Seminar
QUANTUM FIELD THEORY

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Algebraic structure of the renormalization group in the renormalizable QFT theories

Based on the papers

A.L.Kataev, K.S., ArXiv:2404.15856 [hep-th] (INR-TH-2024-003);
N.P.Meshcheriakov, V.V.Shatalova, K.S., Phys.Rev.D **106** (2022) 10, 105011;
JHEP 12 (2023) 097; ArXiv:2405.11557 [hep-th].

Divergences in renormalizable quantum field theory models can be removed by renormalization, e.g.,

$$\alpha_0 = \alpha_0(\alpha(\mu), \ln\Lambda/\mu); \qquad \varphi = Z(\alpha, \ln\Lambda/\mu)\varphi_R,$$

where μ is a renormalization point and Λ is the dimensionful regularization parameter. However, the renormalization procedure is not uniquely defined.

For instance, it is possible to remove divergences in the two-point Green function of the background gauge field by the splitting of the bare coupling constant to the renormalized coupling constant and counterterm either as

$$\frac{1}{\alpha_0} = \frac{1}{\alpha} - \beta_1 \ln\frac{\Lambda}{\mu} + O(\alpha), \qquad \text{or as} \qquad \frac{1}{\alpha_0} = \frac{1}{\alpha'} - \beta_1\Big(\ln\frac{\Lambda}{\mu} + c_1\Big) + O(\alpha'),$$

where c_1 is a finite constant. These two definitions of the renormalized coupling constant are related by the equation

$$\frac{1}{\alpha'} = \frac{1}{\alpha} + \beta_1 c_1 + O(\alpha)$$

or, in other words, by the finite renormalization

$$\alpha' = \alpha - \alpha^2\beta_1 c_1 + O(\alpha^3).$$

Renormalization group functions and their dependence on a renormalization prescription

In general, it is possible to make the finite renormalizations

$$\alpha' = \alpha'(\alpha); \qquad Z'(\alpha', \ln\Lambda/\mu) = z(\alpha) Z(\alpha, \ln\Lambda/\mu).$$

Note that the renormalization group functions (RGFs)

$$\beta(\alpha) \equiv \frac{d\alpha(\mu)}{d\ln\mu}\bigg|_{\alpha_0=\text{const}}; \qquad \gamma(\alpha) \equiv \frac{d}{d\ln\mu}\ln Z(\alpha, \ln\Lambda/\mu)\bigg|_{\alpha_0=\text{const}}.$$

nontrivially change under the finite renormalizations

A. A. Vladimirov, Teor. Mat. Fiz. **25** (1975), 335; Sov. J. Nucl. Phys. **31** (1980), 558; A. A. Vladimirov and D. V. Shirkov, Sov. Phys. Usp. **22** (1979), 860.

$$\beta'(\alpha') = \frac{d\alpha'}{d\alpha}\beta(\alpha); \qquad \gamma'(\alpha') = \frac{d\ln z}{d\alpha}\beta(\alpha) + \gamma(\alpha).$$

The perturbative expansions of RGFs can be written as

$$\beta(\alpha) = \sum_{n=1}^{\infty}\beta_n\alpha^{n+1}; \qquad \gamma(\alpha) = \sum_{n=1}^{\infty}\gamma_n\alpha^n,$$

where the index n numerates an order of the perturbation theory.

As is well known, two first coefficients of the gauge β-function and the first coefficient of the anomalous dimension are scheme independent while the others depend on a specific choice of the renormalization prescription.

$$\beta_1' = \beta_1; \qquad \beta_2' = \beta_2; \qquad \gamma_1' = \gamma_1.$$

Finite renormalizations form an infinite dimensional Lie group, which is investigated in this talk. In particular it is planned to

1. Construct the corresponding Lie algebra.

2. Demonstrate that large finite renormalizations can be obtained with the help of the exponential map.

3. Describe the Abelian subalgebra corresponding to the changes of the renormalization point.

4. Demonstrate that the algebraic structure of the rescaling subalgebra leads to some simple all-loop expressions for the renormalization constants which encode all equations relating coefficients at higher ε-poles and logarithms to the coefficients of RGFs.

Exponential map

Let us now specify the group structure of the renormalization group.
For any Lie group $\mathcal{G}$ in a certain vicinity of the identity element 1 the group element $\hat{\omega} \in \mathcal{G}$ can be presented as the exponential of the corresponding Lie algebra $\mathcal{A}$ element $\hat{a} \in \mathcal{A}$,

$$\hat{\omega} = \exp(\hat{a}).$$

To describe the group structure of the renormalization group, it is sufficient to construct its Lie algebra. First, for simplicity, we consider the finite renormalizations of the coupling constant $\alpha \to \alpha'(\alpha)$. In the infinitesimal form they can be presented as the series

$$\delta\alpha = -\sum_{n=1}^{\infty} a_n \alpha^{n+1} \equiv \sum_{n=1}^{\infty} a_n \hat{L}_n \alpha \equiv \hat{a}\alpha,$$

where a_n are arbitrary (real) constants. The operators $\hat{L}_n$ are the generators of the renormalization group, where $n \geq 1$ due to using of the perturbation theory.

The finite transformations are obtained with the help of the exponential map

$$\alpha' = \hat{\omega}\alpha = \exp\Big(\sum_{n=1}^{\infty} a_n \hat{L}_n\Big)\alpha = \alpha + \delta\alpha + O(a^2).$$

Lie algebra of the renormalization group is determined by the commutation relations of its generators. By definition,

$$\hat{L}_n \alpha \equiv -\alpha^{n+1}.$$

If these generators act on an arbitrary function of α, then

$$\hat{L}_n = -\alpha^{n+1} \frac{d}{d\alpha}.$$

These operators satisfy the commutation relations of the Witt algebra

$$[\hat{L}_n, \hat{L}_m] = (n-m)\hat{L}_{n+m}.$$

However, in the Witt algebra n is an arbitrary positive integer, while in the case under consideration $n \geq 1$. Therefore, the Lie algebra of the renormalization group (for the renormalization of charge) is a subalgebra of the Witt algebra generated by $\hat{L}_n$ with $n \geq 1$. (This is the derived algebra of the Borel subalgebra.) Note that the central extension of the Witt algebra

$$[\hat{L}_n, \hat{L}_m] = (n-m)\hat{L}_{n+m} + \frac{n^3 - n}{12}\delta_{m+n,0}\hat{C}; \qquad [\hat{L}_n, \hat{C}] = 0.$$

is well known as Virasoro algebra.

A representation acting on the Green's functions

Under the finite renormalization of the coupling constant the Green's functions change as

$$G'_R\big(\alpha', p_1/\mu, \ldots, p_n/\mu\big) = z_G(\alpha) G_R\big(\alpha, p_1/\mu, \ldots, p_n/\mu\big).$$

First we consider the case $z_G(\alpha) = 1$. Then the infinitesimal change of the Green's function can be written as

$$G'_R(\alpha, p/\mu) - G_R(\alpha, p/\mu) = \sum_{n=1}^{\infty} a_n T_1(\hat{L}_n) G_R(\alpha, p/\mu) + O(a^2) \equiv \delta G_R + O(a^2),$$

where $T_1(\hat{L}_n)$ are the generators in the considered representation,

$$T_1(\hat{L}_n) G_R = \alpha^{n+1} \frac{\partial}{\partial\alpha} G_R.$$

Note that the argument of the function G'_R is the same as of the function G_R, so that $T_1(\hat{L}_n)\alpha = 0$. Therefore, a different sign is really needed,

$$[T_1(\hat{L}_n), T_1(\hat{L}_m)] G_R = T_1(\hat{L}_n)\, \alpha^{m+1} \frac{\partial}{\partial\alpha} G_R - (n \leftrightarrow m) = \alpha^{m+1} \frac{\partial}{\partial\alpha}\Big(\alpha^{n+1} \frac{\partial}{\partial\alpha} G_R\Big)$$
$$-(n \leftrightarrow m) = (n-m)\alpha^{n+m+1} \frac{\partial}{\partial\alpha} G_R = (n-m) T_1(\hat{L}_{n+m})\, G_R.$$

As well known, the coefficients of the β-function change under the transformations of the renormalization group. Let us construct its generators in this representation. Taking into account that

$$\beta'(\alpha') = \frac{d\alpha'}{d\alpha}\beta(\alpha)$$

we see that under the infinitesimal finite renormalizations

$$\beta'\Big(\alpha - \sum_{n=1}^{\infty} a_n \alpha^{n+1}\Big) = \Big(1 - \sum_{n=1}^{\infty} a_n (n+1)\alpha^n\Big)\beta(\alpha) + O(a^2).$$

Therefore,

$$\beta'(\alpha) - \beta(\alpha) = \sum_{n=1}^{\infty} a_n \Big(\alpha^{n+1}\frac{d\beta(\alpha)}{d\alpha} - (n+1)\alpha^n\beta(\alpha)\Big) + O(a^2)$$

$$\equiv \sum_{n=1}^{\infty} a_n T_2(\hat{L}_n)\beta(\alpha) + O(a^2) \equiv \delta\beta(\alpha) + O(a^2),$$

where the operators $T_2(\hat{L}_n)$ are the generators of considered representation of the renormalization group,

$$T_2(\hat{L}_n)\,\beta(\alpha) = \Big(\alpha^{n+1}\frac{d}{d\alpha} - (n+1)\alpha^n\Big)\beta(\alpha),$$

which satisfy the commutation relations of the Witt algebra.

The non-infinitesimal transformations are obtained using the exponential map, for instance,

$$\alpha' = \exp\Big(\sum_{n=1}^{\infty} a_n \hat{L}_n\Big)\alpha = \exp\Big(-\sum_{n=1}^{\infty} a_n \alpha^{n+1}\frac{d}{d\alpha}\Big)\alpha.$$

Similarly, the expression for β-function in the new renormalization scheme is written in the form

$$\beta'(\alpha) = \exp\Big(\sum_{n=1}^{\infty} a_n T_2(\hat{L}_n)\Big)\beta(\alpha) = \exp\Big(\sum_{n=1}^{\infty} a_n\Big(\alpha^{n+1}\frac{d}{d\alpha} - (n+1)\alpha^n\Big)\Big)\beta(\alpha).$$

Let us verify these exact expressions in the lowest approximations:

$$\alpha' = \alpha - a_1\alpha^2 + (-a_2 + a_1^2)\alpha^3 + \Big(-a_3 + \frac{5}{2}a_1 a_2 - a_1^3\Big)\alpha^4 + O(\alpha^5)$$
$$\equiv \alpha + k_1\alpha^2 + k_2\alpha^3 + k_3\alpha^4 + O(\alpha^5).$$

Similarly, the new β-function in the lowest orders (up to and including the four-loop approximation) takes the form

$$\beta'(\alpha) = \beta_1\alpha^2 + \beta_2\alpha^3 + (\beta_3 + a_1\beta_2 - a_2\beta_1)\alpha^4 + (\beta_4 + 2a_1\beta_3 - 2a_3\beta_1$$
$$+ a_1^2\beta_2 - a_1 a_2\beta_1)\alpha^5 + O(\alpha^6) \equiv \beta_1'\alpha^2 + \beta_2'\alpha^3 + \beta_3'\alpha^4 + \beta_4'\alpha^5 + O(\alpha^6).$$

Non-infinitesimal finite renormalizations of the coupling constant

Expressing the coefficients a_n in terms of k_n the transformations of the β-function coefficients can be written as

$$\beta'_1 = \beta_1; \qquad \beta'_2 = \beta_2; \qquad \beta'_3 = \beta_3 - k_1\beta_2 + k_2\beta_1 - k_1^2\beta_1;$$
$$\beta'_4 = \beta_4 - 2k_1\beta_3 + 2k_3\beta_1 - 6k_1k_2\beta_1 + 4k_1^3\beta_1 + k_1^2\beta_2.$$

They exactly agree with the direct calculations made with the help of the equation

$$\beta'(\alpha') = \frac{d\alpha'}{d\alpha}\beta(\alpha).$$

Also they can equivalently be rewritten in the form

$$k_2 = \frac{\beta'_3 - \beta_3}{\beta_1} + \frac{k_1\beta_2}{\beta_1} + k_1^2;$$
$$k_3 = \frac{\beta'_4 - \beta_4}{2\beta_1} + \frac{k_1(3\beta'_3 - 2\beta_3)}{\beta_1} + \frac{5k_1^2\beta_2}{2\beta_1} + k_1^3,$$

in which they coincide with the expressions presented in

A. L. Kataev and M. D. Vardiashvili, Phys. Lett. B **221** (1989), 377.

Thus, large finite renormalizations can be obtained using the exponential map.

Let us consider a particular case of finite renormalizations changing the renormalization scale

$$\alpha(\mu) \to \alpha'(\mu) \equiv \alpha(\mu'),$$

parametrized by $t = \ln \frac{\mu'}{\mu}$.

These transformations form an Abelian subgroup of the renormalization group.

For the infinitesimal transformations for which μ' is close to μ (or, equivalently, the parameter t is small)

$$\alpha' - \alpha = \beta(\alpha)t + O(t^2) = t\sum_{n=1}^{\infty} \beta_n \alpha^{n+1} + O(t^2) = -t\sum_{n=1}^{\infty} \beta_n \hat{L}_n \alpha + O(t^2).$$

Therefore, these transformations are generated by the operator

$$\hat{L} = -\sum_{n=1}^{\infty} \beta_n \hat{L}_n = -\sum_{n=1}^{\infty} \beta_n \alpha^{n+1} \frac{d}{d\alpha} = \beta(\alpha)\frac{d}{d\alpha},$$

so that the infinitesimal change of a function $f(\alpha)$ is

$$\delta f(\alpha) = \ln\frac{\mu'}{\mu}\,\beta(\alpha)\frac{d}{d\alpha}f(\alpha) = \ln\frac{\mu'}{\mu}\,\hat{L}f(\alpha).$$

The corresponding finite transformations are obtained with the help of the exponential map

S.Groote, J.G.Korner and A.A.Pivovarov, Phys. Rev. D **65** (2002), 036001;
S.V.Mikhailov, JHEP **06** (2007), 009;
A. L. Kataev and S. V. Mikhailov, Phys. Rev. D **91** (2015) no.1, 014007.

$$f(\alpha') = \exp\Big(\ln\frac{\mu'}{\mu}\,\beta(\alpha)\frac{\partial}{\partial\alpha}\Big) f(\alpha).$$

As a check, it is easy to verify that the β-function remains unchanged,

$$T_2(\hat{L})\,\beta(\alpha) = -\sum_{n=1}^{\infty}\beta_n T_2(\hat{L}_n)\,\beta(\alpha) = -\sum_{n=1}^{\infty}\beta_n\Big(\alpha^{n+1}\frac{d}{d\alpha} - (n+1)\alpha^n\Big)\beta(\alpha)$$

$$= -\Big(\beta(\alpha)\frac{d}{d\alpha} - \frac{d\beta(\alpha)}{d\alpha}\Big)\beta(\alpha) = 0.$$

so that

$$\beta'(\alpha) = \exp\Big(tT_2(\hat{L})\Big)\beta(\alpha) = \beta(\alpha).$$

In what follows this subgroup will be used for deriving the all-loop expressions for renormalizations constants which relate them to RGFs.

Let us now investigate the finite renormalizations that include the renormalization of the matter fields and masses. They are determined by the functions $\alpha'(\alpha)$ and $z(\alpha)$. For the infinitesimal finite renormalizations we present $z(\alpha)$ in the form

$$z(\alpha) = 1 - \sum_{n=1}^{\infty} z_n \alpha^n + O(az, z^2).$$

Under this finite renormalization the renormalized fields change as

$$\varphi'_R = z^{-1}(\alpha)\varphi_R = \Big(1 + \sum_{n=1}^{\infty} z_n \alpha^n + O(z^2)\Big)\varphi_R \equiv \varphi_R + \sum_{n=1}^{\infty} z_n \hat{G}_n \varphi_R + O(z^2),$$

where the operators

$$\hat{G}_n \varphi_R \equiv \alpha^n \varphi_R$$

generate the infinitesimal finite renormalizations of the matter fields. Then it is easy to see that they satisfy the commutation relations

$$[\hat{L}_n, \hat{L}_m] = (n-m)\hat{L}_{n+m}; \qquad [\hat{G}_n, \hat{G}_m] = 0; \qquad [\hat{L}_n, \hat{G}_m] = -m\hat{G}_{n+m},$$

where $n, m \geq 1$. The corresponding Jacobi identities can easily be verified.

To construct the explicit expressions for the renormalization group generators in the representation acting on the Green's functions, we use the perturbative expansion of the finite function $z_G(\alpha)$,

$$z_G(\alpha) = 1 - \sum_{n=1}^{\infty} (z_G)_n \alpha^n + O(z_G^2),$$

where $(z_G)_n$ are small parameters. Then, under the infinitesimal finite renormalization a Green's function changes as

$$\delta G_R = \sum_{n=1}^{\infty} a_n \alpha^{n+1} \frac{\partial}{\partial\alpha} G_R - \sum_{n=1}^{\infty} (z_G)_n \alpha^n G_R \equiv \sum_{n=1}^{\infty} \Big(a_n T_1(\hat{L}_n) + (z_G)_n T_1(\hat{G}_n) \Big) G_R.$$

From this equation we obtain that in the considered representation the renormalization group generators are written in the form

$$T_1(\hat{L}_n) = \alpha^{n+1} \frac{\partial}{\partial\alpha}; \qquad T_1(\hat{G}_n) = -\alpha^n.$$

The commutation relations presented above are verified straightforwardly, e.g.,

$$[T_1(\hat{L}_n), T_1(\hat{G}_m)] G_R = \Big(- T_1(\hat{L}_n) \alpha^m - T_1(\hat{G}_m) \alpha^{n+1} \frac{\partial}{\partial\alpha} \Big) G_R$$

$$= -\alpha^{m+n+1} \frac{\partial}{\partial\alpha} G_R + \alpha^{n+1} \frac{\partial}{\partial\alpha} \Big(\alpha^m G_R \Big) = m \alpha^{m+n} G_R = -m T_1(\hat{G}_{m+n}) G_R.$$

From the equations describing how RGFs change under finite renormalization we obtain the infinitesimal transformations

$$\gamma'(\alpha)-\gamma(\alpha)=\delta\gamma(\alpha)+O(a^2,az,z^2)=\sum_{n=1}^{\infty}a_n\alpha^{n+1}\frac{d}{d\alpha}\gamma(\alpha)-\sum_{n=1}^{\infty}z_n n\alpha^{n-1}\beta(\alpha)$$

$$+O(a^2,az,z^2)\equiv\sum_{n=1}^{\infty}\Big(a_nT_2(\hat{L}_n)+z_nT_2(\hat{G}_n)\Big)\gamma(\alpha)+O(a^2,az,z^2).$$

It is convenient to present the corresponding generators in the matrix form

$$T_2(\hat{L}_n)\begin{pmatrix}\beta(\alpha)\\ \gamma(\alpha)\end{pmatrix}=\begin{pmatrix}\alpha^{n+1}\frac{d}{d\alpha}-(n+1)\alpha^n & 0\\ 0 & \alpha^{n+1}\frac{d}{d\alpha}\end{pmatrix}\begin{pmatrix}\beta(\alpha)\\ \gamma(\alpha)\end{pmatrix};$$

$$T_2(\hat{G}_n)\begin{pmatrix}\beta(\alpha)\\ \gamma(\alpha)\end{pmatrix}=\begin{pmatrix}0 & 0\\ -n\alpha^{n-1} & 0\end{pmatrix}\begin{pmatrix}\beta(\alpha)\\ \gamma(\alpha)\end{pmatrix},$$

which produces the explicit expressions for them in the considered representation.

The large transformations of RGFs can be obtained with the help of the exponential map,

$$\begin{pmatrix} \beta'(\alpha) \\ \gamma'(\alpha) \end{pmatrix} = \exp\Big(\sum_{n=1}^{\infty} a_n T_2(\hat{L}_n) + \sum_{n=1}^{\infty} z_n T_2(\hat{G}_n)\Big) \begin{pmatrix} \beta(\alpha) \\ \gamma(\alpha) \end{pmatrix}$$

$$= \exp\left\{ \begin{pmatrix} a_1\Big(\alpha^2\dfrac{d}{d\alpha} - 2\alpha\Big) + a_2\Big(\alpha^3\dfrac{d}{d\alpha} - 3\alpha^2\Big) + \ldots & 0 \\ -z_1 - 2z_2\alpha - 3z_3\alpha^2 + \ldots & a_1\alpha^2\dfrac{d}{d\alpha} + a_2\alpha^3\dfrac{d}{d\alpha} + \ldots \end{pmatrix}\right\} \begin{pmatrix} \beta(\alpha) \\ \gamma(\alpha) \end{pmatrix}.$$

The upper string gives the transformation of the β-function considered earlier. Expanding the lower string in powers of α we derive the expression for the new anomalous dimension. It is convenient to express it in terms of $\alpha'(\alpha)$ and $z(\alpha)$. Taking into account that for the infinitesimal transformations

$$Z'(\alpha - \sum_{n=1}^{\infty} a_n\alpha^{n+1}, \ln\Lambda/\mu) = \Big(1 - \sum_{n=1}^{\infty} z_n\alpha^n\Big) Z(\alpha, \ln\Lambda/\mu) + O(a^2, az, z^2)$$

we obtain

$$Z'(\alpha, \ln\Lambda/\mu) = \Big(1 + \sum_{n=1}^{\infty} a_n\alpha^{n+1}\frac{\partial}{\partial\alpha} - \sum_{n=1}^{\infty} z_n\alpha^n\Big) Z(\alpha, \ln\Lambda/\mu) + O(a^2, az, z^2)$$

$$= \Big(1 - \sum_{n=1}^{\infty} a_n\hat{L}_n - \sum_{n=1}^{\infty} z_n\hat{G}_n + O(a^2, az, z^2)\Big) Z(\alpha, \ln\Lambda/\mu).$$

The all-order expression for $Z'(\alpha,\ln\Lambda/\mu)$ is obtained with the help of the exponential map,

$$Z'(\alpha,\ln\Lambda/\mu) = \exp\Big(-\sum_{n=1}^{\infty} a_n \hat{L}_n - \sum_{n=1}^{\infty} z_n \hat{G}_n\Big) Z(\alpha,\ln\Lambda/\mu).$$

Also we know that

$$z(\alpha) Z(\alpha,\ln\Lambda/\mu) = Z'(\alpha',\ln\Lambda/\mu) = \exp\Big(\sum_{n=1}^{\infty} a_n \hat{L}_n\Big) Z'(\alpha,\ln\Lambda/\mu)$$

$$= \exp\Big(\sum_{n=1}^{\infty} a_n \hat{L}_n\Big) \exp\Big(-\sum_{n=1}^{\infty} a_n \hat{L}_n - \sum_{n=1}^{\infty} z_n \hat{G}_n\Big) Z(\alpha,\ln\Lambda/\mu).$$

Therefore, the all-order expression for $z(\alpha)$ can be presented as

$$z(\alpha) = \exp\Big(\sum_{n=1}^{\infty} a_n \hat{L}_n\Big) \exp\Big(-\sum_{n=1}^{\infty} a_n \hat{L}_n - \sum_{n=1}^{\infty} z_n \hat{G}_n\Big).$$

It can be calculated using the Baker–Campbell–Hausdorff formula

$$e^A e^B = \exp\Big(A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] + \frac{1}{12}[B,[B,A]] + \ldots\Big).$$

With the help of the commutation relations of the renormalization group algebra in the lowest orders this expression can be written as

$$z(\alpha) = 1 - \alpha z_1 + \alpha^2\Big(- z_2 + \frac{1}{2} z_1^2 + \frac{1}{2} a_1 z_1\Big) + \alpha^3\Big(- z_3 + z_1 z_2 - \frac{1}{6} z_1^3$$
$$+\frac{1}{2} a_2 z_1 + a_1 z_2 - \frac{1}{3} a_1^2 z_1 - \frac{1}{2} a_1 z_1^2\Big) + O(\alpha^4) \equiv 1 + l_1\alpha + l_2\alpha^2 + l_3\alpha^3 + O(\alpha^4).$$

Then, using the relations between the coefficients z_n and l_n from the exponential map we obtain

$$\gamma_1' = \gamma_1; \qquad \gamma_2' = \gamma_2 + \beta_1 l_1 - \gamma_1 k_1;$$

$$\gamma_3' = \gamma_3 + \beta_2 l_1 + \beta_1 (2 l_2 - 2 k_1 l_1 - l_1^2) - 2 k_1 \gamma_2 + \gamma_1 (2 k_1^2 - k_2);$$

$$\gamma_4' = \gamma_4 + \beta_3 l_1 + \beta_2 (2 l_2 - l_1^2 - 3 k_1 l_1) + \beta_1 (3 l_3 - 3 l_2 l_1 + l_1^3 - 6 k_1 l_2 + 3 k_1 l_1^2$$
$$+5 k_1^2 l_1 - 2 k_2 l_1) - 3\gamma_3 k_1 + \gamma_2 (5 k_1^2 - 2 k_2) + \gamma_1 (-k_3 - 5 k_1^3 + 5 k_2 k_1).$$

Exactly the same results are obtained directly from the equation

$$\gamma'(\alpha') = \frac{d\ln z(\alpha)}{d\alpha}\beta(\alpha) + \gamma(\alpha).$$

Finite renormalizations changing the renormalization point $\mu \to \mu'$ have the form

$$\alpha(\mu) \to \alpha(\mu') \equiv \alpha'(\mu);$$
$$Z\big(\alpha(\mu), \ln\Lambda/\mu\big) \to Z'\big(\alpha(\mu'), \ln\Lambda/\mu'\big) \equiv z\big(\alpha(\mu)\big) Z\big(\alpha(\mu), \ln\Lambda/\mu\big).$$

For the infinitesimal field transformation they give

$$\varphi_R' - \varphi_R = -\delta z \varphi_R + O(t^2) = -\gamma(\alpha) t\, \varphi_R + O(t^2)$$
$$= -t \sum_{n=1}^{\infty} \gamma_n \alpha^n\, \varphi_R + O(t^2) = -t \sum_{n=1}^{\infty} \gamma_n \hat{G}_n \varphi_R + O(t^2).$$

Therefore, the overall transformation consists of both finite renormalizations of the coupling constant with $a_n = -t\beta_n$ and the finite renormalization of fields with the coefficients $z_n = -t\gamma_n$,

$$\hat{L} = -\sum_{n=1}^{\infty} \Big(\beta_n \hat{L}_n + \gamma_n \hat{G}_n\Big).$$

Invariance of RGFs under the rescaling subgroup

It is possible to verify that, acting on RGFs, the generator of the rescaling subgroup gives 0, so that RGFs remain invariant under this transformation:

$$T_2(\hat{L})\begin{pmatrix}\beta(\alpha)\\ \gamma(\alpha)\end{pmatrix} = -\sum_{n=1}^{\infty}\Big(\beta_n T_2(\hat{L}_n) + \gamma_n T_2(\hat{G}_n)\Big)\begin{pmatrix}\beta(\alpha)\\ \gamma(\alpha)\end{pmatrix}$$

$$= -\begin{pmatrix}\beta(\alpha)\dfrac{d}{d\alpha} - \dfrac{d\beta(\alpha)}{d\alpha} & 0\\ -\dfrac{d\gamma(\alpha)}{d\alpha} & \beta(\alpha)\dfrac{d}{d\alpha}\end{pmatrix}\begin{pmatrix}\beta(\alpha)\\ \gamma(\alpha)\end{pmatrix} = 0,$$

so that

$$\begin{pmatrix}\beta'(\alpha)\\ \gamma'(\alpha)\end{pmatrix} = \exp\Big(\ln\frac{\mu'}{\mu}\, T_2(\hat{L})\Big)\begin{pmatrix}\beta(\alpha)\\ \gamma(\alpha)\end{pmatrix} = \begin{pmatrix}\beta(\alpha)\\ \gamma(\alpha)\end{pmatrix}.$$

Application of the rescaling subgroup: the structure of the renormalization constants

As well known, the coefficients at higher ε-poles are related to the coefficients of RGFs

G. 't Hooft, Nucl. Phys. B **61** (1973), 455.

by the 't Hooft pole equations, see, e.g., the review

D. I. Kazakov, "Radiative Corrections, Divergences, Regularization, Renormalization, Renormalization Group and All That in Examples in Quantum Field Theory," arXiv:0901.2208 [hep-ph].

So do the coefficients at higher powers of logarithms

J. C. Collins, "Renormalization: An Introduction to Renormalization, The Renormalization Group, and the Operator Product Expansion," 1986.

The argumentation based on the algebraic structure of the rescaling subgroup allows to construct beautiful expressions for renormalization constants relating them to the renormalization group functions and give their simple derivation.

It is convenient to consider a case when both ε-poles and logarithms are present in the renormalization constants.

Dimensional regularization with $\Lambda \ne \mu$

In the dimension $D \ne 4$ the gauge coupling constant $\widetilde{\alpha}_0$ has the dimension m^ε and can, therefore, be presented as $\widetilde{\alpha}_0 = \alpha_0 \Lambda^\varepsilon$, where Λ is a parameter with the dimension of mass. Then the renormalization of the coupling constant can be made according to the prescription

$$\alpha_0 = \Big(\frac{\mu}{\Lambda}\Big)^\varepsilon \boldsymbol{\alpha}\, \boldsymbol{Z}_{\boldsymbol{\alpha}}^{-1}(\boldsymbol{\alpha}, \varepsilon^{-1}),$$

where μ is a renormalization point and $\boldsymbol{\alpha}$ is the renormalized gauge coupling.
In the case of using the MS scheme the renormalization constant includes only ε-poles. The $\overline{\mbox{MS}}$-scheme is obtained after the redefinition of the renormalization point

$$\mu \to \frac{\mu\, \exp(\gamma/2)}{\sqrt{4\pi}},$$

where $\gamma \equiv -\Gamma'(1) \approx 0.577$.
Field renormalization constant $\boldsymbol{Z}(\boldsymbol{\alpha}, \varepsilon^{-1})$ is defined by requiring the finiteness of the corresponding renormalized Green's function G_R in the limit $\varepsilon \to 0$,

$$G_R\Big(\boldsymbol{\alpha}, \ln\frac{\mu}{P}\Big) = \lim_{\varepsilon\to 0} \boldsymbol{Z}(\boldsymbol{\alpha}, \varepsilon^{-1})\, G\Big[\Big(\frac{\mu}{P}\Big)^\varepsilon \boldsymbol{\alpha} \boldsymbol{Z}_{\boldsymbol{\alpha}}^{-1}(\boldsymbol{\alpha}, \varepsilon^{-1}), \varepsilon^{-1}\Big].$$

It is convenient to encode ultraviolet divergences in RGFs. D-dimensional RGFs are defined as

$$\boldsymbol{\beta}(\boldsymbol{\alpha},\varepsilon) \equiv \frac{d\boldsymbol{\alpha}(\alpha_0(\Lambda/\mu)^\varepsilon,\varepsilon^{-1})}{d\ln\mu}\Big|_{\alpha_0=\text{const}}; \qquad \boldsymbol{\gamma}(\boldsymbol{\alpha}) \equiv \frac{d\ln \boldsymbol{Z}(\boldsymbol{\alpha},\varepsilon^{-1})}{d\ln\mu}\Big|_{\alpha_0=\text{const}}.$$

Alternatively, the renormalization can be made in the four-dimensional form

$$\frac{1}{\alpha_0} = \frac{Z_\alpha(\alpha,\varepsilon^{-1},\ln\Lambda/\mu)}{\alpha};$$

$$G_R\Big(\alpha,\ln\frac{\mu}{P}\Big) = \lim_{\varepsilon\to 0} Z(\alpha,\varepsilon^{-1},\ln\Lambda/\mu)\, G\Big[\Big(\frac{\Lambda}{P}\Big)^\varepsilon \alpha Z_\alpha^{-1}(\alpha,\varepsilon^{-1},\ln\Lambda/\mu),\varepsilon^{-1}\Big].$$

In this case the renormalization constants Z_α and Z should not contain positive powers of ε, and RGFs are defined as

$$\beta(\alpha) \equiv \frac{d\alpha(\alpha_0,\varepsilon^{-1},\ln\Lambda/\mu)}{d\ln\mu}\Big|_{\alpha_0=\text{const}}; \qquad \gamma(\alpha) \equiv \frac{d\ln Z(\alpha,\varepsilon^{-1},\ln\Lambda/\mu)}{d\ln\mu}\Big|_{\alpha_0=\text{const}}.$$

The D-dimensional RGFs are related to these ones by the equations

$$\boldsymbol{\beta}(\alpha,\varepsilon) = -\varepsilon\alpha + \beta(\alpha); \qquad \boldsymbol{\gamma}(\alpha) = \gamma(\alpha).$$

It is important that $\beta(\alpha)$ and $\gamma(\alpha)$ do not depend on ε and $\ln\Lambda/\mu$. Using this statement it is possible to relate the coefficients at higher ε-poles and higher powers of $\ln\Lambda/\mu$ to the coefficients at simple (ε^{-1}) poles and at the first power of $\ln\Lambda/\mu$, which determine RGFs. For the regularization considered here the corresponding formulas are encoded in the all-order exact equations

N.P.Meshcheriakov, V.V.Shatalova, K.S., JHEP **12** (2023) 097.

$$\frac{\partial \ln Z_\alpha}{\partial \ln\alpha} = 1 - \exp\Big\{\ln\frac{\Lambda}{\mu}\,\frac{\hat\partial}{\partial\ln\alpha}\,\frac{\beta(\alpha)}{\alpha}\Big\}\Big(1-\frac{\beta(\alpha)}{\varepsilon\alpha}\Big)^{-1};$$

$$\Big(\frac{\partial}{\partial\ln\alpha} - S\Big)(Z_\alpha)^S = -S\exp\Big\{\ln\frac{\Lambda}{\mu}\Big(\frac{\hat\partial}{\partial\ln\alpha}-S\Big)\frac{\beta(\alpha)}{\alpha}\Big\}\times\Big(1-\frac{\beta(\alpha)}{\varepsilon\alpha}+S\,\widehat{\int}\frac{d\alpha}{\alpha}\,\frac{\beta(\alpha)}{\varepsilon\alpha}\Big)^{-1};$$

$$\frac{\partial \ln Z}{\partial \ln\alpha} = \frac{\alpha\gamma(\alpha)}{\beta(\alpha)} - \exp\Big\{\ln\frac{\Lambda}{\mu}\,\frac{\hat\partial}{\partial\ln\alpha}\,\frac{\beta(\alpha)}{\alpha}\Big\}\Big[\frac{\alpha\gamma(\alpha)}{\beta(\alpha)}\Big(1-\frac{\beta(\alpha)}{\varepsilon\alpha}\Big)^{-1}\Big],$$

where a hat means that the corresponding operator acts on everything to the right of it.

According to

N.P.Meshcheriakov, V.V.Shatalova, K.S., ArXiv:2405.11557 [hep-th].

the above equations can be simplified and rewritten in the form

$$\ln\alpha_0 = \exp\Big(\ln\frac{\Lambda}{\mu}\,\beta(\alpha)\frac{\partial}{\partial\alpha}\Big)\Big\{-\int\limits_0^\alpha \frac{d\alpha}{\alpha}\,\frac{\beta(\alpha)}{\beta(\alpha)-\varepsilon\alpha} + \ln\alpha\Big\};$$

$$\alpha_0^{-S} = \exp\Big(\ln\frac{\Lambda}{\mu}\,\beta(\alpha)\frac{\partial}{\partial\alpha}\Big)\,\alpha^{-S}\exp\Big\{S\int\limits_0^\alpha \frac{d\alpha}{\alpha}\,\frac{\beta(\alpha)}{\beta(\alpha)-\varepsilon\alpha}\Big\};$$

$$\ln Z - \int\limits_a^\alpha d\alpha\,\frac{\gamma(\alpha)}{\beta(\alpha)} = \exp\Big(\ln\frac{\Lambda}{\mu}\,\beta(\alpha)\frac{\partial}{\partial\alpha}\Big)\Big[\int\limits_0^\alpha d\alpha\,\frac{\gamma(\alpha)}{\beta(\alpha)-\varepsilon\alpha} - \int\limits_a^\alpha d\alpha\,\frac{\gamma(\alpha)}{\beta(\alpha)}\Big],$$

where the constant a in the last equation can be arbitrary.
This form reveals the renormalization group origin of the considered equations.
For this purpose, let us first investigate the case of the cut-off type regularizations, which is obtained by removing ε-poles in the formal limit $\varepsilon\to\infty$.

Transformations of the rescaling subgroup

Finite renormalizations changing of the renormalization scale

$$\alpha(\mu) \to \alpha'(\mu) \equiv \alpha(\mu')$$

are generated by the operator

$$\hat{L} \equiv \beta(\alpha)\frac{d}{d\alpha},$$

so that the infinitesimal change of a function $f(\alpha)$ is

$$\delta f(\alpha) = \ln\frac{\mu'}{\mu}\,\beta(\alpha)\frac{d}{d\alpha}f(\alpha) = \ln\frac{\mu'}{\mu}\,\hat{L}f(\alpha).$$

The corresponding finite transformations with the help of the exponential map

> S.Groote, J.G.Korner and A.A.Pivovarov, Phys. Rev. D **65** (2002), 036001;
> S.V.Mikhailov, JHEP **06** (2007), 009;
> A. L. Kataev and S. V. Mikhailov, Phys. Rev. D **91** (2015) no.1, 014007.

$$f(\alpha') = \exp\left(\ln\frac{\mu'}{\mu}\,\beta(\alpha)\frac{\partial}{\partial\alpha}\right)f(\alpha).$$

Let us first consider a simple case, when a theory is regularized by higher derivatives

A.A.Slavnov, Nucl.Phys. **B31**, (1971), 301; Theor.Math.Phys. **13** (1972) 1064.

(or any other regularization of the cut-off type). Then, a renormalization prescription analogous to minimal subtraction is the HD+MSL scheme

A. L. Kataev, K.S., Nucl. Phys. B **875** (2013), 459.

In this scheme the renormalization constants include only powers of $\ln\Lambda/\mu$. Therefore, choosing $\mu' = \Lambda$ and taking into account that $\alpha'(\mu) = \alpha(\Lambda) = \alpha(\alpha_0, \ln\Lambda/\mu = 0) = \alpha_0$ for an arbitrary function $f(\alpha)$ in the HD+MSL scheme we obtain

$$f(\alpha_0) = \exp\Big(\ln\frac{\Lambda}{\mu}\,\beta(\alpha)\frac{\partial}{\partial\alpha}\Big) f(\alpha).$$

This equation exactly reproduces the above expressions for $\ln Z_\alpha$, $(Z_\alpha)^S$, and $\ln Z$ if

$$f(\alpha_0) = \ln\alpha_0; \qquad f(\alpha_0) = \alpha_0^{-S}; \qquad f(\alpha_0) = \int\limits_a^{\alpha_0} d\alpha\,\frac{\beta(\alpha)}{\gamma(\alpha)},$$

respectively.

Let us now consider a more complicated case of the dimensional regularization with $\Lambda \ne \mu$. In this version of dimensional regularization the renormalization constants contain not only ε-poles, but also powers of $\ln \Lambda/\mu$ and various mixed terms. In the standard case $\mu = \Lambda$ from the above equations we see that

$$\alpha \exp\Big\{ -\int\limits_0^\alpha \frac{d\alpha}{\alpha} \frac{\beta(\alpha)}{\beta(\alpha) - \varepsilon\alpha} \Big\}\Big|_{\mu=\Lambda} = \alpha_0 = \alpha Z_\alpha^{-1}(\alpha, \varepsilon^{-1}, 0).$$

A. V. Ivanov, Zap. Nauchn. Semin. POMI **465** (2017), 147; earlier papers (?).

Using the exponential map for an arbitrary μ we obtain

$$f(\alpha_0) = \exp\Big(\ln\frac{\Lambda}{\mu}\,\beta(\alpha)\frac{\partial}{\partial\alpha}\Big) f\bigg(\alpha \exp\Big\{ -\int\limits_0^\alpha \frac{d\alpha}{\alpha} \frac{\beta(\alpha)}{\beta(\alpha) - \varepsilon\alpha} \Big\}\bigg).$$

In particular, for $f(\alpha_0) = 1/\alpha_0$ this equation gives

$$Z_\alpha(\alpha, \varepsilon^{-1}, \ln\Lambda/\mu) = \alpha \exp\Big(\ln\frac{\Lambda}{\mu}\,\beta(\alpha)\frac{\partial}{\partial\alpha}\Big)\Big(\alpha^{-1} Z_\alpha(\alpha, \varepsilon^{-1}, 0)\Big),$$

where

$$Z_\alpha(\alpha, \varepsilon^{-1}, 0) = \exp\bigg(\int\limits_0^\alpha \frac{d\alpha}{\alpha} \frac{\beta(\alpha)}{\beta(\alpha) - \varepsilon\alpha}\bigg).$$

Let us reveal how this equation is related to the algebraic structure of the renormalization group. For this purpose, first, we note that if the operators A and B satisfy the condition $[[A,B],B]=0$, then

$$e^B e^A e^{-B} = \exp(e^B A e^{-B}) = \exp\Big(A - [A,B]\Big).$$

This identity can be applied for transforming the above expression for Z_α if we take the operators

$$A \to \ln\frac{\Lambda}{\mu}\,\beta(\alpha)\frac{\partial}{\partial\alpha};\qquad B \to \ln\alpha,$$

for which the equation $[[A,B],B]=0$ is evidently valid. As a result, the constant Z_α can be cast in the form

$$Z_\alpha(\alpha,\varepsilon^{-1},\ln\Lambda/\mu) = \exp\Big\{\ln\frac{\Lambda}{\mu}\Big[\beta(\alpha)\frac{\partial}{\partial\alpha} - \frac{\beta(\alpha)}{\alpha}\Big]\Big\} Z_\alpha(\alpha,\varepsilon^{-1},0)$$

$$= \exp\Big\{\ln\frac{\Lambda}{\mu}\Big[\beta(\alpha)\frac{\partial}{\partial\alpha} - \gamma_\alpha(\alpha)\Big]\Big\}\exp\Big(\int\limits_0^\alpha \frac{d\alpha}{\alpha}\,\frac{\beta(\alpha)}{\beta(\alpha)-\varepsilon\alpha}\Big),$$

because $\gamma_\alpha(\alpha) \equiv d\ln Z_\alpha/d\ln\mu = \beta(\alpha)/\alpha$.

A similar equation can be derived for an arbitrary renormalization constant Z. Here we describe its renormalization group derivation.

Under the rescaling transformations

$$Z(\alpha,\varepsilon^{-1},\ln\Lambda/\mu) \to Z'(\alpha,\varepsilon^{-1},\ln\Lambda/\mu) \equiv Z(\alpha,\varepsilon^{-1},\ln\Lambda/\mu').$$

For small $t\equiv \ln\mu'/\mu$ these transformations take the form

$$\delta Z = t\,\frac{\partial Z}{\partial\ln\mu} = t\Big(\frac{dZ}{d\ln\mu} - \beta(\alpha)\frac{\partial Z}{\partial\alpha}\Big) = t\Big(\gamma(\alpha) - \beta(\alpha)\frac{\partial}{\partial\alpha}\Big)Z.$$

The corresponding finite transformations are obtained with the help of the exponential map,

$$Z(\alpha,\varepsilon^{-1},\ln\Lambda/\mu') = \exp\Big\{t\Big[\gamma(\alpha) - \beta(\alpha)\frac{\partial}{\partial\alpha}\Big]\Big\}\,Z(\alpha,\varepsilon^{-1},\ln\Lambda/\mu).$$

Setting $\mu'=\Lambda$ we obtain

$$Z(\alpha,\varepsilon^{-1},\ln\Lambda/\mu) = \exp\Big\{\ln\frac{\Lambda}{\mu}\Big(\beta(\alpha)\frac{\partial}{\partial\alpha} - \gamma(\alpha)\Big)\Big\}\,Z(\alpha,\varepsilon^{-1},0),$$

where $Z(\alpha,\varepsilon^{-1},0)$ can easily be found from the above expressions.

Thus, the all-order expression for the renormalization constant Z_α takes the form

$$Z_\alpha(\alpha,\varepsilon^{-1},\ln\Lambda/\mu) = \exp\Big\{\ln\frac{\Lambda}{\mu}\Big[\beta(\alpha)\frac{\partial}{\partial\alpha} - \frac{\beta(\alpha)}{\alpha}\Big]\Big\} Z_\alpha(\alpha,\varepsilon^{-1},0)$$

$$= \exp\Big\{\ln\frac{\Lambda}{\mu}\Big[\beta(\alpha)\frac{\partial}{\partial\alpha} - \gamma_\alpha(\alpha)\Big]\Big\}\exp\Big(\int\limits_0^\alpha \frac{d\alpha}{\alpha}\,\frac{\beta(\alpha)}{\beta(\alpha)-\varepsilon\alpha}\Big),$$

because the anomalous dimension of the coupling constant is $\gamma_\alpha(\alpha) = \beta(\alpha)/\alpha$. This expression is a particular case of the general equation for an arbitrary renormalization constant

$$Z(\alpha,\varepsilon^{-1},\ln\Lambda/\mu) = \exp\Big\{\ln\frac{\Lambda}{\mu}\Big(\beta(\alpha)\frac{\partial}{\partial\alpha} - \gamma(\alpha)\Big)\Big\} Z(\alpha,\varepsilon^{-1},0)$$

$$= \exp\Big\{\ln\frac{\Lambda}{\mu}\Big(\beta(\alpha)\frac{\partial}{\partial\alpha} - \gamma(\alpha)\Big)\Big\}\exp\Big\{\int\limits_0^\alpha d\alpha\,\frac{\gamma(\alpha)}{\beta(\alpha)-\varepsilon\alpha}\Big\},$$

As a correctness test, we have verified that this equation exactly reproduces the five-loop expression for $\ln Z$ presented in

N.P.Meshcheriakov, V.V.Shatalova, K.S., JHEP 12 (2023) 097.

$$\ln Z = -\alpha\gamma_1\Big(\frac{1}{\varepsilon}+\ln\frac{\Lambda}{\mu}\Big) - \frac{\alpha^2}{2}\Big[\gamma_2\Big(\frac{1}{\varepsilon}+2\ln\frac{\Lambda}{\mu}\Big)+\gamma_1\beta_1\Big(\frac{1}{\varepsilon}+\ln\frac{\Lambda}{\mu}\Big)^2\Big]$$
$$-\frac{\alpha^3}{3}\Big[\gamma_3\Big(\frac{1}{\varepsilon}+3\ln\frac{\Lambda}{\mu}\Big)+\gamma_1\beta_2\Big(\frac{1}{\varepsilon^2}+\frac{3}{\varepsilon}\ln\frac{\Lambda}{\mu}+\frac{3}{2}\ln^2\frac{\Lambda}{\mu}\Big)+\gamma_2\beta_1\Big(\frac{1}{\varepsilon^2}+\frac{3}{\varepsilon}\ln\frac{\Lambda}{\mu}+3\ln^2\frac{\Lambda}{\mu}\Big)$$
$$+\gamma_1\beta_1^2\Big(\frac{1}{\varepsilon}+\ln\frac{\Lambda}{\mu}\Big)^3\Big]$$
$$-\frac{\alpha^4}{4}\Big[\gamma_4\Big(\frac{1}{\varepsilon}+4\ln\frac{\Lambda}{\mu}\Big)+\gamma_1\beta_3\Big(\frac{1}{\varepsilon^2}+\frac{4}{\varepsilon}\ln\frac{\Lambda}{\mu}+2\ln^2\frac{\Lambda}{\mu}\Big)+\gamma_2\beta_2\Big(\frac{1}{\varepsilon}+2\ln\frac{\Lambda}{\mu}\Big)^2$$
$$+\gamma_3\beta_1\Big(\frac{1}{\varepsilon^2}+\frac{4}{\varepsilon}\ln\frac{\Lambda}{\mu}+6\ln^2\frac{\Lambda}{\mu}\Big)+2\gamma_1\beta_1\beta_2\Big(\frac{1}{\varepsilon^3}+\frac{4}{\varepsilon^2}\ln\frac{\Lambda}{\mu}+\frac{5}{\varepsilon}\ln^2\frac{\Lambda}{\mu}+\frac{5}{3}\ln^3\frac{\Lambda}{\mu}\Big)$$
$$+\gamma_2\beta_1^2\Big(\frac{1}{\varepsilon^3}+\frac{4}{\varepsilon^2}\ln\frac{\Lambda}{\mu}+\frac{6}{\varepsilon}\ln^2\frac{\Lambda}{\mu}+4\ln^3\frac{\Lambda}{\mu}\Big)+\gamma_1\beta_1^3\Big(\frac{1}{\varepsilon}+\ln\frac{\Lambda}{\mu}\Big)^4\Big]$$
$$-\frac{\alpha^5}{5}\Big[\gamma_5\Big(\frac{1}{\varepsilon}+5\ln\frac{\Lambda}{\mu}\Big)+\gamma_1\beta_4\Big(\frac{1}{\varepsilon^2}+\frac{5}{\varepsilon}\ln\frac{\Lambda}{\mu}+\frac{5}{2}\ln^2\frac{\Lambda}{\mu}\Big)+\gamma_2\beta_3\Big(\frac{1}{\varepsilon^2}+\frac{5}{\varepsilon}\ln\frac{\Lambda}{\mu}+5\ln^2\frac{\Lambda}{\mu}\Big)$$
$$+\gamma_3\beta_2\Big(\frac{1}{\varepsilon^2}+\frac{5}{\varepsilon}\ln\frac{\Lambda}{\mu}+\frac{15}{2}\ln^2\frac{\Lambda}{\mu}\Big)+2\gamma_2\beta_1\beta_2\Big(\frac{1}{\varepsilon^3}+\frac{5}{\varepsilon^2}\ln\frac{\Lambda}{\mu}+\frac{35}{4\varepsilon}\ln^2\frac{\Lambda}{\mu}+\frac{35}{6}\ln^3\frac{\Lambda}{\mu}\Big)$$
$$+\gamma_4\beta_1\Big(\frac{1}{\varepsilon^2}+\frac{5}{\varepsilon}\ln\frac{\Lambda}{\mu}+10\ln^2\frac{\Lambda}{\mu}\Big)+\gamma_3\beta_1^2\Big(\frac{1}{\varepsilon^3}+\frac{5}{\varepsilon^2}\ln\frac{\Lambda}{\mu}+\frac{10}{\varepsilon}\ln^2\frac{\Lambda}{\mu}+10\ln^3\frac{\Lambda}{\mu}\Big)$$
$$+(2\gamma_1\beta_1\beta_3+\gamma_1\beta_2^2)\Big(\frac{1}{\varepsilon^3}+\frac{5}{\varepsilon^2}\ln\frac{\Lambda}{\mu}+\frac{15}{2\varepsilon}\ln^2\frac{\Lambda}{\mu}+\frac{5}{2}\ln^3\frac{\Lambda}{\mu}\Big)+\gamma_1\beta_1^4\Big(\frac{1}{\varepsilon}+\ln\frac{\Lambda}{\mu}\Big)^5$$
$$+\gamma_2\beta_1^3\Big(\frac{1}{\varepsilon^4}+\frac{5}{\varepsilon^3}\ln\frac{\Lambda}{\mu}+\frac{10}{\varepsilon^2}\ln^2\frac{\Lambda}{\mu}+\frac{10}{\varepsilon}\ln^3\frac{\Lambda}{\mu}+5\ln^4\frac{\Lambda}{\mu}\Big)$$
$$+3\gamma_1\beta_1^2\beta_2\Big(\frac{1}{\varepsilon^4}+\frac{5}{\varepsilon^3}\ln\frac{\Lambda}{\mu}+\frac{55}{6\varepsilon^2}\ln^2\frac{\Lambda}{\mu}+\frac{65}{9\varepsilon}\ln^3\frac{\Lambda}{\mu}+\frac{65}{36}\ln^4\frac{\Lambda}{\mu}\Big)\Big]+O(\alpha^6).$$

The equation obtained allows relating coefficients at higher poles and logarithms. For instance,

$$\ln \boldsymbol{Z_\alpha}(\alpha, 1/\varepsilon) = -\sum_{q=1}^{\infty} \sum_{k_1,k_2,\ldots,k_q=1}^{\infty} \frac{1}{K_q}\, \beta_{k_1}\beta_{k_2}\ldots\beta_{k_q}\, \alpha^{K_q}\, \varepsilon^{-q};$$

$$\ln Z_\alpha(\alpha, \ln\Lambda/\mu) = -\sum_{q=1}^{\infty} \sum_{k_1,k_2,\ldots,k_q=1}^{\infty} \frac{1}{K_q}\cdot \frac{K_q\boldsymbol{!}}{q!}\, \beta_{k_1}\beta_{k_2}\ldots\beta_{k_q}\, \alpha^{K_q} \ln^q \frac{\Lambda}{\mu},$$

where $K_m \equiv \sum\limits_{i=1}^{m} k_i;\quad K_m\boldsymbol{!} \equiv K_1 K_2 \ldots K_m;\quad K_0\boldsymbol{!} \equiv 1.$

Also it is possible to find some features of the renormalization constant structure. The simplest one is the relation between coefficients at simple ε-poles and at the first power of $\ln\Lambda/\mu$,

$$\ln Z_\alpha = -\sum_{L=1}^{\infty} \alpha^L \beta_L \Big(\frac{1}{L\varepsilon} + \ln\frac{\Lambda}{\mu}\Big) + \text{higher poles and logarithms}$$

K. G. Chetyrkin, A. L. Kataev and F. V. Tkachov, "Computation of the α_s^2 Correction Sigma-t $(e^+e^- \to$ Hadrons) in QCD," IYaI-P-0170 (1980).

(There are also a lot of more complicated relations.)

Some relations between poles and logarithms for $\ln Z_\alpha$

Although coefficients at higher poles and logarithms are related in a very nontrivial way, some features can be noted. As an example, here we consider $\ln Z_\alpha$.

For $\ln Z_\alpha$ all terms proportional to $1/\varepsilon^2$, $\varepsilon^{-1}\ln\Lambda/\mu$, and $\ln^2\Lambda/\mu$ are factorized into perfect squares.

$$\ln Z_\alpha = -\sum_{L=1}^{\infty}\alpha^L\beta_L\Big(\frac{1}{L\varepsilon}+\ln\frac{\Lambda}{\mu}\Big) - \frac{1}{L}\sum_{L=2}^{\infty}\alpha^L\sum_{k=1}^{L-1}\beta_k\beta_{L-k}\Big(\frac{1}{\varepsilon}+\frac{L}{2}\ln\frac{\Lambda}{\mu}\Big)^2$$

$+$ higher poles and logarithms.

Moreover, some terms have a rather simple structure

$$\ln Z_\alpha = -\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{\big(\beta_k\alpha^k\big)^m}{mk}\Big(\frac{1}{\varepsilon}+k\ln\frac{\Lambda}{\mu}\Big)^m + \text{the other terms.}$$

Some features have also been found for Z_α, $(Z_\alpha)^S$, and $\ln Z$, for instance,

$$\ln Z = -\sum_{L=1}^{\infty}\frac{\alpha^L}{L}\sum_{k=1}^{L}\gamma_{L-k+1}(\beta_1)^{k-1}\varepsilon^{L-k}\Big(\frac{1}{\varepsilon}+\ln\frac{\Lambda}{\mu}\Big)^L\Big|_{\varepsilon^s\to 0\ \text{for all}\ s>0}$$

$+$ terms containing β_i with $i\ge 2$.

Conclusion

- The infinitesimal finite renormalizations of the gauge coupling constant form a subalgebra of the Witt algebra (the central extension of which is the well-known Virasoro algebra). (The index numerating the generators of the Witt algebra is an arbitrary integer, while the finite renormalizations of the coupling constant are numerated by a positive integer.)
- The commutation relations for the algebra of the infinitesimal finite renormalizations which also involve the matter field renormalizations were also constructed.
- The finite renormalizations which belong to the corresponding Lie group can be obtained in standard way with the help of the exponential map.
- The renormalization group contains an Abelian subgroup which corresponds to the shifts of the renormalization point. Under its transformations RGFs do not change.
- Using the exponential map for the Abelian rescaling subgroup it is possible to derive simple formulas which relate the coefficients at ε-poles, powers of $\ln\Lambda/\mu$, and the mixed terms to the coefficients of RGFs.
- Some of the equations obtained allow relating the coefficients at higher poles and at higher logarithms in a simple way, although at the first sight this relation is highly nontrivial.

Thank you for the attention!