(n_f)}(\alpha_s)} - \frac{\tilde{\gamma}_{j0}}{2\beta_0^{(n_f)}}\right)$$

$$\beta^{(n_f)}(\alpha_s^{(n_f)}) = -\frac{1}{2}\frac{d\log\alpha_s^{(n_f)}}{d\log\mu} = \sum_{L=1}^{\infty}\beta_{L-1}^{(n_f)}\left(\frac{\alpha_s^{(n_f)}}{4\pi}\right)^L$$

$$\beta_0^{(n_f)} = \frac{11}{3}C_A - \frac{4}{3}T_F n_f$$

$$\tilde{\gamma}_j^{(n_f)}(\alpha_s) = \tilde{\gamma}_{j0}\frac{\alpha_s}{4\pi} + \sum_{L=2}^{\infty}\tilde{\gamma}_{j,L-1}^{(n_f)}\left(\frac{\alpha_s}{4\pi}\right)^L$$

$$\frac{f_B}{f_D} = \sqrt{\frac{m_D}{m_B}} \frac{C_{\not{v}}^{(4)}(m_b)}{C_{\not{v}}^{(3)}(m_c)} \tilde{C}^{(3)}(m_c) \left(\frac{\alpha_s^{(4)}(m_b)}{\alpha_s^{(4)}(m_c)} \right)^{\tilde{\gamma}_{j0}/(2\beta_0^{(4)})} \frac{K^{(4)}(\alpha_s^{(4)}(m_b))}{K^{(4)}(\alpha_s^{(4)}(m_c))}$$

$$\times \left[1 + A\left(\frac{1}{m_c} - \frac{1}{m_b} \right) + \mathcal{O}\left(\frac{1}{m_{c,b}^2} \right) \right]$$

- tree-level $C_{\not{v},\Lambda} = -1$, $C_m = 1$
- neglect running of $G_{k,m}$
  and their differences between $n_f = 4$ and 3
- neglect the α_s^2/m_c corrections in $\tilde{C}^{(3)}(m_c)$

$$A = \frac{1}{2}(\bar{\Lambda} - G_k - G_m)$$

$$\frac{f_B}{f_D} = \sqrt{\frac{m_D}{m_B}} x^{-\tilde{\gamma}_{j0}/(2\beta_0^{(4)})} \Bigg\{ 1 + r_1(x-1)a_s$$

$$+ \left[r_{20} + r_{21}(x^2-1) + \frac{r_1^2}{2}(x-1)^2 \right] a_s^2$$

$$+ \Bigg[r_{30} + r_{31}(x^3-1) + \frac{r_1^3}{6}(x-1)^3 + r_1 r_{20}(x-1)$$

$$+ r_1 r_{21}(x-1)(x^2-1) \Bigg] a_s^3$$

$$+ A\left(\frac{1}{m_c} - \frac{1}{m_b}\right) + \mathcal{O}\left(\alpha_s^4, \frac{1}{m_{c,b}^2}\right) \Bigg\}$$

$$a_s = \frac{\alpha_s^{(4)}(m_b)}{4\pi} \qquad \alpha_s^{(4)}(m_b) \approx 0.215 \qquad x = \frac{\alpha_s^{(4)}(m_c)}{\alpha_s^{(4)}(m_b)} \approx 1.63$$

$$r_1 = -c_1 - \frac{\tilde{\gamma}_{j0}}{2\beta_0^{(4)}}\left(\frac{\tilde{\gamma}_{j1}^{(4)}}{\tilde{\gamma}_{j0}} - \frac{\beta_1^{(4)}}{\beta_0^{(4)}}\right) \qquad r_{20} = c_2^{(4)} - c_2^{(3)} + z_2$$

$$r_{21} = -c_2^{(3)} + \frac{c_1^2}{2} + z_2 + \frac{\tilde{\gamma}_{j0}}{4\beta_0^{(4)}}\left[-\frac{\tilde{\gamma}_{j2}^{(4)}}{\tilde{\gamma}_{j0}} + \frac{\beta_1^{(4)}}{\beta_0^{(4)}}\frac{\tilde{\gamma}_{j1}^{(4)}}{\tilde{\gamma}_{j0}} + \frac{\beta_2^{(4)}}{\beta_0^{(4)}} - \left(\frac{\beta_1^{(4)}}{\beta_0^{(4)}}\right)^2\right]$$

$$r_{30} = c_3^{(4)} - c_3^{(3)} - c_1\left(c_2^{(4)} - c_2^{(3)} + d_2\right) + z_3$$

$$r_{31} = -c_3^{(3)} + c_1(c_2^{(3)} - d_2) - \frac{c_1^3}{3} + z_3 + \frac{\tilde{\gamma}_{j0}}{6\beta_0^{(4)}}\left[-\frac{\tilde{\gamma}_{j3}^{(4)}}{\tilde{\gamma}_{j0}} + \frac{\beta_1^{(4)}}{\beta_0^{(4)}}\frac{\tilde{\gamma}_{j2}^{(4)}}{\tilde{\gamma}_{j0}} + \frac{\beta_2^{(4)}}{\beta_0^{(4)}}\frac{\tilde{\gamma}_{j1}^{(4)}}{\tilde{\gamma}_{j0}} - \left(\frac{\beta_1^{(4)}}{\beta_0^{(4)}}\right)^2\frac{\tilde{\gamma}_{j1}^{(4)}}{\tilde{\gamma}_{j0}} + \frac{\beta_3^{(4)}}{\beta_0^{(4)}} - 2\frac{\beta_1^{(4)}}{\beta_0^{(4)}}\frac{\beta_2^{(4)}}{\beta_0^{(4)}} + \left(\frac{\beta_1^{(4)}}{\beta_0^{(4)}}\right)^3\right]$$

$$C_{\not{\psi}}^{(n_f)}(m_Q) = 1 + c_1 \frac{\alpha_s^{(n_f)}(m_Q)}{4\pi} + \sum_{L=2}^{\infty} c_L^{(n_f)} \left(\frac{\alpha_s^{(n_f)}(m_Q)}{4\pi} \right)^L$$

$$\tilde{C}^{(n_f)}(m_Q) = 1 + z_2 \left(\frac{\alpha_s^{(n_f+1)}(m_Q)}{4\pi} \right)^2 + \sum_{L=3}^{\infty} z_L^{(n_f)} \left(\frac{\alpha_s^{(n_f+1)}(m_Q)}{4\pi} \right)^L$$

$$\frac{\alpha_s^{(3)}(m_c)}{4\pi} = \frac{\alpha_s^{(4)}(m_c)}{4\pi} \left[1 + \sum_{L=2}^{\infty} d_L \left(\frac{\alpha_s^{(4)}(m_c)}{4\pi} \right)^L \right]$$

$$
\begin{aligned}
\frac{f_B}{f_D} &= 0.669 \cdot \left[1 + 0.566\frac{\alpha_s^{(4)}(m_b)}{\pi} + 6.176\left(\frac{\alpha_s^{(4)}(m_b)}{\pi}\right)^2\right. \\
&\quad + 99.170\left(\frac{\alpha_s^{(4)}(m_b)}{\pi}\right)^3 \\
&\quad \left. + [\sim 1\,\mathrm{GeV}] \cdot \left(\frac{1}{m_c} - \frac{1}{m_b}\right)\right] \\
&= 0.669 \cdot (1 + 0.039 + 0.029 + 0.032 + [\sim 0.46])
\end{aligned}
$$

Large β_0 limit $b = \beta_0 a_s$, diff $n_f = 4$ and 3 neglected, $\tilde{C}(m_c) = 1$

$$K(\alpha_s(m_b))C_{\not{v}}(m_b) = 1 + \frac{1}{\beta_0}\int_0^\infty du\, e^{-u/b} S(u) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$$S(u) = -3C_F\left[e^{\frac{5}{3}u}\frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)}(1-u-u^2) - \frac{1}{2u}\right]$$

$$\frac{f_B}{f_D} = \sqrt{\frac{m_D}{m_B}} x^{-\tilde{\gamma}_{j0}/(2\beta_0)}$$

$$\times\left[1 + \frac{1}{\beta_0}\int_0^\infty du\big(e^{-u/b} - e^{-u/(xb)}\big)S(u) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\right]$$

$$\times\left[1 + A\left(\frac{1}{m_c} - \frac{1}{m_b}\right) + \mathcal{O}\left(\frac{1}{m_{c,b}^2}\right)\right]$$

$$1 + 0.686\frac{\alpha_s^{(4)}(m_b)}{\pi} + 8.271\left(\frac{\alpha_s^{(4)}(m_b)}{\pi}\right)^2 + 121.97\left(\frac{\alpha_s^{(4)}(m_b)}{\pi}\right)^3$$

$$+ 2567.6\left(\frac{\alpha_s^{(4)}(m_b)}{\pi}\right)^4 + \cdots$$

$$= 1 + 0.047 + 0.039 + 0.039 + 0.056 + \cdots$$

The leading renormalon $u = \frac{1}{2}$

$$\Delta C_{\not{\gamma}}(\mu) = \frac{1}{4}\frac{\Delta\bar{\Lambda}}{m_Q} \qquad \Delta\bar{\Lambda} = -2C_F\frac{e^{5/6}\Lambda_{\overline{\mathrm{MS}}}}{\beta_0}$$

$$\Delta G_k = -\frac{3}{2}\Delta\bar{\Lambda} \qquad \Delta G_m = 2\,\Delta\bar{\Lambda}$$

Principle value $\pm$ theoretical uncertainty

$$[] = 1.077 \pm 0.025 \qquad \frac{f_B}{f_D} = 0.721 \pm 0.016$$

without power corrections

Without large β_0

$$1 + c\alpha_s\left(1 + \sum_{n=1}^{\infty} c_n \alpha_s^n\right) = 1 + c\int_0^{\infty} du\, e^{-u/\alpha_s} S(u)$$

$$S(u) = 1 + \sum_{n=1}^{\infty} c_n \frac{u^n}{n!}$$

Pade

$$S(u) = \frac{1 + p_1 u}{1 + p_2 u} = \frac{1 + 0.917u}{1 - 2.555u} \qquad \text{pole} \qquad u_0 = 0.391$$

$$[] = 1 + 0.566\frac{\alpha_s}{\pi} + 6.176\left(\frac{\alpha_s}{\pi}\right)^2 + 99.17\left(\frac{\alpha_s}{\pi}\right)^3$$

$$+ 2388\left(\frac{\alpha_s}{\pi}\right)^4 + 76699\left(\frac{\alpha_s}{\pi}\right)^5 + \cdots$$

$$[] = 1.053 \pm 0.016 \qquad \frac{f_B}{f_D} = 0.705 \pm 0.010$$

without power corrections

Lattice $f_B = (190.0 \pm 1.3)\,\mathrm{MeV}$, $f_D = (212.0 \pm 0.7)\,\mathrm{MeV}$
$f_B/f_D = 0.896 \pm 0.009$

Conclusion

- $\tilde{\gamma}_j$ is known up to α_s^4 (4 loops)
- Perturbative f_B/f_D is known up to $\mathrm{N^3LL}$