# Grassmannian integral for off-shell gauge invariant amplitudes and Willson line form factors in N=4 SYM.

Leonid Bork in collaboration with
Andrei Onishchenko
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ITEP, BLTP, CFAR VNIIA

# N=4 SYM theory.

- **N=4 SYM - one may hope that this theory is <u>exactly solvable.</u>**
- **Physical content - resembles perturbative part of QCD (massless QED without running of the coupling). <u>Tree amplitudes identical to QCD</u>.**
- **The correlation functions in this theory can be studied in the weak and strong regimes ( via AdS/CFT).**
- **The computation of anomalous dimensions of local operators in N=4 SYM in planar limit can be reduced to the problem of solving some integrable system.**
- **There are numerous results for perturbative expansions of amplitudes (S-matrix) with some results valid in all orders of PT (BDS ansatz for 4,5 points, collinear OPE).**
- **Some perturbative results for generalisations of amplitudes (form factors) with arbitrary number of on-shell states.**
- **N=4 SYM is perfect theoretical laboratory development and tests of <u>new ideas, methods and representations</u> for D=4 gauge theories**

# N=4 SYM theory.

- **Why we know so much about N=4 SYM now ?**
- **Proper variables - helicity spinors:**

Rev. in BernDixonKosower 96

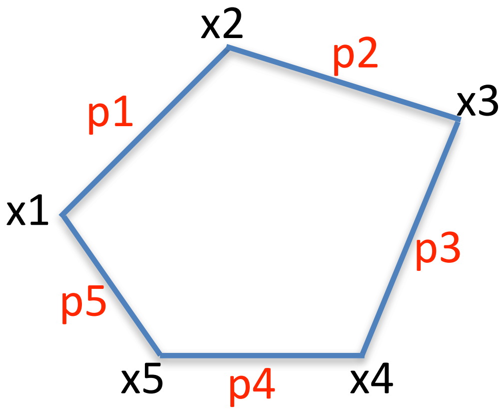
$$p_\mu^{(i)} \mapsto (\sigma^\mu)_{\alpha\dot\alpha} p_\mu^{(i)} = \lambda_\alpha^{(i)} \tilde\lambda_{\dot\alpha}^{(i)} \qquad \lambda_\alpha \in SL(2,\mathbb{C})$$

$$\epsilon^{\alpha\beta}\lambda_\alpha^{(i)}\lambda_\beta^{(j)} \equiv \langle ij\rangle = \sqrt{(p_i+p_j)^2} e^{i\phi_{ij}} = \sqrt{s_{ij}} e^{i\phi_{ij}}, \;\; \phi_{ij} \in \mathbb{R} \quad (\langle ij\rangle)^* \equiv [ij]$$

- **and momentum twistors:**

$$Z_i^M = \begin{pmatrix} \lambda_i^\alpha \\ \mu_i^{\dot\alpha} \end{pmatrix}, \;\; \mu_i^{\dot\alpha} = x_i^{\alpha\dot\alpha}\lambda_{\alpha i}$$

$$p_i^{\alpha\dot\alpha} = x_i^{\alpha\dot\alpha} - x_{i-1}^{\alpha\dot\alpha}$$




Hoges ~10

# N=4 SYM theory.

- **Why we know so much about N=4 SYM now ?**
- **Colour decomposition:**

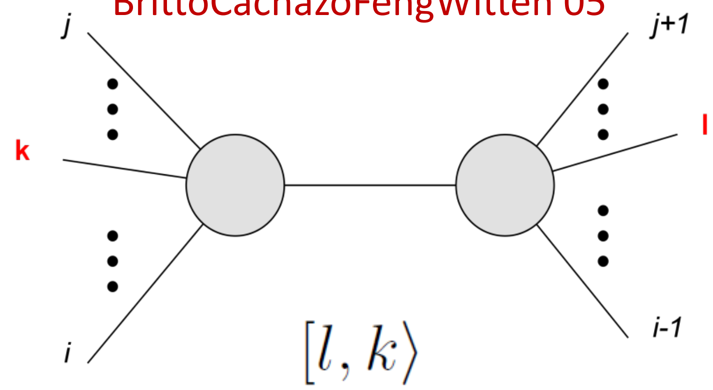
Rev. in BernDixonKosower 96

$$\mathcal{A}_n^{(l)}(p_1^{\lambda_1}...p_n^{\lambda_n}) = g^{n-2}(g^2N_c)^l \sum_{\sigma\in S_n/Z_n} Tr[\sigma(T^{a_1}...T^{a_n})]A_n(\sigma(p_1^{\lambda_1}...p_n^{\lambda_n})) + O(1/N_c)$$

- **And BCFW recursion:**

BrittoCachazoFengWitten 05

$$p_{(l)} = \lambda_{(l)}\tilde{\lambda}_{(l)} : \lambda_{(l)} \mapsto \lambda_{(l)} - z\lambda_{(k)}$$
$$p_{(k)} = \lambda_{(k)}\tilde{\lambda}_{(k)} : \tilde{\lambda}_{(k)} \mapsto \tilde{\lambda}_{(k)} + z\tilde{\lambda}_{(l)}$$




- **Also other methods which relies on general analytical properties of amplitudes (and the "on-shell" objects) rather then on some Lagrangian formulation et.s.**

# N=4 SYM theory.

- <u>**Representation where all properties of amplitudes are manifest**</u>
- **Relations between different BCFW representations.**

Example:

$$\frac{A_6^{NMHV}}{A_6^{MHV}} = [1,2,3,4,5] + [1,2,3,5,6] + [1,3,4,5,6] \quad \frac{A_6^{NMHV}}{A_6^{MHV}} = \mathbb{P}\left([1,2,3,4,5] + [1,2,3,5,6] + [1,3,4,5,6]\right)$$

$$[a,b,c,d,e] = \frac{\hat{\delta}^4(\langle a,b,c,d\rangle\chi_e + \text{cycl.})}{\langle a,b,c,d\rangle\langle b,c,d,e\rangle\langle c,d,e,a\rangle\langle d,e,a,b\rangle\langle e,a,b,c\rangle} \qquad \langle i,j,k,l\rangle = \epsilon_{ABCD} Z_i^A Z_j^B Z_k^C Z_l^D$$

- **Yangian invariance of tree level amplitudes.**
- **Absence of spurious poles in amplitudes represented via BCFW recursion.**
- **Relation between coefficients before master integrals (leading singularities).**
- **ets.**

# Integral over Grassmannian for amplitudes.

One of such remarkable ideas is representations of amplitudes and leading singularities in N=4 SYM in terms of Grassmannian integral and development of on-shell diagram formalism and geometrical interpretation ("amplituhidron"):

On shell information about external particles

$$L_n^k[\Gamma] = \int_\Gamma \frac{d^{k\times n}C}{\mathrm{Vol}[GL(k)]} \frac{\delta^{k\times 2}\left(C\cdot\tilde{\lambda}\right)\delta^{k\times 4}\left(C\cdot\tilde{\eta}\right)\delta^{(n-k)\times 2}\left(C^{\perp}\cdot\lambda\right)}{(1\cdots k)\cdots(n-1\;n\cdots k-2)(n\;1\cdots k-1)}$$

Hodges ~08
Arkani Hamed *et-al* 09

$$A^{tree}_{k,n} = L_n^k[\Gamma_{tree}], \quad C\cdot C^{\perp} = \mathbf{1}$$

C - is nXk matrix - a point in Grassmannian

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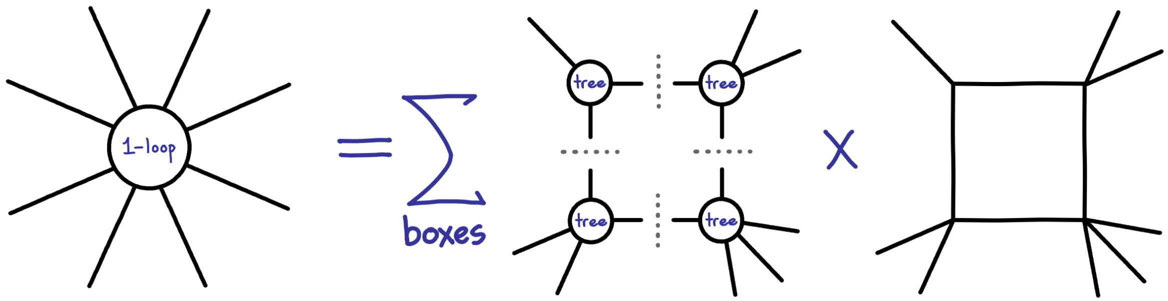
This is multidimensional integral over multiple complex variables, which can be computed by residues. Different choices of integration contour gives different BCFW representations for <u>tree amplitudes</u> and <u>leading singularities</u> to all loop order (!). Also this is the most general form of rational Yangian invariant. And do not forget about <u>twistor strings</u>!

# About leading singularities …

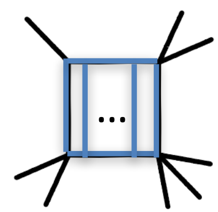
1-loop example. Roughly speaking leading singularities are coefficients before master integrals in loop corrections to the amplitudes.




l-loop example:

pictures
from th 0907.5418

$$\sum_{\boxplus}\left(\int_{\Gamma_{\boxplus}} \frac{d^{(k-2)(n-k-2)}\tau}{(12\ldots k)\ldots(n1\ldots k-1)}\,\delta^4(C\cdot\tilde{\eta})\right)\times$$



# Grassmannian integral, on shell diagrams and (decorated) permutations.

$$\int_\Gamma \frac{d^{n\times k}C_{al}}{Vol[GL(k)]}\frac{1}{M_1 \ldots M_n}\prod_{a=1}^{k}\delta^2\left(\sum_{l=1}^{n}C_{al}\tilde{\lambda}_l\right)\delta^4\left(\sum_{l=1}^{n}C_{al}\eta_l\right)\times$$
$$\times \prod_{b=k+1}^{n}\delta^2\left(\sum_{l=1}^{n}\tilde{C}_{al}\lambda_l\right).$$

There is one to one correspondence between the following objects:

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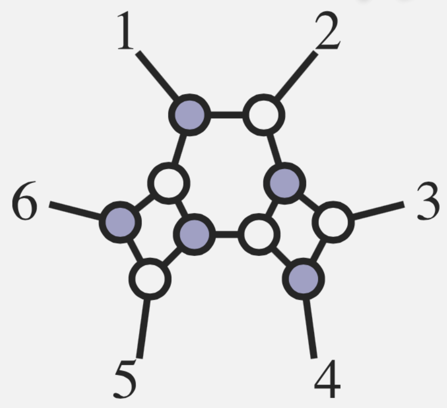
picture from th 1212.5605

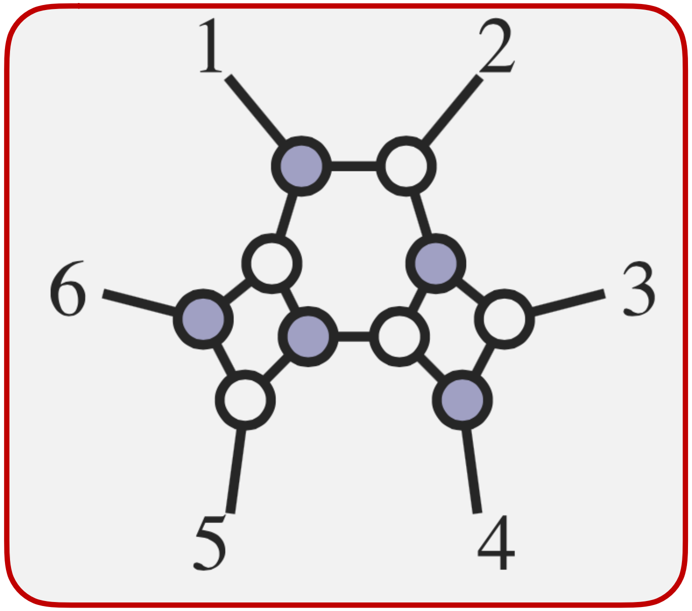



$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 5 & 3 & 2 & 6 & 1 & 4 \end{pmatrix}$$

# Grassmannian integral, on shell diagrams and (decorated) permutations.

$$A_{3,0}(\{\lambda_i,\tilde{\lambda}_i,\eta_i\}) = \int \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\,\delta^2\left(\tilde{\lambda}_1+\alpha_1\tilde{\lambda}_3\right)\delta^2\left(\tilde{\lambda}_2+\alpha_2\tilde{\lambda}_3\right)\times\delta^2\left(\lambda_3+\alpha_1\lambda_1+\alpha_2\lambda_2\right)\times$$
$$\times\quad \hat{\delta}^4\left(\eta_1+\alpha_1\eta_3\right)\hat{\delta}^4\left(\eta_2+\alpha_2\eta_3\right),$$

$$A_{3,-1}(\{\lambda_i,\tilde{\lambda}_i,\eta_i\}) = \int \frac{d\beta_1}{\beta_1}\frac{d\beta_2}{\beta_2}\,\delta^2\left(\lambda_1+\beta_1\lambda_3\right)\delta^2\left(\lambda_2+\beta_2\lambda_3\right)\times\delta^2\left(\tilde{\lambda}_3+\beta_1\tilde{\lambda}_1+\beta_2\tilde{\lambda}_2\right)\times$$
$$\times\quad \hat{\delta}^4\left(\eta_3+\beta_1\eta_1+\beta_2\eta_2\right).$$

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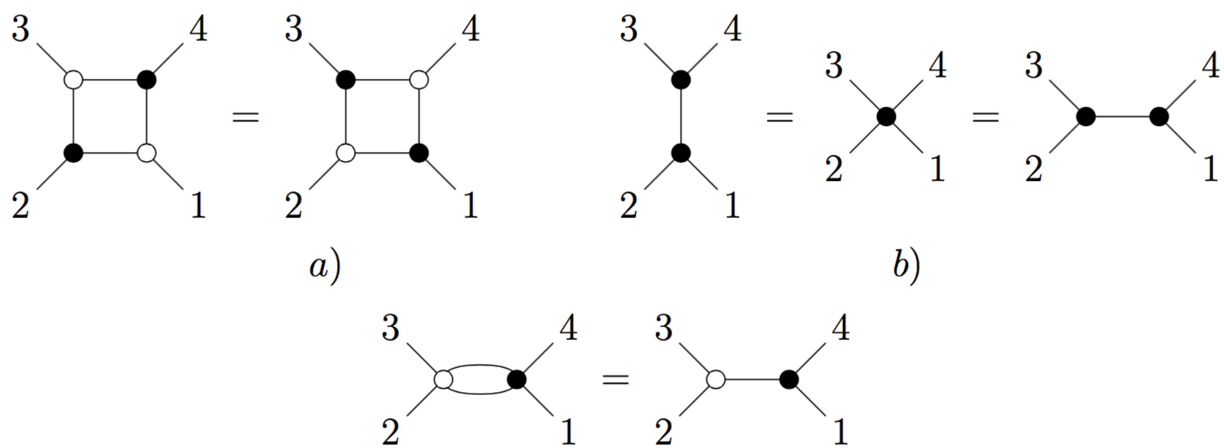
picture
from th 1212.5605

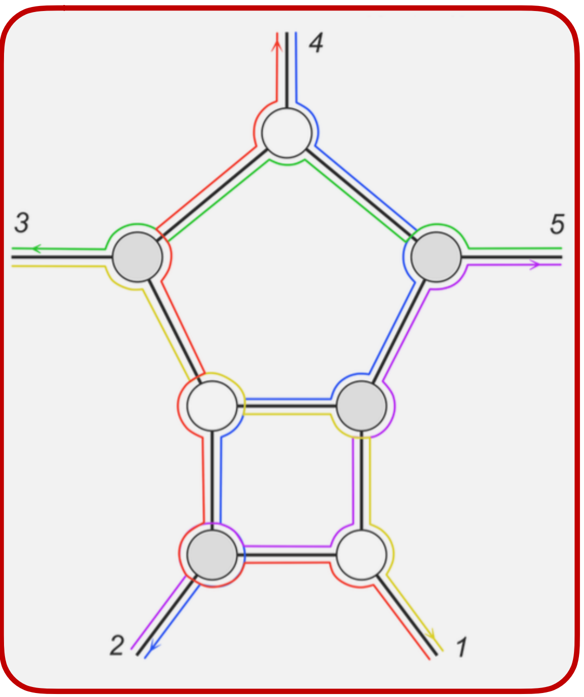
# Grassmannian integral, on shell top-cell diagram.

- **Every Grassmannian integral can be reduced to:**

$$\oint \frac{f_1}{f_1} \oint \frac{f_2}{f_2} \cdots \oint \frac{f_d}{f_d}\, \delta^{k\times 2}(C(f_i)\cdot\tilde{\lambda})\delta^{k\times 4}(C(f_i)\cdot\tilde{\eta})\delta^{(n-k)\times 2}(C^{\perp}(f_i)\cdot\lambda)$$

- **On shell diagrams are equivalent after:**




Top-cell for NMHV_5 Arkani Hammed *et-al* 12

# Grassmannian integral and form factors.

- **In general it is likely that:**

$$\Omega^k_{n+2}[\Gamma] = \sum_i \int_\Gamma \frac{d^{k\times n+2}C}{\mathrm{Vol}[GL(k)]} \; Reg^{(i)} \cdot \frac{\delta^{k\times 2}\left(C\cdot\tilde{\lambda}\right)\delta^{k\times 4}\left(C\cdot\tilde{\eta}\right)\delta^{(n+2-k)\times 2}\left(C^{\perp}\cdot\lambda\right)}{(1\cdots k)\cdots(n-1\;n\cdots k-2)(n\;1\cdots k-1)}$$

$$\text{Form factor}^{tree}_n = \Omega^k_{n+2}[\Gamma_{tree}], \quad C\cdot C^{\perp} = \mathbf{1}$$

Wilhelm, Frassek, Meidinger, Nandan 15

- **For example for stress tensor operator with light like momenta:**

$$\Omega^{(k)}_n[\Gamma] = \sum_{j=4}^{k+1}\int_\Gamma \frac{d^{n+1\times k}C_{al}}{Vol[GL(k)]}\,\frac{Reg^{R,(k)}_j}{M_1\ldots M_{n+1}}\,\delta^{4|4}(1,\ldots,j-1,q,j,\ldots,n) + \int_\Gamma \frac{d^{n+1\times k}C_{al}}{Vol[GL(k)]}\,\frac{Reg^{L,(k)}_n}{M_1\ldots M_{n+1}}\,\delta^{4|4}(1,\ldots,n,q),$$

Brandhuber&Co 16

$$Reg^{R,(k)}_j = \langle q1\rangle\frac{(kk+2\;3\ldots k)}{(13\ldots j-1\;j+1\ldots n+1)} + \sum_{i=j}^{k+1}\langle qi\rangle\frac{(13\;4\ldots i\;i+2\ldots k+2)}{(13\ldots j-1\;j+1\ldots n+1)}$$

$$Reg^{L,(k)}_n = \langle q1\rangle\frac{(nn+1\;3\ldots k)}{(1n\;3\ldots k)},$$

Bork Onishchenko 15

# Gauge invariant off-shell amplitudes - Willson line form factors.

- **One can consider the following interesting objects.**

$$\mathcal{W}_p^c(k) = \int d^4x e^{ix\cdot k} \mathrm{Tr}\left\{\frac{1}{\pi g} t^c \, \mathcal{P} \exp\left[\frac{ig}{\sqrt{2}} \int_{-\infty}^{\infty} ds \, p\cdot A_b(x+sp) t^b\right]\right\}$$

$$k_T^\mu(q) = k^\mu - x(q)p^\mu \quad \text{with} \quad x(q) = \frac{q\cdot k}{q\cdot p} \;\; \text{and} \;\; q^2 = 0$$

$$k_T^\mu(q) = -\frac{\kappa}{2}\frac{\langle p|\gamma^\mu|q]}{[pq]} - \frac{\kappa^*}{2}\frac{\langle q|\gamma^\mu|p]}{\langle qp\rangle} \quad \text{with} \quad \kappa = \frac{\langle q|\not{k}|p]}{\langle qp\rangle}, \;\; \kappa^* = \frac{\langle p|\not{k}|q]}{[pq]}$$

$$\mathcal{A}_{m+n}\left(1,\ldots,m,(m+1)^*\ldots(n+m)^*\right) = \langle k_1,\epsilon_1,c_1;\ldots;k_m,\epsilon_m,c_m|\mathcal{W}_{p_{m+1}}^{c_{m+1}}(k_{m+1})\ldots\mathcal{W}_{p_{n+m}}^{c_{n+m}}(k_{n+m})|0\rangle$$

Lipatov Hammeren Kotko …

# Gauge invariant off-shell amplitudes - Willson line form factors.

- **By the way it is also possible to consider:**

$$\mathcal{A}_{0+n}\left(1^* \ldots n^*\right) = \langle 0|\mathcal{W}^{c_1}_{p_1}(k_1)\ldots\mathcal{W}^{c_n}_{p_n}(k_n)|0\rangle$$


$= \frac{-g^{\mu\nu}}{k^2},$

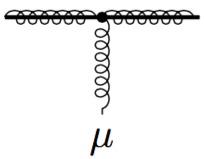
$= \frac{1}{2p\cdot k},$



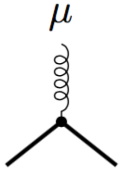
$= \frac{1}{\not k}$



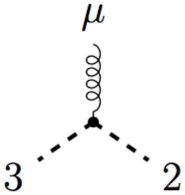
$= \frac{1}{k^2},$




$= \sqrt{2}p^\mu,$

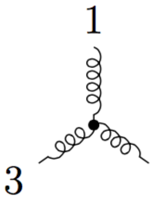



$= \frac{1}{\sqrt{2}}\sigma^\mu,$




$= \frac{1}{\sqrt{2}}(p_2 - p_3)^\mu,$

Hammeren Kotko ~14



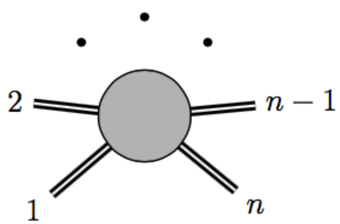

$$= \frac{1}{\sqrt{2}}\left[(k_1-k_2)^{\mu_3}g^{\mu_1\mu_2} + (k_2-k_3)^{\mu_1}g^{\mu_2\mu_3} + (k_3-k_1)^{\mu_2}g^{\mu_3\mu_1}\right].$$

# Gauge invariant off-shell amplitudes - Willson line form factors.

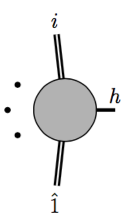
- **Also version of BCFW recursion exists:**

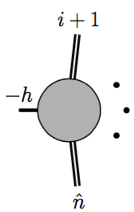
$$\hat{k}_i^\mu(z) \equiv k_i^\mu + z e^\mu = x_i(p_j) p_i^\mu - \frac{\kappa_i - [ij]z}{2}\frac{\langle i|\gamma^\mu|j]}{[ij]} - \frac{\kappa_i^*}{2}\frac{\langle j|\gamma^\mu|i]}{\langle ji\rangle},$$

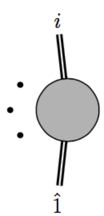
$$\hat{k}_j^\mu(z) \equiv k_j^\mu - z e^\mu = x_j(p_i) p_j^\mu - \frac{\kappa_j}{2}\frac{\langle j|\gamma^\mu|i]}{[ji]} - \frac{\kappa_j^* + \langle ij\rangle z}{2}\frac{\langle i|\gamma^\mu|j]}{\langle ij\rangle}.$$

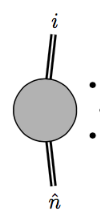


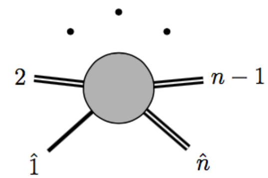

$$= \sum_{i=2}^{n-2}\sum_h \mathbb{A}_{i,h} + \sum_{i=2}^{n-1}\mathbb{B}_i + \mathbb{C} + \mathbb{D},$$

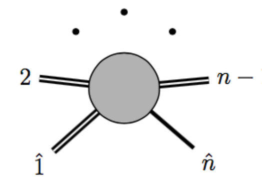
$\mathbb{A}_{i,h} =$ 

$\frac{1}{k_{1,i}^2}$ 


$\mathbb{B}_i =$ 

$\frac{1}{2p_i \cdot k_{i,n}}$ 


$\mathbb{C} = \frac{1}{\kappa_1}$ 


$\mathbb{D} = \frac{1}{\kappa_n^*}$ 


Kotko, Hammeren ~14

# Gauge invariant off-shell amplitudes - Willson line form factors.

- **Examples of answers:**

Kotko, Hammeren, Bork, Onishchenko ~14, 16

$$A^*_{2,3+1}(1^+2^+3^-4^*) = \frac{1}{\kappa^*}\frac{\langle 3p\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 3p\rangle\langle p1\rangle}.$$

$$A^*_{2,3+1}(1^-2^-3^+4^*) = \frac{1}{\kappa}\frac{[3p]^4}{[12][23][3p][p1]}$$

$$A^*_{3,4+1}(1^+2^+3^-4^-5^*) = \frac{1}{\kappa^*}\frac{\langle p|3+4|2]^3}{\langle p1\rangle[23][34]p^2_{2,4}\langle 1|2+3|4]} + \frac{1}{\kappa}\frac{\langle 3|1+2|p]^3}{\langle 12\rangle\langle 23\rangle[4p]p^2_{1,3}\langle 1|2+3|4]}.$$

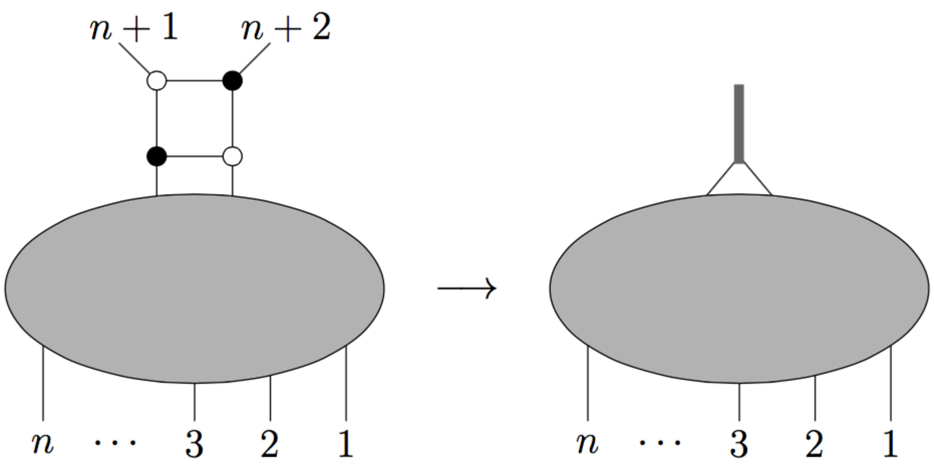
$$A^*_{3,0+3}(1^*,2^*,3^*) = (1+\mathbb{P}'+\mathbb{P}'^2)\frac{1}{\kappa^*_1\kappa_3}\frac{\langle p_1p_2\rangle^3[p_2p_3]^3}{\langle p_2|k_1|p_2]\langle p_2|k_1|p_3]\langle p_1|k_3|p_2]};$$

k is related to total helicity of on shell particles:
total h. = m+2n -2k
m is the number of on-shell particles
n is the number of off-shell once.

# … what if ?

- **"Minimal off-shell amplitude":**

$$A^*_{2,2+1}(1^*,2,3) = \frac{1}{\kappa_1^*}\prod_{A=1}^{4}\frac{\partial}{\partial\tilde{\eta}_p^A}\left[\frac{\delta^4(k+\lambda_2\tilde{\lambda}_2+\lambda_3\tilde{\lambda}_3)\delta^8(\lambda_p\tilde{\eta}_p+\lambda_2\tilde{\eta}_2+\lambda_3\tilde{\eta}_3)}{\langle p2\rangle\langle 23\rangle\langle 3p\rangle}\right]$$




- **Replace set of vertexes in proper on-shell diagram with "Minimal off-shell amplitude".**

- **Or modify integrand such that:**

$$\Omega^k_{n+2}[\Gamma_{tree}]\Big|_{\xi\mapsto\epsilon\xi} = \frac{1}{\kappa^*}L^k_{n+1}[\Gamma'_{tree}] + O(\epsilon)$$

# Conjecture and examples.

$$\Omega^k_{n+2}[\Gamma] = \int_\Gamma \frac{d^{k\times(n+2)}C'}{\mathrm{Vol}[GL(k)]} Reg. \frac{\delta^{k\times 2}\left(C'\cdot\underline{\underline{\tilde{\lambda}}}\right)\delta^{k\times 4}\left(C'\cdot\underline{\underline{\tilde{\eta}}}\right)\delta^{(n+2-k)\times 2}\left(C'^{\perp}\cdot\underline{\underline{\lambda}}\right)}{(1\cdots k)\cdots(n+1\cdots k-2)(n+2\;1\cdots k-1)}$$

with

$$Reg. = \frac{\langle\xi p\rangle}{\kappa^*}\frac{(n+2\;1\cdots k-1)}{(n+1\;1\cdots k-1)},$$

$$\begin{array}{llll}
\underline{\underline{\lambda}}_i = \lambda_i, & i=1,\ldots n, & \underline{\underline{\lambda}}_{n+1} = \lambda_p, & \underline{\underline{\lambda}}_{n+2} = \xi \\
\underline{\underline{\tilde{\lambda}}}_i = \tilde{\lambda}_i, & i=1,\ldots n, & \underline{\underline{\tilde{\lambda}}}_{n+1} = \dfrac{\langle\xi|k}{\langle\xi p\rangle}, & \underline{\underline{\tilde{\lambda}}}_{n+2} = -\dfrac{\langle p|k}{\langle\xi p\rangle} \\
\underline{\underline{\tilde{\eta}}}_i = \tilde{\eta}_i, & i=1,\ldots n, & \underline{\underline{\tilde{\eta}}}_{n+1} = \tilde{\eta}_p, & \underline{\underline{\tilde{\eta}}}_{n+2} = 0.
\end{array}$$

$$A^*_{k,n+1} \quad = \quad \Omega^k_{n+2}[\Gamma_{k,n+2}]$$

# Conjecture and examples.

- **One can work also in momentum twistor variables:**

$$\frac{A^*_{k,n+1}}{A^*_{2,n+1}} = \omega^k_{n+2}[\Gamma_{k,n+2}],$$

$$\omega^k_{n+2}[\Gamma] = \int_\Gamma \frac{d^{(k-2)\times(n+2)}D}{\mathrm{Vol}[GL(k-2)]} \frac{1}{1+\frac{\langle p\xi\rangle}{\langle p1\rangle}\frac{(n+2\ 2\ \ldots\ k-2)}{(1\ \ldots\ k-2)}} \frac{\delta^{4(k-2)|4(k-2)}(D\cdot\mathcal{Z})}{(1\ \ldots\ k-2)\ \ldots\ (n+2\ \ldots\ k-3)}.$$

- **Examples:**

$$A^*_{3,3+1} = \frac{\langle\xi p\rangle}{\kappa^*}\int \frac{d^{3\times5}C'}{\mathrm{Vol}[GL(3)]}\frac{(512)}{(412)}\frac{\delta^6(C'\cdot\underline{\tilde{\lambda}})\delta^8(C\cdot\underline{\tilde{\eta}})\delta^4(C'^{\perp}\cdot\underline{\lambda})}{(123)(234)(345)(451)(512)}$$

$$C' = \begin{pmatrix} 1 & 0 & 0 & c'_{14}=\frac{[15]}{[45]} & c'_{15}=\frac{[14]}{[45]} \\ 0 & 1 & 0 & c'_{24}=\frac{[25]}{[45]} & c'_{25}=\frac{[24]}{[45]} \\ 0 & 0 & 1 & c'_{34}=\frac{[35]}{[45]} & c'_{35}=\frac{[34]}{[45]} \end{pmatrix}$$

$$[i5] = \frac{[i|k|p\rangle}{\langle\xi p\rangle},\quad [i4]=\frac{[i|k|\xi\rangle}{\langle\xi p\rangle},\quad [45]=\frac{k^2}{\langle\xi p\rangle}$$

$$k^2 = \kappa^*\kappa,\quad k|p\rangle = |p]\kappa^*.$$

$$A^*_{3,3+1} = \delta^4(p_1+p_2+p_3+k)\frac{\kappa^3}{[1p][p3][32][21]}\prod_{i=1}^{3}\delta^4\left(\tilde{\eta}_i + \frac{[pi]}{\kappa}\tilde{\eta}_p\right)$$

# Conjecture and examples.

$$\Omega^k_{4+1}[\Gamma] = \frac{\langle \xi p \rangle}{\kappa^*} \int_\Gamma \frac{d^{3\times 6}C'}{\mathrm{Vol}[GL(3)]} \frac{(612)}{(512)} \frac{\delta^6(C' \cdot \underline{\tilde{\lambda}})\delta^8(C \cdot \underline{\tilde{\eta}})\delta^4(C'^{\perp} \cdot \underline{\lambda})}{(123)(234)(345)(456)(561)(612)}$$

$$\Omega^3_{4+2}[\Gamma_{135}]\Big|_{\tilde{\eta}_3^4 \tilde{\eta}_4^4 \tilde{\eta}_5^4} = \{1\} + \{3\} + \{5\} = A^*_{3,4+1}(1^+2^+3^-4^-5^*)$$

- **One can work also in momentum twistor variables:**

$$\omega^3_{4+2}[\Gamma_{246}] = \frac{1}{1 + \frac{\langle p\xi \rangle}{\langle p1 \rangle}\frac{\langle 1345 \rangle}{\langle 3456 \rangle}}[13456] + \frac{1}{1 + \frac{\langle p\xi \rangle}{\langle p1 \rangle}\frac{\langle 1235 \rangle}{\langle 2356 \rangle}}[12356] + [12345] = \frac{A^*_{3,4+1}}{A^*_{2,4+1}}$$

- **Closed solution:**

$$\omega^3_{n+2}[\Gamma_{3,n+2}] = \frac{A^*_{3,n+1}}{A^*_{2,n+1}} \qquad \omega^3_{n+2}[\Gamma_{3,n+2}] = \sum_{i<j}^{n+2} c_j[1\; i-1\; i\; j-1\; j].$$

$$c_{n+2} = \frac{1}{1 + \frac{\langle p\xi \rangle}{\langle p1 \rangle}\frac{\langle 1\; i{-}1\; i\; n{+}1 \rangle}{\langle i{-}1\; i\; n{+}1\; n{+}2 \rangle}}, \text{ and } c_j = 1 \text{ if } j < n+2.$$

# What about multiple Willson line operators?

$$\mathcal{A}_{m+n}\left(1,\ldots,m,(m+1)^{*}\ldots(n+m)^{*}\right) = \langle k_1,\epsilon_1,c_1;\ldots;k_m,\epsilon_m,c_m|\mathcal{W}_{p_{m+1}}^{c_{m+1}}(k_{m+1})\ldots\mathcal{W}_{p_{n+m}}^{c_{n+m}}(k_{n+m})|0\rangle$$

- **From general considerations possible answer may look like:**

$$\Omega^{k}_{m+2n}[\Gamma] = \int_{\Gamma}\frac{d^{k\times(m+2n)}C}{\mathrm{Vol}[GL(k)]}Reg.(m+1,\ldots,m+n)\times$$

$$\times\frac{\delta^{k\times 2}\left(C\cdot\underline{\underline{\tilde{\lambda}}}\right)\delta^{k\times 4}\left(C\cdot\underline{\underline{\tilde{\eta}}}\right)\delta^{(m+2n-k)\times 2}\left(C^{\perp}\cdot\underline{\underline{\lambda}}\right)}{(1\cdots k)\cdots(m\cdots m+k-1)(m+1\cdots m+k)\cdots(m+2n\cdots k-1)}$$

$$\underline{\underline{\lambda}}_i=\lambda_i,\quad i=1,\ldots m,\quad \underline{\underline{\lambda}}_{m+2j-1}=\lambda_{p_j},\quad \underline{\underline{\lambda}}_{m+2j}=\xi_j,\quad j=1,\ldots n,$$

$$\underline{\underline{\tilde{\lambda}}}_i=\tilde{\lambda}_i,\quad i=1,\ldots m,\quad \underline{\underline{\tilde{\lambda}}}_{m+2j-1}=\frac{\langle\xi_j|k_{m+j}}{\langle\xi_j p_j\rangle},\quad \underline{\underline{\tilde{\lambda}}}_{m+2j}=-\frac{\langle p_j|k_{m+j}}{\langle\xi_j p_j\rangle},\quad j=1,\ldots n,$$

$$\underline{\underline{\tilde{\eta}}}_i=\tilde{\eta}_i,\quad i=1,\ldots m,\quad \underline{\underline{\tilde{\eta}}}_{m+2j-1}=\tilde{\eta}_{p_j},\quad \underline{\underline{\tilde{\eta}}}_{m+2j}=0,\quad j=1,\ldots n,$$

- **But what about Reg. function ?**

# What about multiple Willson line operators?

- **From general considerations possible answer may look like:**

$$\Omega^k_{m+2n}[\Gamma] = \int_\Gamma \frac{d^{k\times(m+2n)}C}{\mathrm{Vol}[GL(k)]} Reg.(m+1,\ldots,m+n) \times$$

$$\times \frac{\delta^{k\times 2}(C\cdot\underline{\underline{\tilde{\lambda}}})\,\delta^{k\times 4}(C\cdot\underline{\underline{\tilde{\eta}}})\,\delta^{(m+2n-k)\times 2}(C^{\perp}\cdot\underline{\underline{\lambda}})}{(1\cdots k)\cdots(m\cdots m+k-1)(m+1\cdots m+k)\cdots(m+2n\cdots k-1)}$$

$$\underline{\underline{\lambda}}_i = \lambda_i, \quad i=1,\ldots m, \quad \underline{\underline{\lambda}}_{m+2j-1} = \lambda_{p_j}, \quad \underline{\underline{\lambda}}_{m+2j} = \xi_j, \quad j=1,\ldots n,$$

$$\underline{\underline{\tilde{\lambda}}}_i = \tilde{\lambda}_i, \quad i=1,\ldots m, \quad \underline{\underline{\tilde{\lambda}}}_{m+2j-1} = \frac{\langle\xi_j|k_{m+j}}{\langle\xi_j p_j\rangle}, \quad \underline{\underline{\tilde{\lambda}}}_{m+2j} = -\frac{\langle p_j|k_{m+j}}{\langle\xi_j p_j\rangle}, \quad j=1,\ldots n,$$

$$\underline{\underline{\tilde{\eta}}}_i = \tilde{\eta}_i, \quad i=1,\ldots m, \quad \underline{\underline{\tilde{\eta}}}_{m+2j-1} = \tilde{\eta}_{p_j}, \quad \underline{\underline{\tilde{\eta}}}_{m+2j} = 0, \quad j=1,\ldots n,$$

$$Reg.(m+1,\ldots,m+n) = \prod_{j=1}^{n} Reg(j+m),$$

$$Reg.(j+m) = \frac{\langle\xi_j p_j\rangle}{\kappa_j^*}\frac{(2j+m\ \ 2j+1+m\cdots 2j+k-1+m)}{(2j-1+m\ \ 2j+1+m\cdots 2j+k-1+m)}$$

# What about multiple Willson line operators?

- **Different verifications of the given above formula was performed. (-/-g^*g^*), arbitrary number of + gluons and g^*,g^* etc. <u>It is working!!</u>**
- **Some examples:**

$$\Omega^3_{1+4} = \int \frac{d^{3\times 5}C}{\mathrm{Vol}[GL(3)]} Reg.(2) Reg.(3) \frac{\delta^{3\times 2}\left(C\cdot \underline{\underline{\tilde{\lambda}}}\right)\delta^{3\times 4}\left(C\cdot \underline{\underline{\tilde{\eta}}}\right)\delta^{2\times 2}\left(C^{\perp}\cdot \underline{\underline{\lambda}}\right)}{(123)(234)(345)(451)(512)}$$

$$Reg.(2) = \frac{\langle p_2\xi_2\rangle}{\kappa_2^*}\frac{(345)}{(245)},\;\; Reg.(3) = \frac{\langle p_3\xi_3\rangle}{\kappa_3^*}\frac{(512)}{(412)}$$

$$\begin{aligned} \frac{\partial^4}{\partial\tilde{\eta}_1^4}\prod_{j=2}^{3}\frac{\partial^4}{\partial\tilde{\eta}_{p_j}^4}\Omega^3_{1+4} &= \frac{\langle p_2\xi_2\rangle\langle p_3\xi_3\rangle}{\kappa_2^*\kappa_3^*}\frac{[p_2p_3]^3\kappa_2^{*3}\kappa_3^{*3}\langle p_2\xi_2\rangle^{-3}\langle p_3\xi_3\rangle^{-3}}{[1p_2]\kappa_2^*\langle p_2\xi_2\rangle^{-1}\kappa_2\kappa_2^*\langle p_2\xi_2\rangle^{-1}\kappa_3\kappa_3^*\langle p_3\xi_3\rangle^{-1}[p_31]\kappa_3^*\langle p_3\xi_3\rangle^{-1}} \\ &= \frac{1}{\kappa_2\kappa_3}\frac{[p_2p_3]^3}{[1p_2][p_31]}. \end{aligned}$$

# What about multiple Willson line operators?

- **Most exiting once. Correlation function <WWW>:**

$$\Omega^3_{0+6}[\Gamma] = \int_\Gamma \frac{d^{3\times 6}C}{\mathrm{Vol}[GL(3)]} \prod_{i=1}^{3} Reg.(i) \frac{\delta^{3\times 2}\left(C\cdot \underline{\underline{\tilde{\lambda}}}\right)\delta^{3\times 4}\left(C\cdot \underline{\underline{\tilde{\eta}}}\right)\delta^{3\times 2}\left(C^{\perp}\cdot \underline{\underline{\lambda}}\right)}{(123)(234)(345)(456)(561)(612)},$$

with

$$Reg.(1) = \frac{\langle p_1\xi_1\rangle}{\kappa_1^*}\frac{(234)}{(134)},\ Reg.(2) = \frac{\langle p_2\xi_2\rangle}{\kappa_2^*}\frac{(456)}{(356)},\ Reg.(3) = \frac{\langle p_3\xi_3\rangle}{\kappa_3^*}\frac{(612)}{(512)}.$$

$$\begin{aligned}\{1\} &= \frac{\langle p_1\xi_1\rangle\langle p_2\xi_2\rangle\langle p_3\xi_3\rangle}{\kappa_1^*\kappa_2^*\kappa_3^*}\frac{\langle p_1p_2\rangle^3[p_2p_3]^3\kappa_2^{*3}\kappa_3^{*3}\langle p_2\xi_2\rangle^{-3}\langle p_3\xi_3\rangle^{-3}}{\kappa_3\kappa_3^*\langle p_3\xi_3\rangle^{-2}\langle p_1\xi_1\rangle\kappa_3^*\kappa_2^{*2}\langle p_2\xi_2\rangle^{-2}\langle p_2|k_1|p_2]\langle p_2|k_1|p_3]\langle p_1|k_3|p_2]} \\ &= \frac{1}{\kappa_1^*\kappa_3}\frac{\langle p_1p_2\rangle^3[p_2p_3]^3}{\langle p_2|k_1|p_2]\langle p_2|k_1|p_3]\langle p_1|k_3|p_2]}.\end{aligned}$$

$$\prod_{j=1,2,3}\frac{\partial^4}{\partial\tilde{\eta}^4_{p_j}}\ \Omega^3_{0+6}[\Gamma_{135}] = \{1\} + \{3\} + \{5\} = A^*_{3,0+3}(g_1^*, g_2^*, g_3^*)$$

# What about multiple Willson line operators?

- **Closed solutions:**

$$\frac{A^*_{3,m+2}(\Omega_1,\ldots,\Omega_m,(m+1)^*,(m+2)^*)}{A^*_{2,m+2}(\Omega_1,\ldots,\Omega_m,(m+1)^*,(m+2)^*)} = \omega^3_{m+4}[\Gamma_{tree}] = \sum_{i<j} c_{ij}[1i-1ij-1j].$$

$$c_{ij} = 0,\ \text{if there are no } m+3 \text{ among } i-1, i, j-1, j.$$

$$\begin{aligned}
c_{m+3j} &= 0, \\
c_{i,m+3} &= \frac{1}{1+\frac{\langle p_{m+1}\xi_{m+1}\rangle}{\langle p_{m+1}p_{m+2}\rangle}\frac{\langle 1i{-}1im{+}2\rangle}{\langle 1i{-}1im{+}3\rangle}}, \\
c_{i,m+4} &= \frac{1}{1+\frac{\langle p_{m+2}\xi_{m+2}\rangle}{\langle p_{m+2}1\rangle}\frac{\langle 1i{-}1im{+}3\rangle}{\langle i{-}1im{+}3m{+}4\rangle}}, \\
c_{m+2,m+4} &= \frac{1}{1+\frac{\langle p_{m+1}\xi_{m+1}\rangle}{\langle p_{m+1}p_{m+2}\rangle}\frac{\langle 1m{+}1m{+}3m{+}4\rangle}{\langle 1m{+}1m{+}2m{+}4\rangle}}\frac{1}{1+\frac{\langle p_{m+2}\xi_{m+2}\rangle}{\langle p_{m+2}1\rangle}\frac{\langle 1m{+}1m{+}2m{+}3\rangle}{\langle m{+}1m{+}2m{+}3m{+}4\rangle}}.
\end{aligned}$$

k=3 is related to total helicity of
on shell particles:
total h. = m+2n -2k
m is the number of on-shell particles
n is the number of off-shell once.

# What about multiple Willson line operators?

- **What we have left behind: <u>Relation to integrable systems</u>.**

$$\mathcal{S}_+(n, i-1, i) = \frac{\langle ni \rangle}{\langle n i-1 \rangle \langle i-1 i \rangle},$$

$$\mathcal{S}_-(n, i-1, i) = \frac{1}{[n i-1][i-1 i][in]^3} \hat{\delta}^4(\eta_n [i i-1] + \eta_i [n i-1] + \eta_{i-1}[in])$$

$$(\mathcal{S}_+(n1n-1))^* X|_{sub.} = R(0)_{n1} R(0)_{nn-1} X \delta^2(\lambda_n),$$

$$\mathcal{S}_-(n1n-1) X|_{sub.} = R(0)_{n1} R(0)_{12} X \delta^2(\tilde{\lambda}_n) \hat{\delta}^4(\eta_n).$$

The subtleties with (pseudo)vacuum state for form factors. Form factors (in general) are eigenvectors of Tr(M) while amplitudes are eigenvectors of M.

Where R's are R-matrixes of **gl**(4|4) spin chain (Chicerin,Derkachov,Krichner,Staudacher et.all 13-14):

$$R_{12}(u) = \int \frac{dz}{z^{1-u}} exp[-z(\mathbf{p}_1 \mathbf{x}_2)], \qquad \text{(B.146)}$$

Where $\mathbf{x}_i = (\lambda_i, \partial/\partial\tilde{\lambda}_i, \partial/\partial\eta_i)$ and $\mathbf{p}_i = (\partial/\partial\lambda_i, -\tilde{\lambda}_i, -\eta_i)$. $L(u)$ matrix is given by

$$L(u) = u\mathbb{I} + \mathbf{x} \otimes \mathbf{p}, \qquad \text{(B.147)}$$

# Conclusions.

- **The set of variables and methods which initially was associated with on-shell scattering amplitudes can be successfully applied to all type of gauge invariant objects in N=4 SYM ???**
- **Twistor string description ?**
- **Proper description in terms of integrable system.**
- **More loops/legs !!**
- **Chern-Simons theory generalisation.**
- **…**

# Twistor String theory origins of Grassmannian representation.

- **Let's consider the following worldsheet model:**

$$S = \frac{1}{2\pi}\int_{\Sigma} W\cdot\bar\partial Z - Z\cdot\bar\partial W + aZ\cdot W$$

- *"ambiTwistor string theory"*

- **Where our lambda's are now worldsheet fields:**

$$Z = (\lambda_\alpha, \mu^{\dot\alpha}, \chi^r) \in \mathbb{T}$$

$$W = (\tilde\mu, \tilde\lambda, \tilde\chi) \in \mathbb{T}^*$$

Masson&Co (1404.6219v2) ~14

- **Then one can define the following vertex operators:**

$$\mathcal{V}_a = \int \frac{\mathrm{d}s_a}{s_a}\bar\delta^{2|\mathcal{N}}(\lambda_a - s_a\lambda|\eta_a - s_a\chi)\mathrm{e}^{is_a[\mu\,\tilde\lambda_a]}J\cdot t_a$$

$$\widetilde{\mathcal{V}}_a = \int \frac{\mathrm{d}s_a}{s_a}\bar\delta^{2}(\tilde\lambda_a - s_a\tilde\lambda)\mathrm{e}^{is_a(\langle\tilde\mu\,\lambda_a\rangle+\tilde\chi_r\eta_a^r)}J\cdot t_a$$

# Twistor String theory origins of Grassmannian representation.

- **Then one can consider the correlation function of vertex operators:**

$$\mathcal{A} = \left\langle \widetilde{\mathcal{V}}_1 \ldots \widetilde{\mathcal{V}}_k \mathcal{V}_{k+1} \ldots \mathcal{V}_n \right\rangle$$

$$\mathcal{A} = \int \frac{1}{\mathrm{Vol\,GL}(2,\mathbb{C})} \prod_{a=1}^{n} \frac{\mathrm{d}s_a\,\mathrm{d}\sigma_a}{s_a(\sigma_a - \sigma_{a+1})} \prod_{i=1}^{k} \bar{\delta}^2(\tilde{\lambda}_i - s_i \tilde{\lambda})$$

- **N^kMHV amplitude in N=4 SYM.**

$$\prod_{p=k+1}^{n} \bar{\delta}^{2|\mathcal{N}}(\lambda_p - s_p\lambda(\sigma_p), \eta_p - s_p\chi(\sigma_p)).$$

- **In fact here are whole family of twistor string theories with correlation of the form:**

$$\int_\Sigma d^{2n-4}\zeta\, \mu(\zeta) \prod_\alpha \delta^{4|4}(C^\star_{\alpha a}(\zeta_I)\mathcal{W}_a)$$

Witten ~04
Bercovits ~04
Masson&Co ~13
…

- **Which are related to tree amplitudes in N=4 SYM and in N=8 SUGRA.**