Motivic approach to the evaluation of Feynman integrals

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based on [arXiv:1309.5865], [arXiv:1406.2664], [arXiv:1601.08181] and work in progress

Spencer Bloch, Matt Kerr, Charles Doran

Scattering amplitudes are essential tools to understand a variety of physical phenomena from gauge theory to classical and quantum gravity

A convenient approach is to use modern unitarity methods for expanding the amplitude on a basis of integral functions

$$A^{\mathrm{L-loop}} = \sum_{i \in \mathcal{B}(L)} \mathrm{coeff}_i \, \mathrm{Integral}_i + \mathrm{Rational}$$

What are the intrinsic properties of amplitudes of QFT? How much can we understand about the amplitudes without having to compute them?

- What are the generic constraints on the integral coefficients?
- What are the elements of the basis of integral functions?

The loop amplitudes

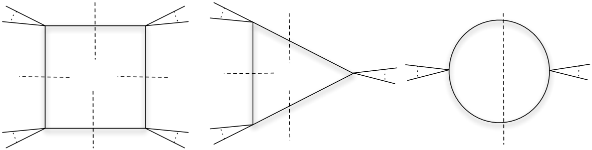
Integration by part considerations indicate the *existence* of a *finite* basis of (master) integral functions $\mathcal{B}(L)$ at each loop order [cf. Weinzierl's talk]

$$A^{\mathrm{L-loop}} = \sum_{i\in\mathcal{B}(L)} \mathrm{coeff}_i\, \mathrm{Integral}_i + \mathrm{Rational}$$

- dimension of the basis at $L \geqslant 2$ loop is not known
- Construction of the basis is still a major open question

The amplitude simplification

- The integral functions are the box, triangle, bubble, tadpole around $D = 4$ dimensions



$$\log(1-z) = \int_0^z d\log(1-t); \qquad \mathrm{Li}_2(z) = -\int_0^z \log(1-t)\, d\log t$$

$$\text{for } z \in \mathbb{C}\backslash[1,\infty[$$

The amplitude simplification

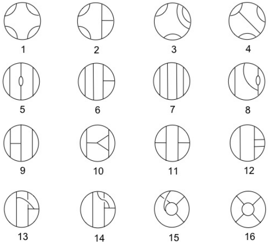
This allows to characterize in a simple way one-loop amplitudes in various gauge theory

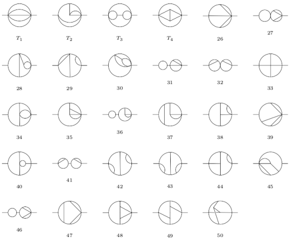
- Only boxes for $\mathcal{N} = 4$ SYM
- No triangle property of $\mathcal{N} = 8$ SUGRA [Bern, Carrasco, Forde, Ita, Johansson; Bjerrum-Bohr, Vanhove]
- Only box for QED multi-photon amplitudes with $n \geqslant 8$ photons [Badger, Bjerrum-Bohr, Vanhove]

Master integrals basis

It is crucial to know the basis of Master integrals for many physical problems

- Two-loop integral for Higgs processes, fishnet graphs [Chicherin et al.]
- Ultraviolet divergences in (maximal) supergravity at 5 loops
- Computation of the Post-Newtonian corrections to the gravitational potential





Master integrals basis

- A 'traditional' approach consists in deriving Integration by Part identities (IBP) using Laporta's method and then bring the system is the form [cf Weinzeirl's talk]

$$\frac{d}{dx}\begin{pmatrix} I_{\dots}(x) \\ \vdots \end{pmatrix} = A(x,\epsilon)\begin{pmatrix} I_{\dots}(x) \\ \vdots \end{pmatrix}$$

- For a nice form for $A(x,\epsilon) = \epsilon A(x)$ one gets multiple polylogarithms [Henn]

$$df(x) = \sum_{i=1}^{r} \frac{a_i}{x - x_i} f(x)$$

- Deriving this system can be expensive numerically. And the resolution not obvious if A is not of the nice form as used by [Henn]

Master integrals basis

- The previous system of equation has the geometry interpretation of being a connection associated to a given cohomology: this is the Gauss-Manin connection

- It was shown by [Bloch, Esnault, Kreimer; Bloch, Kreimer; Schnetz, Brown] that the 1-loop triangle integral involves only mixed Tate Hodge structure.

What about functions beyond MPL ?

We want to design a method based on the geometry of the graph that gets an intrinsic meaning to the differential equation and the basis of master integrals

Feynman Integrals: parametric representation

Any Feynman integrals with L loops and n propagators

$$I_\Gamma = \int \frac{\prod_{i=1}^{L} d^D\ell_i}{\prod_{i=1}^{n} d_i^{\nu_i}}$$

has the parametric representation

$$I_\Gamma = \Gamma(\nu - \frac{LD}{2}) \int_{x_i \geqslant 0} \frac{\mathcal{U}^{\nu-(L+1)\frac{D}{2}}}{(\mathcal{U}\sum_i m_i^2 x_i - \mathcal{V})^{\nu - L\frac{D}{2}}} \delta(x_n = 1) \prod_{i=1}^{n} \frac{dx_i}{x_i^{1-\nu_i}}$$

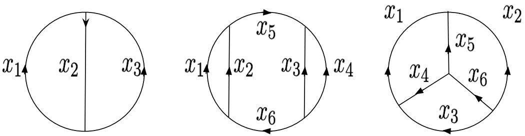
The Symanzik polynomials $\mathcal{U}$ and $\mathcal{V}$ are homogeneous in the $x_1, \dots, x_n$

- $\mathcal{U}$ is of degree L in $\mathbb{P}^{n-1}$
- $\mathcal{V}$ of degree $L+1$ in $\mathbb{P}^{n-1}$

What are the Symanzik polynomials?

$$I_\Gamma = \Gamma(\nu - \frac{LD}{2}) \int_{x_i \geqslant 0} \frac{\mathcal{U}^{\nu-(L+1)\frac{D}{2}}}{(\mathcal{U}\sum_i m_i^2 x_i - \mathcal{V})^{\nu - L\frac{D}{2}}} \delta(x_n = 1) \prod_{i=1}^{n} \frac{dx_i}{x_i^{1-\nu_i}}$$

$\mathcal{U} = \det \Omega$ determinant of the period matrix of the graph $\Omega_{ij} = \oint_{C_i} v_j$



(a) (b) (c)

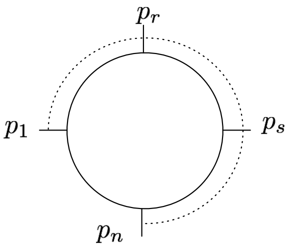
$$\Omega_{2(a)} = \begin{pmatrix} x_1 + x_3 & x_3 \\ x_3 & x_2 + x_3 \end{pmatrix}; \qquad \Omega_{3(b)} = \begin{pmatrix} x_1 + x_2 & x_2 & 0 \\ x_2 & x_2 + x_3 + x_5 + x_6 & x_3 \\ 0 & x_3 & x_3 + x_4 \end{pmatrix}$$

$$\Omega_{3(c)} = \begin{pmatrix} x_1 + x_4 + x_5 & x_5 & x_4 \\ x_5 & x_2 + x_5 + x_6 & x_6 \\ x_4 & x_6 & x_3 + x_4 + x_6 \end{pmatrix}$$

What are the Symanzik polynomials?

$$I_\Gamma \propto \int_0^\infty \frac{\delta(1-x_n)}{(\sum_i m_i^2 x_i - \mathcal{V}/\mathcal{U})^{n-L\frac{D}{2}}} \frac{1}{\mathcal{U}^{\frac{D}{2}}} \prod_{i=1}^n \frac{dx_i}{x_i^{1-\nu_i}}$$

$\mathcal{V}/\mathcal{U} = \sum_{1\leqslant r\leqslant s\leqslant n} p_r \cdot p_s G(x_r/T_r, x_s/T_s; \Omega)$ sum of Green's function



$$G^{1-loop}(\alpha_r, \alpha_s; L) = -\frac{1}{2}(\alpha_s - \alpha_r) + \frac{1}{2}\frac{(\alpha_r - \alpha_s)^2}{T}.$$

The geometry of a Feynman graph

The homogeneous polynomial of n variables and degree $L+1$ completely characterises the Feynman graph and its integral

$$\Phi_\Gamma = \mathcal{U} \times (\sum_{i=1}^{n} m_i^2 x_i) - \mathcal{V}$$

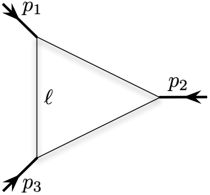
- We can recover both Symanzik polynomials
- Determines the graph topology
  - the number of propagators is the number of variables n
  - the loop order is the degree minus one $L = \deg(\Phi_\Gamma) - 1$
  - Number of vertices $v = 1 + n - L$ from Euler characteristic

From parametric representation to graph

The most general quadric polynomial in $\mathbb{P}^2$

$$W_{2,3}(x_1, x_2, x_3) = \sum_{\substack{i_1+i_2+i_3=2\\ i_r\geqslant 0}} w_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3}$$

The graph has $n = 3$ propagators, $L = 1$ loop, $v = 3$ vertices
This can only be a triangle graph



$$p_1 + p_2 + p_3 = 0; \qquad p_i^2 \neq 0$$

$$\Phi_\triangleright = (x_1 + x_2 + x_3)(m_1^2 x_1 + m_2^2 x_2 + m_3^2 x_3) - (p_1^2 x_2 x_3 + p_2^2 x_1 x_3 + p_3^2 x_1 x_2)$$

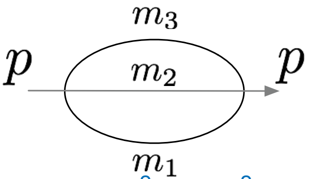
From parametric representation to graph

The most general cubic in $\mathbb{P}^2$

$$W_{3,3} = \sum_{\substack{i_1+i_2+i_3=3 \\ i_r \geqslant 0}} w_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3}$$

The graph has $n = 3$ propagators, $L = 2$ loops, $v = 2$ vertices

This can only be a sunset graph



$$\Phi_{\ominus} = (x_1x_2 + x_1x_3 + x_2x_3)(m_1^2x_1 + m_2^2x_2 + m_3^2x_3) - p^2x_1x_2x_3$$

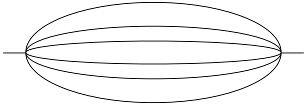
From parametric representation to graph

The most general polynomial of degree n in $\mathbb{P}^{n-1}$

$$W_{n,n} = \sum_{\substack{i_1+\cdots+i_n=n\\ i_r\geqslant 0}} w_{i_1,\ldots,i_n} x_1^{i_1}\cdots x_n^{i_n}$$

The graph has n propagators, $L=n-1$ loops, $v=2$ vertices

This can only be a n-loop sunset graphs



$$\Phi_n = \prod_{i=1}^n x_i \sum_{i=1}^n x_i^{-1} \sum_{i=1}^n m_i^2 x_i - p^2 \prod_{i=1}^n x_i$$

In general several graphs can occur in particular planar and non-planar topologies

Feynman integral and periods

$$I_\Gamma = \Gamma(\nu - \frac{LD}{2}) \int_{\Delta_n} \Omega_\Gamma; \qquad \Omega_\Gamma := \frac{\mathcal{U}^{\nu-(L+1)\frac{D}{2}}}{\Phi_\Gamma(x_i)^{\nu-L\frac{D}{2}}} \prod_{i=1}^{n-1} \frac{dx_i}{x_i^{1-\nu_i}}$$

Ω_Γ algebraic differential form on the complement of the graph hypersurface

$$\Omega_\Gamma \in H^{n-1}(\mathbb{P}^{n-1} \backslash X_\Gamma) \qquad X_\Gamma := \{\Phi_\Gamma(x_i) = 0, x_i \in \mathbb{P}^{n-1}\}$$

The domain of integration is the simplex Δ_n

$$\Delta_n := \{x_1 \geqslant 0, \ldots, x_n \geqslant 0 | [x_1, \ldots, x_n] \in \mathbb{P}^{n-1}\}$$

Feynman integral and periods

$$I_\Gamma = \Gamma(\nu - \frac{LD}{2}) \int_{\Delta_n} \Omega_\Gamma; \qquad \Omega_\Gamma := \frac{\mathcal{U}^{\nu-(L+1)\frac{D}{2}}}{\Phi_\Gamma(x_i)^{\nu - L\frac{D}{2}}} \prod_{i=1}^{n-1} \frac{dx_i}{x_i^{\nu_i - 1}}$$

The domain of integration is the simplex Δ_n

$$\Delta_n := \{x_1 \geqslant 0, \ldots, x_n \geqslant 0 | [x_1, \ldots, x_n] \in \mathbb{P}^{n-1}\}$$

with boundary contained in the normal crossings divisor

$$\partial\Delta_n \subset \mathcal{I}_n := \{x_1 \cdots x_n = 0\}$$

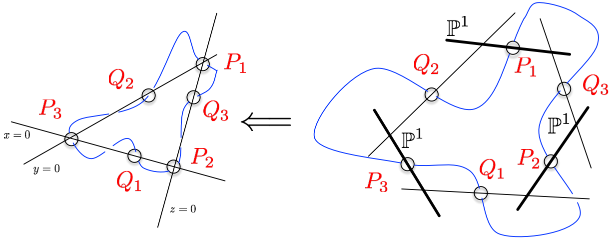
But $\partial\Delta_n \cap X_\Gamma \neq \emptyset$ therefore $\Delta_n \notin H_{n-1}(\mathbb{P}^{n-1} \backslash X_\Gamma)$

This is resolved by looking at the relative cohomology

$$H^\bullet(\mathbb{P}^{n-1} \backslash X_\Gamma; \mathcal{I}_n \backslash \mathcal{I}_n \cap X_\Gamma)$$

Feynman integral and periods

Δ_n and X_Γ are separated by performing a series of iterated blowups of the complement of the graph hypersurface [Bloch, Esnault, Kreimer]



The Feynman integral *are* periods of the relative cohomology after performing the appropriate blow-ups

$$H^{n-1}(\widetilde{\mathbb{P}^{n-1}}\backslash\widetilde{X_F};\widetilde{\Delta_n}\backslash\widetilde{\Delta_n}\cap\widetilde{X_\Gamma})$$

Feynman integral and periods

- In QFT one is interested in the $\epsilon = (D - D_c)/2$ (e.g. $D_c = 4$) expansion of the Feynman integral

$$I_\Gamma = \sum_{i \geqslant -n} c_i \, \epsilon^i$$

- The c_i are numerical periods [Belkale, Brosnan; Kontsevich, Zagier; Bogner, Weinzierl]

$$\mathfrak{M} := H^\bullet(\widetilde{\mathbb{P}^{n-1}} \backslash \widetilde{X_F}; \widetilde{\Pi_n} \backslash \widetilde{\Pi_n} \cap \widetilde{X_\Gamma})$$

- The QFT questions: numbers of master integrals for amplitudes, their differential equations are now reformulated in a cohomological framework

When physics and mathematics meet

The central questions about amplitudes in QFT can be reformulated as Riemann-Hilbert problem for periods

- Compute period explicitly
  - *Numerically or by series expansion in the physical region*
- Derive the local monodromy
  - *unitarity of the S-matrix*
- Construct a complete system of differential equations
  - *Relate this to the integration-by-part method used in QCD*
- Understand the new class of special functions that are needed
  - *What is needed beyond beyond elliptic multiple polylogarithm?*

Differential equation

$$\mathfrak{M}(s_{ij}, m_i) := H^\bullet(\widetilde{\mathbb{P}^{n-1}} \backslash \widetilde{X_F}; \widetilde{\Pi_n} \backslash \widetilde{\Pi_n} \cap \widetilde{X_\Gamma})$$

Since Ω_Γ varies when one changes the kinematic variables s_{ij} one needs to study a variation of (mixed) Hodge structure

Consequently the Feynman integral will satisfy a differential equation

$$L_{PF}\, I_\Gamma = S_\Gamma$$

The Picard-Fuchs operator will arise from the study of the variation of the differential in the cohomology when kinematic variables change

Generically there is an inhomogeneous term $S_\Gamma \neq 0$

The GZK approach

Consider the homogeneous polynomial of degree L in $\mathbb{P}^{n-1}$

$$P(z_1,\dots,z_r) = \sum_{a_1,\dots,a_{n-1}} z_{a_1,\cdots,a_{n-1}} \prod_{i=1}^{n-1} x_i^{a_i}$$

with $\mathbf{a} = (a_1,\dots,a_{n-1})$ and $A = (\mathbf{a}_1,\cdots,\mathbf{a}_r)$ finite subset of $\mathbb{Z}^r$
For every vector $(\ell_1,\dots,\ell_r) \in \mathbb{Z}^r$ such that

$$\ell_1 + \cdots + \ell_r = 0, \qquad \ell_1\mathbf{a}_1 + \cdots + \ell_r\mathbf{a}_r = 0$$

then holds the differential equation

$$\left(\prod_{l_i>0} \partial_{z_i}^{l_i} - \prod_{l_i<0} \partial_{z_i}^{-l_i}\right) \int_{|x_1|=\cdots=|x_{n-1}|=1} \frac{1}{P(z_1,\dots,z_r)} \prod_{i=1}^{n-1} \frac{dx_i}{x_i} = 0$$

The GZK approach

[Gel'fand, Zelevinsky, Kapranov] have shown this is true for

$$\int_{|x_1|=\cdots=|x_{n-1}|=1} \prod_i P(z_1,\ldots,z_r)^{m_i} \prod_{i=1}^{n-1} x^{\beta_i} \frac{dx_i}{x_i}$$

The GZK system for a function Φ of r variables $z_1,\cdots,z_r$ and a vector $\mathbf{c}\in\mathbb{C}^n$ and r elements $\{\mathbf{a}_1,\ldots,\mathbf{a}_r\}\in\mathbb{Z}^n$

- for every $(\ell_1,\ldots,\ell_r)\in\{(\ell_1,\cdots,\ell_r)\in\mathbb{Z}^r|\sum_{i=1}^r \ell_i\mathbf{a}_i=0\}$ there is one differential operator

$$\left(\Box_\ell := \prod_{\ell_i>0} \partial_{z_i}^{\ell_i} - \prod_{\ell_i<0} \partial_{z_i}^{-\ell_i}\right)\Phi = 0$$

- an system of n differential equation (includes the Euler operator)

$$\left(\mathbf{a}_1 z_1 \frac{\partial}{\partial z_1} + \cdots + \mathbf{a}_r z_r \frac{\partial}{\partial z_r} - \mathbf{c}\right)\Phi = 0$$

The GZK approach: consequences

1. The maximal cut integral is always killed by the GZK operator

$$\pi_P = \int_{|x_1|=\cdots=|x_{n-1}|=1} \frac{1}{P(z_1,\dots,z_r)} \prod_{i=1}^{n-1} \frac{dx_i}{x_i}$$

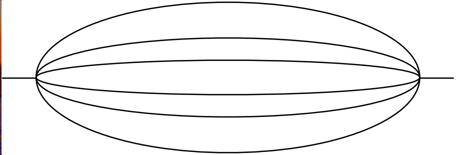
2. The generic solution of GZK system are the hypergeometric series

$$\Phi_{\mathbb{L},\gamma}(z_1,\cdots,z_r) = \sum_{(\ell_1,\dots,\ell_r)\in\mathbb{L}} \prod_{j=1}^{r} \frac{z_j^{\gamma_j+\ell_j}}{\Gamma(\gamma_j+\ell_j+1)}$$

with $\mathbb{L} = \{(\ell_1,\dots,\ell_r) \in \mathbb{Z} | \sum_{i=1}^r \ell_i \mathbf{a}_i = 0\}$ with $\ell_1+\cdots+\ell_r=0$ and $(\gamma_1,\dots,\gamma_r)\in\mathbb{C}^r$

3. In general for a well choosen $\ell\in\mathbb{L}$ the differential operator factorizes a piece giving the Picard-Fuchs operator

The sunset family



The sunset family

This talk will be focused on the special families of n-loop sunset graphs

$$\Phi_n = \prod_{i=1}^{n} x_i \sum_{i=1}^{n} x_i^{-1} \sum_{i=1}^{n} m_i^2 x_i - p^2 \prod_{i=1}^{n} x_i$$

- This family is a nice and important playground for understanding relations between Feynman integrals and periods
- This family leads to interesting motives : not mixed Tate, non trivial extensions
- Surprisingly rich: interesting Hodge structure, mirror symmetry
- For $p^2 = m_1^2 = \cdots = m_n^2$ [Broadhurst] found that special values of these sunset Feynman integrals are given by L-function evaluated in the critical band

The sunset family

The graph polynomials

$$\Phi_n = \prod_{i=1}^{n} x_i \sum_{i=1}^{n} x_i^{-1} \sum_{i=1}^{n} m_i^2 x_i - p^2 \prod_{i=1}^{n} x_i$$

- ϕ_n has a reflexive Newton polytope $\Delta \subset \mathbb{R}^{n-1}$.
- Its polar part Δ° has only integral points in $\mathbb{R}^{n+1}$
- Δ° is associated to a noncompact toric Fano n-fold $\mathbb{P}_\Delta$

The sunset graphs lead to 1-parameter families of Calabi–Yau hypersurfaces in toric Fano n-folds

The two-loop sunset integral

We consider the sunset integral in two Euclidean dimensions

$$\mathcal{I}^2_\ominus = \int_{\Delta_3} \Omega_\ominus; \qquad \Delta_3 := \{[x:y:z] \in \mathbb{P}^2 | x \geqslant 0, y \geqslant 0, z \geqslant 0\}$$

- The sunset integral is the integration of the 2-form

$$\Omega_\ominus = \frac{zdx \wedge dy + xdy \wedge dz + ydz \wedge dx}{(m_1^2 x + m_2^2 y + m_3^2 z)(xz + xy + yz) - p^2 xyz} \in H^2(\mathbb{P}^2 - \mathcal{E}_{p^2})$$

- The sunset family of open elliptic curve

$$\mathcal{E}_{p^2} = \{(m_1^2 x + m_2^2 y + m_3^2 z)(xz + xy + yz) - p^2 xyz = 0\}$$

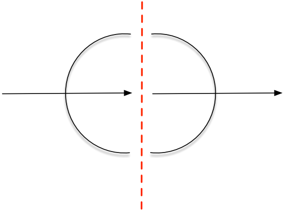
- For $m_1 = m_2 = m_3$ we have a modular curve $\mathcal{E}_{p^2} \simeq X_1(6)$

The differential operator: from the period

The analytic period of the elliptic curve around $p^2 \sim \infty$ has the same integrand as the Feynman integral but we have just changed the domain of integration

$$\pi_0(p^2) := \int_{|x|=|y|=1} \Omega_\ominus$$

This is the imaginary part or the maximal cut of the amplitude



$$\Im\mathrm{m}(\mathcal{I}_\ominus(p^2)) = \oint_C \prod_{i=1}^{3} \delta(\ell_i^2 - m_i^2)\, \delta(\ell_1 + \ell_2 + p)\, d^2\ell_1 d^2\ell_2$$

The other period is $\pi_1(s) = \log(s)\,\pi_0(s) + \varpi_1(s)$ with $\varpi_1(s)$ analytic is obtained by looking at different unitarity cut cutting less lines [Primo, Tancredi]

The differential operator: from the period

The integral is the analytic period of the elliptic curve around $p^2 \sim \infty$

$$\pi_0(p^2) := -\sum_{n\geqslant 0} \frac{1}{(p^2)^{n+1}} \left(\sum_{n_1+n_2+n_3=n} \left(\frac{n!}{n_1!n_2!n_3!}\right)^2 \prod_{i=1}^{3} m_i^{2n_i} \right)$$

From the series expansion we can deduce the Picard-Fuch differential operator (the system has maximal unipotent monodromy [Lian, Todorov, Yau])

$$L_{\ominus}\pi_0(p^2) = 0$$

- With this method one easily derives the PF at all loop order for the all equal mass sunset and show the order(PF)=loop [Vanhove]
- Gives for the L-loop sunset PF of order L for all equal masses and $2L$ for all different masses [Vanhove; to appear]

The differential equation

By general consideration we know that since the integrand is a top form we have

$$L_\Gamma I_\Gamma = \int_{\Delta_n} d\beta_\Gamma = -\int_{\partial\Delta_n} \beta_\Gamma = \mathcal{S}_\Gamma \neq 0$$

Writing the differential equation as $\delta_s := s\frac{d}{ds}$ $s = 1/p^2$

$$\left(\delta_s^2 + q_1(s)\delta_s + q_0(s)\right)\left(\frac{1}{s} I_\ominus(s)\right) = \mathcal{Y}_\ominus + \sum_{i=1}^{3} \log(m_i^2) c_i(s)$$

The differential equation

Using works from [del Angel,Müller-Stach] and [Doran, Kerr] we know that when rank of the D-module system of differential equations that $\mathcal{Y}_\ominus$ is the Yukawa coupling

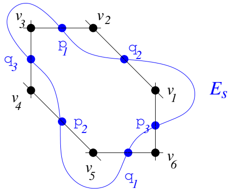
$$\mathcal{Y}_\ominus := \int_{\mathcal{E}(p^2)} \Omega_\ominus \wedge s\frac{d}{ds}\Omega_\ominus = \frac{2s^2 \prod_{i=1}^4 \mu_i - 4s\sum_i m_i^2 + 6}{\prod_{i=1}^4(\mu_i^2 s - 1)}$$

The Yukawa coupling is the Wronskian of the Picard-Fuchs operator and only depends on the form of the Picard-Fuchs operator

$$\mathcal{Y}_\ominus = s\,\det\begin{pmatrix} \pi_0(s) & \pi_1(s) \\ \frac{d}{ds}\pi_0(s) & \frac{d}{ds}\pi_1(s)\end{pmatrix}$$

So far all we got can be deduced from the graph polynomial, and the associated Picard-Fuchs operator.

The differential equation



$$\left(\delta_s^2 + p_1(s)\delta_s + p_0(s)\right)\left(\frac{1}{s}\, I_\ominus(s)\right) = y_\ominus + \sum_{i=1}^{3} \log(m_i^2)\, c_i(s)$$

The mass dependent log-terms come from derivative of partial elliptic integrals on globally well-defined algebraic 0-cycles arising from the punctures on the elliptic curve [Bloch, Kerr,Vanhove]

$$c_1(s) = \frac{d}{ds}\int_{q_2}^{q_3} \Omega_\ominus$$

They are rational function by construction.

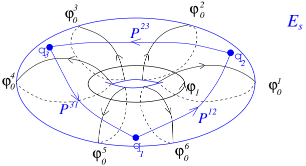
The 2-loop sunset integral as elliptic dilogarithm

The integral divided by a period of the elliptic curve is a function defined on the punctured torus [Bloch, Kerr,Vanhove]

$$\mathcal{J}_\ominus \equiv \frac{i\varpi_r}{\pi}\left(\mathcal{L}_2\left\{\frac{X}{Z},\frac{Y}{Z}\right\}+\mathcal{L}_2\left\{\frac{Z}{X},\frac{Y}{X}\right\}+\mathcal{L}_2\left\{\frac{X}{Y},\frac{Z}{Y}\right\}\right) \mod \text{period}$$

- ϖ_r is the elliptic curve period which is real on the line $0 < p^2 < (m_1+m_2+m_3)^2$
- The sunset integral is the regulator period (with tame Milnor symbol) in the K_2 of the elliptic curve [Bloch, Vanhove]

The 2-loop sunset integral as elliptic dilogarithm



$P_1 = [1,0,0]$; $\quad Q_1 = [0, -m_3^2, m_2^2]$; $\quad x(P_1)x(Q_1) = -1$

$P_2 = [0,1,0]$; $\quad Q_2 = [-m_3^2, 0, m_1^2]$; $\quad x(P_2)x(Q_2) = -1$

$P_3 = [0,0,1]$; $\quad Q_3 = [-m_2^2, m_1^2, 0]$; $\quad x(P_3)x(Q_3) = -1$

Representing the ratio of the coordinates on the sunset cubic curve as functions on $\mathcal{E}_\ominus \simeq \mathbb{C}^\times/q^{\mathbb{Z}}$

$$\frac{X}{Z}(x) = \frac{\theta_1(x/x(Q_1))\theta_1(x/x(P_3))}{\theta_1(x/x(P_1))\theta_1(x/x(Q_3))} \qquad \frac{Y}{Z}(x) = \frac{\theta_1(x/x(Q_2))\theta_1(x/x(P_3))}{\theta_1(x/x(P_2))\theta_1(x/x(Q_3))}$$

$\theta_1(x)$ is the Jacobi theta function

$$\theta_1(x) = q^{\frac{1}{8}} \frac{x^{1/2} - x^{-1/2}}{i} \prod_{n\geqslant 1} (1-q^n)(1-q^n x)(1-q^n/x)\,.$$

The 2-loop sunset integral as elliptic dilogarithm

$$\mathcal{L}_2\left\{\frac{X}{Z},\frac{Y}{Z}\right\} = -\int_{x_0}^{x} \log\left(\frac{X}{Z}(y)\right)\, d\log y$$

Since

$$\begin{aligned}\int \log(\theta_1(x))\, d\log x &= \sum_{n\geqslant 1}\int (\mathrm{Li}_1(q^n x) + \mathrm{Li}_1(q^n/x) + \mathrm{cste})\, d\log(x)\\ &= \sum_{n\geqslant 1}(\mathrm{Li}_2(q^n x) - \mathrm{Li}_2(q^n/x)) + \mathrm{cste}\,\log(x)\end{aligned}$$

The 2-loop sunset integral as elliptic dilogarithm

We find

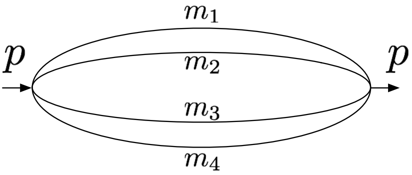
$$\mathcal{J}_{\ominus}(s) \equiv \frac{i\varpi_r}{\pi}\left(\hat{E}_2\left(\frac{x(P_1)}{x(P_2)}\right)+\hat{E}_2\left(\frac{x(P_2)}{x(P_3)}\right)+\hat{E}_2\left(\frac{x(P_3)}{x(P_1)}\right)\right) \mod \text{periods}$$

where

$$\hat{E}_2(x)=\sum_{n\geqslant 0}\left(\mathrm{Li}_2\left(q^n x\right)-\mathrm{Li}_2\left(-q^n x\right)\right)-\sum_{n\geqslant 1}\left(\mathrm{Li}_2\left(q^n/x\right)-\mathrm{Li}_2\left(-q^n/x\right)\right).$$

Close to the form given by [Brown, Levin]. See as well [Adams, Bogner, Weinzeirl]

The three-loop sunset graph: integral



We look at the 3-loop sunset graph in $D = 2$ dimensions

- The Feynman parametrisation is given by

$$I^2_{\ominus}(m_i; K^2) = \int_{x_i \geqslant 0} \frac{1}{(m_4^2 + \sum_{i=1}^3 m_i^2 x_i)(1 + \sum_{i=1}^3 x_i^{-1}) - K^2} \prod_{i=1}^{3} \frac{dx_i}{x_i}$$

three-loop sunset graph: differential equation

For the all equal mass case the geometry of the 3-loop sunset graph is a $K3$ surface (Shioda-Inose family for $\Gamma_1(6)^{+3}$) with Picard number 19 and discriminant of Picard lattice is 6

$$(m^2+\sum_{i=1}^{3} m^2 x_i)(1+\sum_{i=1}^{3} x_i^{-1})\prod_{i=1}^{3} x_i - p^2\prod_{i=1}^{3} x_i = 0$$

The $t = p^2/m^2$ Picard-Fuchs equation

$$\Big(t^2(t-4)(t-16)\frac{d^3}{dt^3}+6t(t^2-15t+32)\frac{d^2}{dt^2} + (7t^2-68t+64)\frac{d}{dt}+t-4\Big)J^2_{\circleddash}(t) = -4!$$

- One miracle is that this picard-fuchs operator is the symmetric square of the picard-fuchs operator for the sunset graph [Verrill]

three-loop sunset graph: solution

- It is immediate to use the Wronskian method to solve the differential equation [Bloch, Kerr, Vanhove]

$$m^2\, I^2_{\ominus}(t) = 40\pi^2 \log(q)\, \varpi_1(\tau)$$
$$-48\varpi_1(\tau)\left(24\mathcal{L}i_3(\tau,\zeta_6) + 21\,\mathcal{L}i_3(\tau,\zeta_6^2) + 8\mathcal{L}i_3(\tau,\zeta_6^3) + 7\mathcal{L}i_3(\tau,1)\right)$$

with $\mathcal{L}i_3(\tau, z)$ [Zagier; Beilinson, Levin]

$$\mathcal{L}i_3(\tau,z) := \mathrm{Li}_3(z) + \sum_{n\geqslant 1}\left(\mathrm{Li}_3(q^n z) + \mathrm{Li}_3\left(q^n z^{-1}\right)\right)$$
$$-\left(-\frac{1}{12}\log(z)^3 + \frac{1}{24}\log(q)\log(z)^2 - \frac{1}{720}(\log(q))^3\right).$$

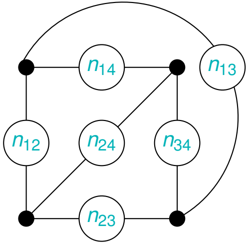
- The 3-loop sunset integral is a regulator period of a motivic class of the K_3 of the the $K3$ surface [Bloch, Kerr, Vanhove]

Modular graph function in closed string theory

String theory gives interesting new class of elliptic multiple logarithm [Broedel, Mafra, Matthes, Schlotterer] has shown the appearance of iterated elliptic integrals in open string presented by [Weinzierl] in his talk

In closed string theory, i.e. on the torus one finds interesting new modular functions

$$I_\Gamma(q) \;=$$



Modular graph function in closed string theory

$$I_\Gamma(q) = \prod_{k=2}^{4} \int_\Sigma \frac{d^2 z_k}{2\pi\tau_2} \prod_{1\leqslant i<j\leqslant 4} G_{1-\text{loop}}(z_j - z_i|\tau)^{n_{ij}}$$

$G_{1-\text{loop}}(z|\tau)$ is the (Arakelov) Green function on the elliptic curve

$$G_{1-\text{loop}}(z|\tau) = -\log\left|\frac{\theta_1(z|\tau)}{\eta(\tau)}\right|^2 - \frac{\pi(z-\bar{z})^2}{2\Im\mathrm{m}(\tau)}$$

or using single value elliptic 1-log $\zeta = e^{2\pi i(v+u\tau)} = q^u e^{2\pi i v}$

$$G_{1-\text{loop}}(z|\tau) = 2\Re\mathrm{e}\Big(\sum_{n\geqslant 0}\mathrm{Li}_1\,(q^n\zeta) + \sum_{n\geqslant 1}\mathrm{Li}_1\,(q^n/\zeta)\Big) + 2\pi\tau_2\left(u^2 - u + \frac{1}{6}\right)$$

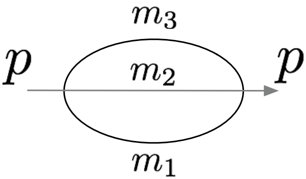
Modular graph function in closed string theory

The lattice momentum space Feynman representation [Green, Vanhove; Green, Russo, Vanhove; Green, d'Hoker, Vanhove]

$$I_\Gamma(q) = \sum_{p_1,\dots,p_w \in \mathbb{Z}\tau+\mathbb{Z}}^{\prime} \prod_{\alpha=1}^{w} \frac{\tau_2}{\pi |p_\alpha|^2} \prod_{i=1}^{N} \delta\left(\sum_{\alpha=1}^{w} p_\alpha\right) .$$

Mirror Symmetry

sunset



sunrise

three-loop sunset graph

- The solution in terms of elliptic trilogarithm is special the three mass cases
- In the general mass situation other function can appear?
- What kind function can arise in the general mass case?

An important insight if to consider all the parameters together in the same geometrical setup

Giving power to the masses

One way to get geometric is to treat all the parameters at the same footing

[Broadhurst] fascinating Bessel representation realises that

$$I_n(p^2, \underline{m}) = 2^{n-1} \int_0^1 I_0(\sqrt{p^2}t) \prod_{i=1}^{n} K_0(m_i t)\, dt$$

Can we do the same geometrically?

Yes we use toric geometry

The sunset integrals at infinite momentum

Around $1/s = p^2 = \infty$ the sunset Feynman has the expansion

$$\mathcal{J}_\ominus(s) = -\pi_0 \left(3R_0^3 + \sum_{\substack{\ell_1+\ell_2+\ell_3=\ell>0 \\ (\ell_1,\ell_2,\ell_3)\in\mathbb{N}^3\backslash(0,0,0)}} \ell(1-\ell R_0) N_{\ell_1,\ell_2,\ell_3} \prod_{i=1}^{3} m_i^{2\ell_i} e^{\ell R_0} \right) .$$

where the Kähler parameters are $Q_i = m_i^2 Q_0$ and $Q_0 = e^{R_0}$ is the logarithmic Mahler measure defined by

$$R_0 := i\pi - \int_{|x|=|y|=1} \log(\Phi_\ominus(x,y)/(xy)) \frac{d\log x d\log y}{(2\pi i)^2} .$$

This is related to the holomorphic $\pi_0(s)$ period near $s = 1/p^2 = 0$

$$\pi_0 = s\frac{dR_0(s)}{ds}$$

The sunset integrals at infinite momentum

One discovers that one can rewrite the mass dependence as follows

$$\mathcal{J}_\ominus(s) \sim -\pi_0 \left(\sum_{\substack{n_1+n_2+n_3+n_4=n\geqslant 1\\ n_i\geqslant 0}} d_{n_1,n_2,n_3,n_4} \mathrm{Li}_3\left(Q_1^{n_1} Q_2^{n_2} Q_3^{n_3} Q_4^{n_4}\right) \right) .$$

where the Kähler parameters are $Q_i = m_i^2 e^{R_0}$

$$d_{0,0,0,1} = d_{1,0,0,1} = 1, \qquad d_{1,0,1,1} = 2, \qquad d_{1,1,1,1} = d_{2,1,1,1} = 3,$$
$$d_{2,1,1,2} = 4, d_{4,3,4,4} = 286, \qquad d_{4,4,4,4} = -192,$$

This matches the expansion given in [Huang, Klemm, Poretschkin] The numbers N_{ℓ_1,ℓ_2,ℓ_3} (and d_{n_1,n_2,n_3,n_4}) are local Gromov-Witten expressed in terms of the virtual integer number of degree ℓ rational curves by

$$N_{\ell_1,\ell_2,\ell_3} = \sum_{d|\ell_1,\ell_2,\ell_3} \frac{1}{d^3} n_{\frac{\ell_1}{d},\frac{\ell_2}{d},\frac{\ell_3}{d}} .$$

The sunset mirror symmetry

- The sunset elliptic curve is embedded into a singular compactification X_0 of the local Hori-Vafa 3-fold

$$Y := \{1 - s(m_1^2 x + m_2^2 y + m_3^2)(1 + x^{-1} + y^{-1}) + uv = 0\} \subset (\mathbb{C}^*)^2 \times \mathbb{C}^2 ,$$

  limit of a family of elliptically-fibered CY 3-folds X_z
- The base given by $\Phi_\ominus$ is a toric del Pezzo surface of degree 6
- We have an isomorphism of A- and B-model $\mathbb{Z}$-variation of Hodge structure

$$H^3(\mathrm{X}_{z_0}) \cong H^{even}(\mathrm{X}^\circ_{Q_0}) ,$$

  and taking (the invariant part of) limiting mixed Hodge structure on both sides yields

the sunset Feynman integral given by the second regulator period of the motivic cohomology class is identified to the local Gromov-Witten prepotential for the 3-fold X

Mirror symmetry for elliptically fibered CY 3-fold

- In the degeneration limit the Yukawa coupling CY 3-fold X leads to the local Yukawa of the sunset elliptic curve

$$Y_{ijk} = \int_X \tilde{\Omega} \wedge \nabla_{\delta_i \delta_j \delta_k} \tilde{\Omega} \Longrightarrow Y^{\mathrm{loc}}_{0ij} \propto Y_{\ominus} = \int \Omega_{\ominus} \wedge \nabla_{\frac{d}{ds}} \Omega_{\ominus}$$

The holomorphic prepotential of [Huang, Klemm, Poretschkin]

$$F(Q_1, Q_2, Q_3, Q_4) = \frac{c_{ijk} t^i t^j t^k}{3!} + \frac{c_{ij}}{2!} t^i t^j + c_i t^i + c + \sum_{\beta \in H_2(M,\mathbb{Z})} n_0^{\beta} \, \mathrm{Li}_3(Q^{\beta})$$

is mapped to the sunset integral with the identification of the Kähler parameter $Q_r = \exp(2\pi i t_r) = m_r^2 Q_0$ for $r = 1, 2, 3$ [Klemm private communication]

$$m_1^2 = \frac{(Q_1 Q_2 Q_4)^{\frac{1}{3}}}{Q_2^{\frac{2}{3}}}; \, m_2^2 = \frac{(Q_1 Q_3 Q_4)^{\frac{1}{3}}}{Q_3^{\frac{2}{3}}}; \, m_3^2 = \frac{(Q_1 Q_2 Q_4)^{\frac{1}{3}}}{Q_4^{\frac{2}{3}}}; \, Q_0 = (Q_1 Q_2 Q_3 Q_4)^{\frac{1}{3}}$$

Mirror symmetry for higher sunset integrals

At higher-loop loop the geometry is more intricate

- Nice connection with toric construction [work in progress with Charles Doran])
- Need to go beyond the smoothness hypothesis for $K_{\mathbb{P}_\Delta}$ used in [Lian, Todorov, Yau]
- Need to extend the construction of the motivic cohomology classes and the regulator period of [Doran, Kerr]
- Have an completly automatic implementation in Sage

The construction gives new way for computing amplitudes in QFT

- Efficient method for deriving Picard-Fuchs equation for Feynman integral in geometrical way
- Should help with the integration by part method and fix the ambiguities in the definition of the loop momentum