Multi-loop calculations in $\varphi^4 + \varphi^6$ theory

A.V. Trenogin, PhD student

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Saint Petersburg State University

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$\varphi^4 + \varphi^6$ theory

Model

Model

$O(n)$ φ^4 model ($d = 4 - 2\varepsilon$):

$$S_0(\varphi) = \frac{1}{2}\partial_i\varphi_a\partial_i\varphi_a + \frac{\tau_0}{2}\varphi_a\varphi_a + \frac{g_0}{4!}(\varphi_a\varphi_a)^2,$$

where $\varphi = \{\varphi_a, a = 1, \ldots, n\}$ – n-component order parameter, τ_0 and g_0 – parameters.

$O(n)$ $\varphi^4 + \varphi^6$ model (generalization of the $O(n)$ φ^6 model) ($d = 3 - 2\varepsilon$):

$$S_0(\varphi) = \frac{1}{2}\partial_i\varphi_a\partial_i\varphi_a + \frac{\tau_0}{2}\varphi_a\varphi_a + \frac{\lambda_0}{4!}(\varphi_a\varphi_a)^2 + \frac{g_0}{6!}(\varphi_a\varphi_a)^3,$$

where λ_0 – one more parameter.

Models difference (mean-field theory)

$O(n)$ $\varphi^4 + \varphi^6$ model:

$$S_0(\varphi) = \frac{1}{2}\partial_i\varphi_a\partial_i\varphi_a + \frac{\tau_0}{2}\varphi_a\varphi_a + \frac{\lambda_0}{4!}(\varphi_a\varphi_a)^2 + \frac{g_0}{6!}(\varphi_a\varphi_a)^3,$$

$$\lambda_0 = \bar{\lambda}_0\tau_0{}^{\phi}.$$

	φ^4	$\varphi^4 + \varphi^6$ (φ^6)
logarithmic dim	4	3
asymptotic	critical	if $d = 3$: 1. $\phi > \frac{1}{2} \to$ purely tricritical (φ^6); 2. $\phi < \frac{1}{2} \to$ modified critical (φ^4 in $d = 4 - 2\varepsilon$); 3. $\phi = \phi_t = \frac{1}{2} \to$ combined tricritical ($\varphi^4 + \varphi^6$).

Tricritical exponents

φ^6 (the same as in φ^4):

α – the exponent of the specific heat;

β and $1/\delta$ – different order parameter exponents;

γ – the susceptibility exponent;

ν – the exponent of the correlation length;

η – the Fisher exponent (the critical-point correlation exponent).

$\varphi^4 + \varphi^6$ (additional to φ^6):

ϕ_t – crossover exponent (the limiting value of the ϕ when both interactions (φ^4 and φ^6) are significant).

	ϕ_t	[Lewis and Adams 1978]
ϕ_t notation:	φ	[J. S. Hager 2002]
	α_0	[Vasil'ev 2004]

Renormalization

Renormalized action:

$$S_R(\varphi) = \frac{(Z_1\Delta + Z_2\tau + Z_5\lambda^2)}{2}\varphi^2 + \frac{Z_4\lambda\mu^{2\varepsilon}}{4!}\varphi^4 + \frac{Z_3 g\mu^{4\varepsilon}}{6!}\varphi^6,$$

where

$$\hat{\varphi} = Z_\varphi \hat{\varphi}_R; \qquad Z_1 = Z_\varphi^2; \qquad Z_4 = Z_\lambda Z_\varphi^4;$$

$$\tau_0 = Z'_\tau \tau = Z_\tau \tau + \bar{Z}\lambda^2; \qquad Z_2 = Z_\tau Z_\varphi^2; \qquad Z_5 = \bar{Z} Z_\varphi^2.$$

$$g_0 = Z_g g \mu^{4\varepsilon}; \qquad Z_3 = Z_g Z_\varphi^6;$$

$$\lambda_0 = Z_\lambda \lambda \mu^{2\varepsilon}.$$

A.N. Vasil'ev notations [Vasil'ev 2004]

$\varphi^4 + \varphi^6$ theory

Realization of our computer program

Program structure

1. generation of graphs (with a possibility to draw them);
2. counterterm (KR') operation;
3. symmetry factors and $O(n)$ factors;
4. renormalization constants;
5. RG functions.

Main used libraries

We use Python and the main libraries are

- GraphState – to define a graph structure;
- GTopology – to generate all necessary graph topologies;
- Graphine – to manipulate graphs;
- Decomposition – to calculate using Sector Decomposition;
- graphviz – to draw graphs;
- ginac – to calculate symbolically;
- uncertainties – to work with numbers with uncertainties;
- dataset – to store data.

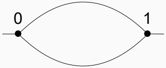
1. Graph generation

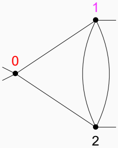
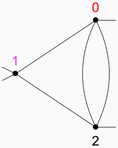
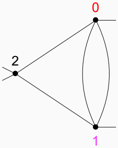
Nickel notation:

e – external edge;

graph with k nodes – *edges into 0 node*$|\cdots|$*edges into k-1 node*|.

Examples:

*e*11|*e*| corresponds to 

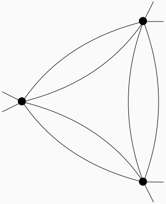
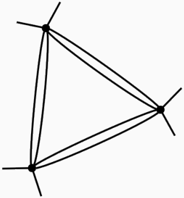
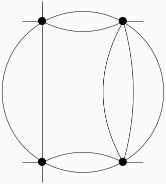
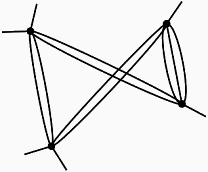
*ee*12|*e*22|*e*|	*e*122|*ee*2|*e*|	*e*112|*e*2|*ee*|
		

Nickel index – minimal Nickel notation (*ee*12|*e*22|*e*| for the previous one).

We use Nickel index to avoid a problem of different notations of the same graph.

1. Graph drawing

Examples:

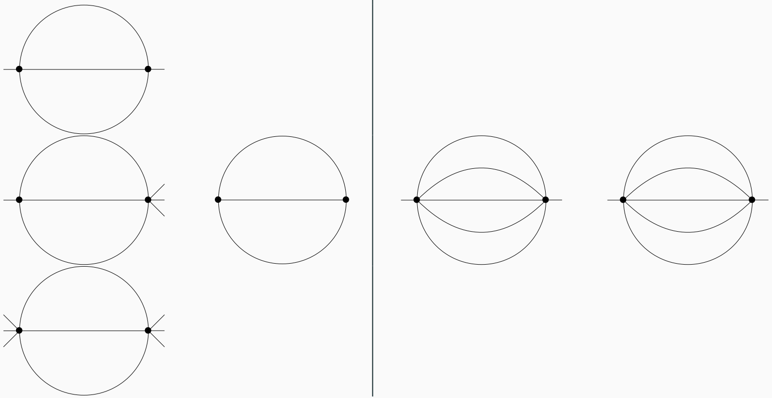
graph index	manual	program
*ee*1122|*ee*22|*ee*|		
*ee*1122|*ee*33|*e*333|*e*|		

2. Counterterm (KR') operation structure

a. define a corresponding graph for a diagram;
b. choose an IR-safe rearrangement;
c. factorize the graph into irreducible ones;
d. calculate the irreducible graphs;
e. calculate the factorized graph;
f. calculate the counterterm.

Corresponding graphs – to calculate graphs with the same internal structure once:

logarithmically divergent graph	corr. graph	quadratically divergent graph	corr. graph



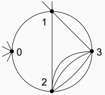
2a. Corresponding graph/database

2 tables (of the database):

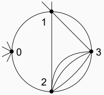
1. graph: corresponding graph;
2. corresponding graph: KR'[corresponding graph].

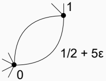
2b. IR-safe rearrangement

There are 2 ways to get possible IR-safe rearrangements:

1. 

→ all possible arrangements: (0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3) → IR-safe rearrangements: (0, 1), (0, 2), (0, 3).

2. 

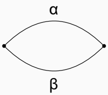
→ replace subgraphs with 2 boundary nodes by edges: 

→ all possible arrangements: (0, 1) → IR-safe rearrangements: (0, 1).

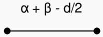
We use the second way.

After that we sort the IR-safe rearrangements and choose the best ones (which allow us to calculate the graph in the easiest way).

IR-unsafe rearrangements could be taken into account too if instead of R' operation use $R^{*\prime}$ operation (we use R' yet).

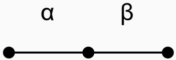


$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{d\vec{k}}{k^{2\alpha}(k-p)^{2\beta}} = \frac{1}{(4\pi)^{d/2}} \frac{G(\alpha,\beta)}{p^{2(\alpha+\beta-d/2)}} \sim$$

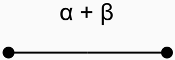


,

$$G(\alpha,\beta) = \frac{\Gamma(d/2-\alpha)\Gamma(d/2-\beta)\Gamma(\alpha+\beta-d/2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(d-\alpha-\beta)}.$$



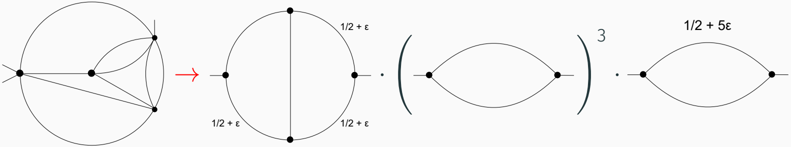
$=$



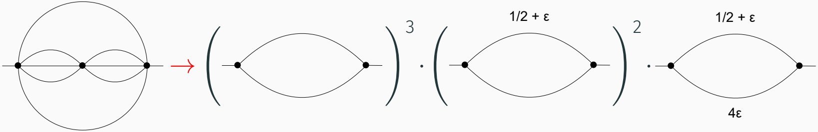
2c. Graph factorization/examples

Irreducible graph – a graph that cannot be simplified using G-functions.

1.

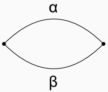


2.



2d. Calculation of irreducible graphs/methods

If a graph is one-loop we use

G-functions: 

$$= \frac{1}{(4\pi)^{d/2}} \frac{G(\alpha,\beta)}{p^{2(\alpha+\beta-d/2)}},$$

$$G(\alpha,\beta) = \frac{\Gamma(d/2-\alpha)\Gamma(d/2-\beta)\Gamma(\alpha+\beta-d/2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(d-\alpha-\beta)}$$

else

Sector Decomposition.

2d. Calculation of irreducible graphs/procedure

If a graph is one-loop:

1. calculate using G-functions;
2. save the result as an analytical Laurent series with required number of terms.

Otherwise:

1. unload the graph in a special file;
2. numerically calculate graphs from the file using Sector Decomposition (required number of Laurent series terms with required precision);
3. present the numerically calculated graph in the form of formal analytical Laurent series (the graph $\to \sum_{-\infty}^{\text{required term}} a_i \varepsilon^i$, where a_i are auxiliary variables that correspond to the numerical values of the graph);
4. save the correspondences ($a_i \to$ numerical value) and the formal analytical Laurent series.

2d. Calculation of irreducible graphs/database

2 tables:

1. graph: analytical representation (in the form of a Laurent series with required number of terms);
2. auxiliary variable (a_i): corresponding numerical value with uncertainty.

Using a_i:

- solves the incompatibility problem of ginac and uncertainties libraries;
- does not increase uncertainties in final results.

ginac – for symbolic calculation;
uncertainties – for working with numbers with uncertainties.

2ef. Last two points of KR' operation

a. define a corresponding graph for a diagram;
b. choose an IR-safe rearrangement;
c. factorize the graph into irreducible ones;
d. calculate the irreducible graphs;
e. calculate the factorized graph; (using the irreducible ones)
f. calculate the counterterm. (Minimal Subtraction (MS) scheme)

3. Calculation of symmetry factors and $O(n)$-factors

- to calculate symmetry factors we use the function symmetry_coefficient from Graphine;
- to calculate $O(n)$-factors we use FORM (a symbolic manipulation system). For a graph we:
  1. create a .frm file based on the Nickel notation;
  2. run the file using FORM;
  3. get the result from the generated .txt file;
  4. save the result in the special database table.

All work with FORM is done exclusively through Python.

4 & 5. Renormalization constants and RG functions

$$\text{counterterms} = f(a_i, \varepsilon, n) \xrightarrow{a_i \to \text{numerical values}} f(\varepsilon, n)$$

$\downarrow$ analytically

$$\text{renormalization constants} = f(a_i, \varepsilon, n) \xrightarrow{a_i \to \text{numerical values}} f(\varepsilon, n)$$

$\downarrow$ analytically

$$\text{RG functions} = f(a_i, \varepsilon, n) \xrightarrow{a_i \to \text{numerical values}} f(\varepsilon, n)$$

Analytical and numerical results are stored in different tables.

$\varphi^4 + \varphi^6$ theory

Results

Previous results/tricritical exponents

[1]: η – ε^3 (1 six-loop graph with 2 external edges in φ^6 theory, six-loop contribution into Z_φ) and ϕ_t – ε^2;

[2]: ν – ε^3 (six-loop contribution into Z_τ) and confirmed ε^3 term in η;

[3]: calculated full 3 order:

- ϕ_t – calculated ε^3 term;
- confirmed ε^2 and ε^3 terms.

[1] Lewis and Adams 1978, "Tricritical behavior in two dimensions. II. Universal quantities from the ϵ expansion".

[2] Hager and Schäfer 1999, "Θ-point behavior of diluted polymer solutions: Can one observe the universal logarithmic corrections predicted by field theory?"

[3] J. S. Hager 2002, "Six-loop renormalization group functions of $O(n)$-symmetric ϕ^6-theory and ϵ-expansions of tricritical exponents up to ϵ^3".

Previous results/problems of the article[4]

The article contains two problems:

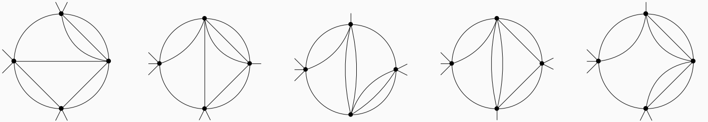
1. The intermediate calculation is written with mistakes (for example w_R does not nullify $\beta(w_R)$). Nevertheless the final values for the tricritical exponents correspond to Z_i given in the paper.

2. Z_4 and Z_6 are calculated incorrectly.
We are unable to identify the problem due to lack of information presented in the paper.

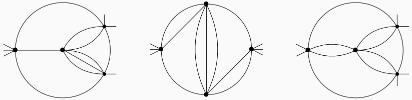
[4] J. S. Hager 2002, "Six-loop renormalization group functions of $O(n)$-symmetric ϕ^6-theory and ϵ-expansions of tricritical exponents up to ϵ^3".

Previous results/graph counterterms in an article[5]

The article contains counterterms of 5 one-loop reducible



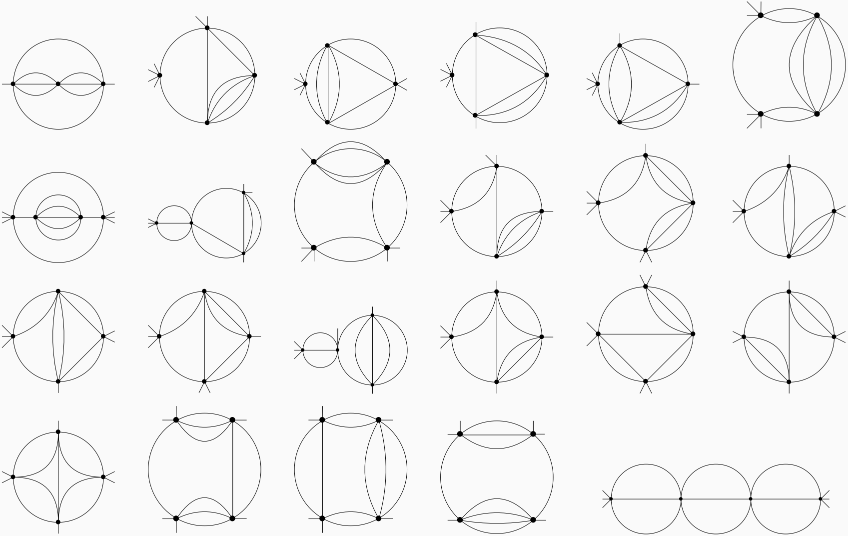
and 3 complex



graphs, where a one-loop reducible graph is a graph that can be easily calculated analytically.

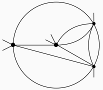
[5] Jack and Jones 2020, "Anomalous dimensions for ϕ^n in scale invariant $d = 3$ theory".

One-loop reducible graphs, 6 loops



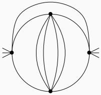
Complex graphs, 6 loops

Complex graphs – graphs that are not one-loop reducible.



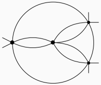
$$\frac{29.13318(6)}{\varepsilon}$$

$$\frac{8(\pi^2\beta(2)+24\beta(4))}{9\varepsilon}\ (IBP)$$



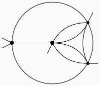
$$\frac{1.3333318(14)}{\varepsilon^2}-\frac{5.33336(3)}{\varepsilon}$$

$$\frac{4}{3\varepsilon^2}-\frac{16}{3\varepsilon}\ (IBP)$$



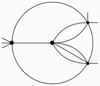
$$\frac{32.46970(7)}{\varepsilon}$$

$$\frac{\pi^4}{3\varepsilon}\ (IBP)$$



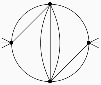
$$-\frac{3.289883(9)}{\varepsilon^2}+\frac{6.72312(16)}{\varepsilon}$$

$$-\frac{\pi^2}{3\varepsilon^2}+\frac{\pi^2(1+\ln 4)-14\zeta(3)}{\varepsilon}\ (IBP)$$



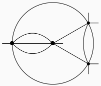
$$\frac{0.16666664(28)}{\varepsilon^3}-\frac{2.000003(6)}{\varepsilon^2}+\frac{1.89371(10)}{\varepsilon}$$

$$\frac{1}{6\varepsilon^3}-\frac{2}{\varepsilon^2}-\frac{8(\pi^2-12)}{9\varepsilon}\ (IBP)$$



$$\frac{0.3333331(3)}{\varepsilon^3}-\frac{2.666661(4)}{\varepsilon^2}+\frac{2.66667(6)}{\varepsilon}$$

$$\frac{1}{3\varepsilon^3}-\frac{8}{3\varepsilon^2}+\frac{8}{3\varepsilon}\ (R^*)$$



$$-\frac{1.644932(2)}{\varepsilon^2}+\frac{4.18125(4)}{\varepsilon}$$

$$-\frac{\pi^2}{6\varepsilon^2}+\frac{\pi^2(5+\ln 4)-42\zeta(3)}{3\varepsilon}\ (IBP)$$

Second line is results of M. Kompaniets and A. Pikelner, Unpublished

Results/tricritical exponents, $O(n)$

$$\eta = (2.66667 + 2n + 0.333333n^2)\frac{\varepsilon^2}{(22+3n)^2} + (33797.3 + 33534.1n + 10838.6n^2 + 1385.63n^3 + 64.232n^4 + 0.822467n^5)\frac{\varepsilon^3}{(22+3n)^4};$$

$$\nu = 0.5 + (10.6667 + 8n + 1.33333n^2)\frac{\varepsilon^2}{(22+3n)^2} + (86891.3 + 82490.4n + 24328.3n^2 + 2518.52n^3 + 56.9602n^4 - 0.411234n^5)\frac{\varepsilon^3}{(22+3n)^4};$$

$$\phi_t = 0.5 + (6-n)\frac{\varepsilon}{22+3n} + (-47927.4 - 20941.2n - 2312.87n^2 - 8.39119n^3 + 2.4674n^4)\frac{\varepsilon^2}{(22+3n)^3} + (4.726074(15)\cdot 10^8 + 3.191107(10)\cdot 10^8 n + 8.107993(26)\cdot 10^7 n^2 + 9692087(31)n^3 + 538116.4(1.6)n^4 + 11367.367(28)n^5 + 203.17798(17)n^6 + 6.08807n^7)\frac{\varepsilon^3}{(22+3n)^5}.$$

Results/ϕ_t difference

Our result:

$$\phi_t = 0.5 + (6-n)\frac{\varepsilon}{22+3n} + (-47927.4 - 20941.2n - 2312.87n^2 - 8.39119n^3 + 2.4674n^4)\frac{\varepsilon^2}{(22+3n)^3} + (4.726074(15)\cdot 10^8 + 3.191107(10)\cdot 10^8 n + 8.107993(26)\cdot 10^7 n^2 + 9692087(31)n^3 + 538116.4(1.6)n^4 + 11367.367(28)n^5 + 203.17798(17)n^6 + 6.08807n^7)\frac{\varepsilon^3}{(22+3n)^5}.$$

Result of the article [J. S. Hager 2002]:

$$\phi_t = 0.5 + (6-n)\frac{\varepsilon}{22+3n} + (-47927.4 - 20941.2n - 2312.87n^2 - 8.39119n^3 + 2.4674n^4)\frac{\varepsilon^2}{(22+3n)^3} + (5.82218\cdot 10^8 + 4.01209\cdot 10^8 n + 1.04251\cdot 10^8 n^2 + 1.26915\cdot 10^7 n^3 + 702497n^4 + 13218.9n^5 + 158.765n^6 + 6.08807n^7)\frac{\varepsilon^3}{(22+3n)^5}.$$

Results/tricritical exponents, $n = 1$ and $n = 2$

$$n = 1:$$

$$\eta = 0.008\varepsilon^2 + 0.203829\varepsilon^3;$$

$$\nu = 0.5 + 0.032\varepsilon^2 + 0.50249\varepsilon^3;$$

$$\phi_t = 0.5 + 0.2\varepsilon - 4.55599\varepsilon^2 + 90.42328(28)\varepsilon^3. \qquad (112.751\varepsilon^3)$$

$$n = 2:$$

$$\eta = 0.0102041\varepsilon^2 + 0.254385\varepsilon^3;$$

$$\nu = 0.5 + 0.0408163\varepsilon^2 + 0.60234\varepsilon^3;$$

$$\phi_t = 0.5 + 0.142857\varepsilon - 4.51389\varepsilon^2 + 88.41604(28)\varepsilon^3. \qquad (111.261\varepsilon^3)$$

Comparison with previous results

Coincidence:

- η and ν completely coincided with the results of the article[6];
- First two ε terms of the ϕ_t coincided with the results of the article[7];
- The article[8] contains counterterms of 8 graphs (5 simple and 3 complex). The results are in full agreement with ours.

Difference:

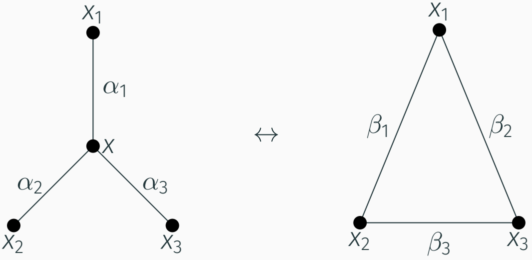
- ε^3 term of the ϕ_t tricritical exponent differs from the result presented in the work[7].

[6] Hager and Schäfer 1999, "Θ-point behavior of diluted polymer solutions: Can one observe the universal logarithmic corrections predicted by field theory?"
[7] J. S. Hager 2002, "Six-loop renormalization group functions of $O(n)$-symmetric ϕ^6-theory and ϵ-expansions of tricritical exponents up to ϵ^3".
[8] Jack and Jones 2020, "Anomalous dimensions for ϕ^n in scale invariant $d = 3$ theory".

Star-triangle transformation

Uniqueness relation

Star-triangle transformation

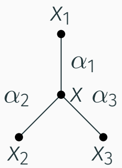


In math form:

$$\int_{\mathbb{R}^d} \frac{d\vec{x}}{(x_1 - x)^{2\alpha_1}(x_2 - x)^{2\alpha_2}(x_3 - x)^{2\alpha_3}} \leftrightarrow \frac{1}{(x_1 - x_2)^{2\beta_1}(x_1 - x_3)^{2\beta_2}(x_2 - x_3)^{2\beta_3}}.$$

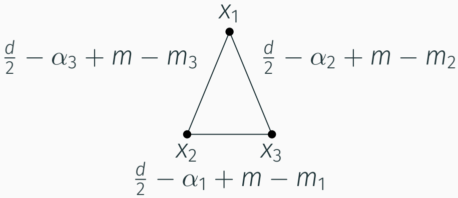
Uniqueness relation [Kazakov, 1984]

$\alpha_1 + \alpha_2 + \alpha_3 = d + m, \quad m \in \mathbb{N} \cup \{0\}$:



$$= \frac{\pi^{\frac{d}{2}} m!}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \sum_{\substack{m_1+m_2+m_3=m \\ 0 \le m_i \le m}} C(m_1, m_2, m_3)$$

$\cdot$



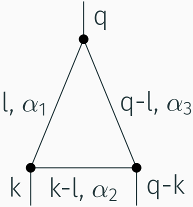
,

$$C(m_1, m_2, m_3) = \frac{\Gamma(\frac{d}{2} - \alpha_1 + m - m_1)}{m_1!} \cdot \frac{\Gamma(\frac{d}{2} - \alpha_2 + m - m_2)}{m_2!} \cdot \frac{\Gamma(\frac{d}{2} - \alpha_3 + m - m_3)}{m_3!}, \quad m_1, m_2, m_3 \in \mathbb{N} \cup \{0\}.$$

Star-triangle transformation

Attempt to get more general relation

Impulse representation:



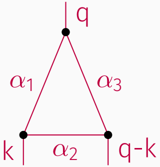
$$= \int\limits_{\mathbb{R}^d} \frac{d\vec{l}}{l^{2\alpha_1}(k-l)^{2\alpha_2}(q-l)^{2\alpha_3}} = \frac{\pi^{\frac{d}{2}}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}$$

$$\int\limits_0^\infty d\tau_1 \int\limits_0^\infty d\tau_2 \int\limits_0^\infty d\tau_3 \; \tau_1^{\frac{d}{2}-\alpha_3-1}\tau_2^{\frac{d}{2}-\alpha_1-1}\tau_3^{\frac{d}{2}-\alpha_2-1}(\tau_1\tau_2+\tau_2\tau_3+\tau_1\tau_3)^{\alpha_1+\alpha_2+\alpha_3-d}$$

$$\cdot \exp(-k^2\tau_1-(k-q)^2\tau_2-q^2\tau_3) \xrightarrow{\alpha_1+\alpha_2+\alpha_3-d=m\in\mathbb{N}\cup\{0\}} \text{uniqueness relation.}$$

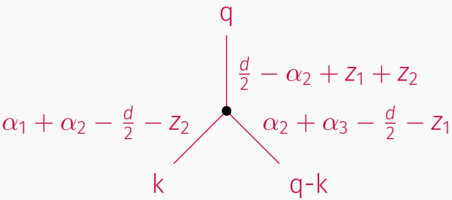
Using Mellin-Barnes representation

$Re(d - \alpha_1 - \alpha_2 - \alpha_3) > 0$:



$$= \frac{\pi^{\frac{d}{2}}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \frac{1}{\Gamma(d - \alpha_1 - \alpha_2 - \alpha_3)}$$

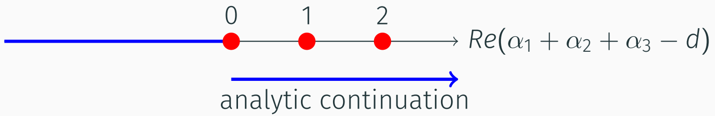
$$\int_{c-i\infty}^{c+i\infty} \frac{dz_1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{dz_2}{2\pi i} \; C(z_1, z_2)$$



,

$$C(z_1, z_2) = \Gamma(-z_1)\Gamma(-z_2)\Gamma(d - \alpha_1 - \alpha_2 - \alpha_3 + z_1 + z_2)\Gamma(\alpha_1 + \alpha_2 - \frac{d}{2} - z_2)$$
$$\cdot \Gamma(\alpha_2 + \alpha_3 - \frac{d}{2} - z_1)\Gamma(\frac{d}{2} - \alpha_2 + z_1 + z_2).$$

Domain of validity

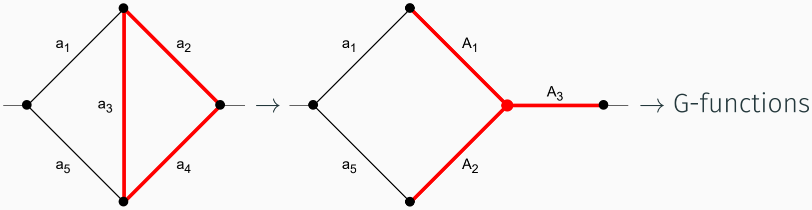


,

red – uniqueness relation,

blue – using Mellin-Barnes representation.

Test diagram



Test diagram/results

Known result [Bierenbaum & Weinzierl, 2003]:

$$I_1(a_1,a_2,a_3,a_4,a_5) = \frac{1}{\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)\Gamma(d-a_2-a_3-a_4)} \int\limits_{c-i\infty}^{c+i\infty} \frac{dz_1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \frac{dz_2}{2\pi i}$$

$$\frac{\Gamma(-z_1)\Gamma(\frac{d}{2}-a_3-a_4-z_1)\Gamma(\frac{d}{2}-a_1+z_1)}{\Gamma(a_1-z_1)} \cdot \frac{\Gamma(-z_2)\Gamma(\frac{d}{2}-a_2-a_3-z_2)\Gamma(\frac{d}{2}-a_5+z_2)}{\Gamma(a_5-z_2)}$$

$$\cdot \frac{\Gamma(a_1+a_5-\frac{d}{2}-z_1-z_2)\Gamma(a_3+z_1+z_2)\Gamma(a_2+a_3+a_4-\frac{d}{2}+z_1+z_2)}{\Gamma(d-a_1-a_5+z_1+z_2)}.$$

Calculated result:

$$I_2(a_1,a_2,a_3,a_4,a_5) = \frac{1}{\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)\Gamma(d-a_2-a_3-a_4)} \int\limits_{c-i\infty}^{c+i\infty} \frac{dz_1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \frac{dz_2}{2\pi i}$$

$$\cdot \frac{\Gamma(-z_1)\Gamma(a_3+a_4-\frac{d}{2}-z_1)\Gamma(d-a_3-a_4-a_5+z_1)}{\Gamma(a_3+a_4+a_5-\frac{d}{2}-z_1)}$$

$$\cdot \frac{\Gamma(-z_2)\Gamma(a_2+a_3-\frac{d}{2}-z_2)\Gamma(d-a_1-a_2-a_3+z_2)}{\Gamma(a_1+a_2+a_3-\frac{d}{2}-z_2)}$$

$$\cdot \frac{\Gamma(a_1+a_2+2a_3+a_4+a_5-\frac{3d}{2}-z_1-z_2)\Gamma(\frac{d}{2}-a_3+z_1+z_2)\Gamma(d-a_2-a_3-a_4+z_1+z_2)}{\Gamma(2d-a_1-a_2-2a_3-a_4-a_5+z_1+z_2)}.$$

Test diagram/results when $a_i = 1$

1. $d = 3 - 2\varepsilon$:
   $$I_1(1,1,1,1,1) = I_2(1,1,1,1,1) = \frac{\pi}{\Gamma(-2\varepsilon)}\left(\frac{1}{\varepsilon^2} + \frac{2}{\varepsilon} + \mathcal{O}(\varepsilon)\right)^9;$$
2. $d = 4 - 2\varepsilon$:
   $$I_1 = I_2 = \frac{\mathcal{O}(\varepsilon)}{\Gamma(1-2\varepsilon)};$$
3. $d = 6 - 2\varepsilon$:
   $$I_1 = I_2 = -\frac{1}{\Gamma(3-2\varepsilon)}\left(\frac{1}{6\varepsilon^2} + \frac{1}{2\varepsilon} + \mathcal{O}(\varepsilon)\right)^{10}.$$

[9] Kotikov and Teber 2018, "Multi-loop techniques for massless Feynman diagram calculations".
[10] Vasil'ev 2004, The Field Theoretic Renormalization Group in Critical Behavior Theory and Stochastic Dynamics.

Conclusion

1. - We have performed six-loop calculation of the tricritical exponents of the $O(n)$-symmetric $\varphi^4 + \varphi^6$ theory using presented computer program;
   - Both η and ν tricritical exponents completely coincided with the results of the work[11];
   - ϕ_t tricritical exponent differs from the result presented in the work[11];
   - TODO: 8-loop calculations in the $O(n)$-symmetric $\varphi^4 + \varphi^6$ theory.
2. We have presented star-triangle transformation through Mellin-Barnes integrals.

[11] J. S. Hager 2002, "Six-loop renormalization group functions of $O(n)$-symmetric ϕ^6-theory and ϵ-expansions of tricritical exponents up to ϵ^3".

Thank you!