Relativistic dynamics without proper time

Edward Kapuścik, Poland

2016

Abstract

It is shown that it is possible to describe motions in a relativistically covariant way in terms of the coordinate time without using the notion of the proper time. For completeness we consider motions of Galilean and Einsteinian both subluminal and superluminal particles. The presented approach can easily be generalized to more general models of spacetime.

Introduction

In Special Relativity motions of mass points are customarily described in terms of the notion of the proper time τ related to the coordinate time t by the relation

$$d\tau = \sqrt{1 - \frac{\vec{v}^2(t)}{c^2}}dt, \quad (1)$$

where $\vec{v}(t)$ is the velocity of the considered mass point. In spite of the fruitfullnes of such an approach the use of the proper time formalism has also a serious disadvantages: **first**, for nonuniform motions (in particular for oscillatory changing velocities) the proper time does not provide a parameter which uniformly increases with coordinate time and **second**, the proper time does not allow to use it for many particle systems because the proper times for each particle are different. In addition, the proper time for nonuniform motions coincide with the coordinate time in continuously changing inertial reference what makes it difficult to visualize the motion.

It is also clear that the proper time approach cannot be applied to tachyons because for superluminal speeds the proper time becomes imaginary and consequently it cannot be used as a parameter labelling the position of the tachyon on its trajectory.
In the present paper we shall show how to describe relativistic motions using only the coordinate time without any reference to the proper time. The clue to this goal is the velocity tensor introduced in [2]. For simplicity, we restrict here the considerations to the two-dimensional spacetime only. The passage to higher dimensional spacetimes is straghtforward but technically more involved

.

The talk is organized as follows. First, we recapitulate the basic properties of the velocity tensors. Then, we shall explicitly construct such tensors for the Galilean an Minkowskian space times. In the later case we shall consider both the subluminal and superluminal motions. Finally, we shall consider the dynamical equations of motions which directly generalize the standard Newton dynamical equation.

Basic properties of the velocity tensors

In the formalism described in [2] the velocity tensors $V^{\mu}_{\nu}(\vec{v})$ are used as functions of the standard three-dimensionl velocity $\vec{v}$. In terms of this tensors the standard kinematic relation

$$d\vec{x} = \vec{v}dt \tag{2}$$

is written in the covariant form as

$$V^{\mu}_{\nu}(\vec{v})dx^{\nu} = 0, \tag{3}$$

where $dx^{\nu}(\nu = 0, 1, 2, 3)$ denote the infinitesimal displacements along the trajectory of the particle.

The general construction of the velocity tensors goes as follows. First, we use the tensorial transformation rule for the matrix V with matrix elements V^{μ}_{ν} given by the components of the velocity tensor. This rule reads

$$V'(\vec{v'}) = SV(\vec{v})S^{-1}, \tag{4}$$

where S is the matrix with matrix elements S^{μ}_{ν} fixed by the transformation rule for spacetime coordinates

$$dx'^{\mu} = S^{\mu}_{\nu} dx^{\nu}. \tag{5}$$

Second, we use the additional conditions for the matrix V of the form [2]

$$Tr_j V = 0 \tag{6}$$

for $j = 1, ...n$, where n is the dimension of spacetime and Tr_j denote the sums of the diagonal minors of order j of the matrix V. Conditions (6) ensure that all eigenvalues of the matrix V are equal to zero because under the conditions (6) the characteristic equation for the eigenvalues λ reduces to the simple equation

$$\lambda^n = 0. \tag{7}$$

The eigenvalue equation (3) provides then an unique eigenvector dx^μ.

Third, we assume that matrix elements of $V(\vec{v})$ are form-invariant functions of the velocity $\vec{v}$ and therefore

$$V'(\vec{v}) = V(\vec{v}). \tag{8}$$

This relation together with (4) provides us functional equations for finding the matrix elements of V. These functional equations obviously have the form

$$V(\vec{v'}) = SV(\vec{v})S^{-1}. \tag{9}$$

We shall now illustrate this method on the examples of the Galilean and Einsteinian two-dimensional spacetimes. The generalization to four dimensional spacetime is straightforward but a little bit more tedious.

Examples of velocity tensors

The Galilean transformations of spacetime coordinates

$$t' = t, \qquad x' = x + ut \tag{10}$$

lead to the following form of the matrix S:

$$S_G(u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \tag{11}$$

where u is the relative velocity of the observer tight to the primed reference frame with respect to the observer tight to the unprimed reference frame.

Then (9) gives the functional equation

$$V_G(v+u) = S_G(u)V_G(v)S_G^{-1}(u) = S_G(u)V_G(v)S_G(-u). \quad (12)$$

Taking the unprimed reference frame as the rest frame for the particle we must put $v = 0$ and we get

$$V_G(u) = S_G(u)V_G(0)S_G(-u). \quad (13)$$

Finally, renaming the velocity u as v we get

$$V_G(v) = S_G(v)V_G(0)S_G(-v). \quad (14)$$

In the rest frame equation (3) gives two equations

$$V_0^0(0)dt + V_1^0(0)dx = 0 \tag{15}$$

and

$$V_0^1(0)dt + V_1^1(0)dx = 0. \tag{16}$$

But in the rest frame of the particle along its trajectory $dx = 0$ for arbitrary dt and therefore

$$V_0^0(0) = V_0^1(0) = 0. \tag{17}$$

The conditions (6) for the two-dimensional spacetime reduce to only two conditions

$$Tr\ V(v) = 0 \tag{18}$$

and

$$det\ V(v) = 0. \tag{19}$$

From these conditions and (17) we get then that also $V^1_1(0) = 0$. Below we shall see that it is convenient to normalize the remaining free matrix element $V^0_1(0)$ to -1. Thus we have

$$V_G(0) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \tag{20}$$

and from (13) we finally get the Galilean velocity tensor in the form

$$V_G(v) = \begin{pmatrix} v & -1 \\ v^2 & -v \end{pmatrix}. \tag{21}$$

Clearly with such velocity tensor the standard equation (2) follows from the equation (3).

In the case of the Einsteinian two-dimensional spacetime from the standard Lorentz transformations

$$t' = \frac{t + \frac{u}{c^2}x}{\sqrt{1 - \frac{u^2}{c^2}}},$$

$$x' = \frac{x + ut}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{22}$$

we have

$$S_L(u) = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \begin{pmatrix} 1 & \frac{u}{c^2} \\ u & 1 \end{pmatrix} \tag{23}$$

and from equations (9) for the Einsteinian velocity tensor V_E we get the functional equation

$$V_E\left(\frac{v+u}{1+\frac{vu}{c^2}}\right) = S_L(u)V_E(v)S_L^{-1}(-u). \tag{24}$$

Again, assuming that the unprimed reference frame is the rest frame of the particle we get the relation (14) with $S_G(v)$ replaced by $S_L(v)$. In the rest frame the velocity tensor $V_E(0)$ is exactly the same as in the Galilean case because the same conditions must be satisfied as for the Galilean case. Thus finally we get

$$V_E(v) = \frac{1}{1-\frac{v^2}{c^2}}\begin{pmatrix} v & -1 \\ v^2 & -v \end{pmatrix}. \tag{25}$$

It is easy to check that with such velocity tensor from (3) equation (2) also follows.

The presented formalism may also be applied to motions with superluminal speeds. The physics of tachyons is relatively poorely developed. The main reason for that is the widely spread but erroneous opinion that the existence of tachyons contradicts the main principles of Special Relativity. Such opinions are based on the unjustified statement that the speed of light is the maximal speed allowed by Special Relativity[3]. As a matter of fact the speed of light is only the invariant speed respected by all inertial observers[4].

The physics of tachyons began with the paper by G. Feinberg [5] who formally introduced the imaginary mass in standard relativistic expressions for momentum and energy for objects moving with superluminal speeds. As a result Feinberg arrived to the following expressions for energy and momentum of tachyons

$$E = \frac{\mu}{\sqrt{\frac{v^2}{c^2} - 1}},$$

$$\vec{p} = \frac{\mu \vec{v}}{\sqrt{\frac{v^2}{c^2} - 1}}.$$

Although G. Feinberg argued that the existence of tachyons does not contradict Special Relativity the fact that they enter physics through complex numbers infected their life from the very beginning.

Fortunately, recently [6, 7] we have shown that there exists a quite natural way to introduce both sub- and superluminal objects into the framework of Special Relativity. The new approach is based on the most general linear transformations which preserve the invariant magnitude of the velocity c. In the present paper we shall however not follow this approach but we shall argue in the framework of the standard Special Relativity based on the Lorentz transformation.

To construct the velocity tensor for tachyons we must use the Lorentz transformation between the reference frame in which the tachyon moves with an infinite speed and the reference frame in which its speed is equal to w. This transformation leads to the following form of the matrix $S(w)$ [6, 7]

$$S(w) = \frac{1}{\sqrt{1 - \frac{c^2}{w^2}}} \begin{pmatrix} 1 & \frac{1}{w} \\ \frac{c^2}{w} & 1 \end{pmatrix} \tag{26}$$

and to the composition law of tachyonic speeds

$$w_{12} = \frac{w_1 w_2 + c^2}{w_1 + w_2}. \tag{27}$$

The tachyonic velocity tensor therefore satisfies the functional equation

$$V\left(\frac{w_1 w_2 + c^2}{w_1 + w_2}\right) = S(w_1)V(w_2)S^{-1}(w_1) = S(w_1)V(w_2)S(-w_1). \tag{28}$$

In the limit $w_2 \to \infty$ we get

$$V(w) = S(w)V(\infty)S(-w). \tag{29}$$

In the reference frame in which the tachyon moves with an infinite speed from (3) we have

$$V_0^0(\infty)dt + V_1^0(\infty)dx = 0 \tag{30}$$

and

$$V_0^1(\infty)dt + V_1^1(\infty)dx = 0. \tag{31}$$

Tachyons with infinite speeds in any finite time pass infinite distances. Therefore, equations (30) and (31) may be satisfied only for

$$V_1^0(\infty) = V_1^1(\infty) = 0. \tag{32}$$

From the traceless condition for velocity tensors we get then also that

$$V^0_0(\infty) = 0. \tag{33}$$

Normalizing $V^1_0(\infty) = +1$ we finally get

$$V_T(\infty) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{34}$$

With such velocity tensor in the reference frame where tachyons move with infinite speeds the time stops along their trajectories because from (31) it follows that

$$dt = 0. \tag{35}$$

With $V_T(\infty)$ of the form (34) the tachyonic velocity tensor $V_T(w)$ has the form

$$V_T(w) = \frac{1}{1-\frac{c^2}{w^2}} \begin{pmatrix} \frac{1}{w} & -\frac{1}{w^2} \\ 1 & -\frac{1}{w} \end{pmatrix}. \tag{36}$$

This tensor also gives equations (2).
As it was shown in [6, 7] the main difference between subluminal and superluminal objects consists just in the existence of rest frames for the former objects and the nonexistence of such frames for the latter.

Dynamical equations

The dynamical equation of motion we shall write in the form

$$\partial_\mu V^\mu_\nu(\vec{v}) = I_\nu, \tag{37}$$

where I_ν describes the influence of the environment on the moving object. It is clear that (37) is the only covariant form which generalize the standard Newton equation

$$\frac{d\vec{v}(t)}{dt} = \frac{1}{M}\vec{F}(t), \tag{38}$$

where M is the mass of the particle and $\vec{F}(t)$ is the acting force. Below, we shall elaborate the meaning of the notion of I_ν and its relation to the standard force $\vec{F}(t)$.

For this purpose we shall apply the dynamical equation (37) using the velocity tensors derived above. We begin with the Galilean velocity tensor (21) with the time dependent velocity $v(t)$. Since the components of the velocity tensor depends only on the time coordinate equation (37) reduces to two equations

$$\frac{dv(t)}{dt} = I_0(t). \tag{39}$$

and

$$I_1(t) = 0. \tag{40}$$

From equation (39) it is clear that the time component of the influence is related to the customary force $F(t)$ in the form

$$I_0(t) = \frac{1}{M} F(t), \tag{41}$$

where M is the mass of the particle.

For the relativistic subluminal particles with the velocity tensor (25) equation (37) gives

$$\frac{d}{dt}\left(\frac{v(t)}{1-\frac{v^2(t)}{c^2}}\right)=l_0(t) \tag{42}$$

and the corresponding equation for the space component

$$-\frac{d}{dt}\left(\frac{1}{1-\frac{v^2(t)}{c^2}}\right)=l_1(t) \tag{43}$$

In the simplest case of constant in time $I_0(t) = I$ we can integrate equation (42) and solve the result with respect to $v(t)$. In this way we get

$$v(t) = c^2 \frac{\sqrt{1 + 4\frac{(It+\Gamma)^2}{c^2}} - 1}{2(It + \Gamma)} = \frac{2(It + \Gamma)}{1 + \sqrt{1 + \frac{4(It+\Gamma)^2}{c^2}}} < c, \qquad (44)$$

where

$$\Gamma = \frac{v_0}{1 - \frac{v_0^2}{c^2}} - It_0 \qquad (45)$$

and v_0 is the initial velocity at time t_0. It is easy to see that $v(t)$ is always less than the speed of light and in the limit $t \to \infty$ we get $v(t) \to \pm c$, where the sign depends on the sign of I.

This result is to be compared with the case of a standard relativistic particle moving under the influence of a constant force [1] for which we have

$$v_{rel}(t) = \frac{Ft/M}{\sqrt{1 + \left(\frac{Ft}{Mc}\right)^2}}. \tag{46}$$

Here M is the mass of the particle, F the standard nonrelativistic force and the initial condition is such that $\Gamma = 0$ (i.e. at $t_0 = 0$ we put $v_0 = 0$). For large values of t formulas (44) and (46) coincide for $\Gamma = 0$ provided $I = F/2M$.

Finally, we pass to the motion of tachyons for which the velocity tensor has the form (36) and equation (37) gives

$$\frac{d}{dt}\left(\frac{w(t)}{w^2(t)-c^2}\right) = l_0(t) \tag{47}$$

and

$$-\frac{d}{dt}\left(\frac{1}{w^2(t)-c^2}\right) = l_1(t). \tag{48}$$

For a constant in time $l_0(t)$ we get

$$w(t) = \frac{1 + \sqrt{1 + 4c^2(lt + \Gamma)^2}}{2(lt + \Gamma)} > c, \tag{49}$$

where

$$\Gamma = \frac{w_0}{w_0^2 - c^2} - lt_0 \tag{50}$$

and w_0 is the initial velocity at time t_0.

Conclusions The main result of the present paper consists in the proof that in the framework of Special Relativity motion may be described exactly in the same way as it is done in the Galilean physics. The formalism is fully covariant under corresponding relativity groups and maximally simple. The extension of the customary description of motion to tachyons is possible because we do not use the notion of a proper time which is necessary in the standard formalism of Special Relativity but which for nonuniform motions is physically very difficult to measure due to its interpretation as time in continuously changing reference frames. Moreover, proper time is meaningless for superluminal motions. Our formalism uses only coordinate time in the reference frame in which the motion is described.

The presented approach can easily be extended to motions in spacetimes with symmetries given by more general transformations than the described above. As a matter of fact, there is no restriction on the transformation matrices S in (4) so the formalism may be applied also in the framework of General Relativity.
It is also worth to note an interesting application of the presented formalism to particle physics[8].

- V. A. Ugarov; *Special Theory of Relativity*, Nauka, Moscow 1977 (in Russian)
- E. Kapuścik and T. Lanczewski; *Physics of Atomic Nuclei* **72** (2009) 809-812
- See, for example, S. Hawking; *A Brief History of Time*, Bantam Books, 1988
- S. Liberati, S. Sonego and M. Visser; *Annals of Physics* **296**(2002)167-185
- G. Feinberg; *Phys. Review*, **159**, 1967, 1089-1105
- E. Kapuścik; *Physics of Atomic Nuclei* **74** (2011) 919-922
- E. Kapuścik; *Condensed Matter Physics* **13** (2010) 43102: 1-8; arXiv:1010.5886v1[physics.gen-ph]
- E. Kapuścik and R. Orlicki; *Ann. Phys. (Berlin)* **523** (2011)