Loops, Triangles
and the Optical Conductivity of Graphene

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Based on works with Anatoly V. Kotikov

PRD **87** 087701 (2013), arXiv:1302.3939
PRD **89** 065038 (2014), arXiv:1312.2430
EPL **107** 57001 (2014), arXiv:1407.7501
TMP, arXiv:1602.01962 [hep-th] (short review)

Outline

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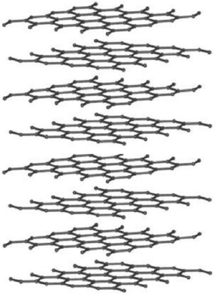
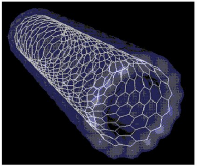
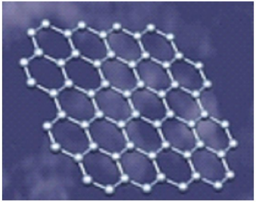
Allotropes of carbon







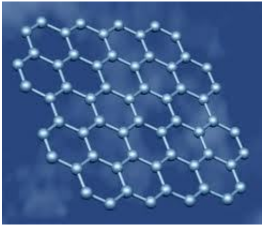
Graphite (3D)

Fullerene (0D)
1985

Nanotube (1D)
1991

Graphene (2D)
2004

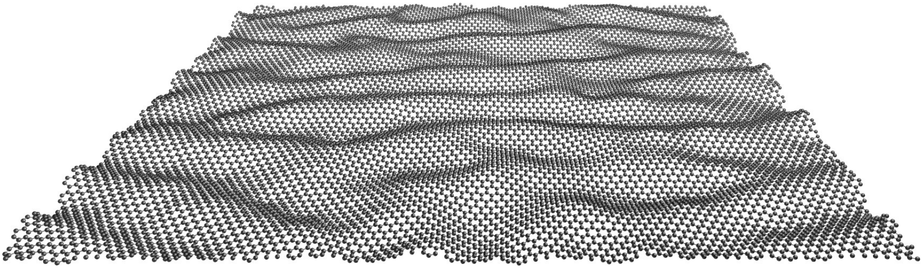
Why graphene interesting?



- **Graphene: one of the most 2D system** (thickness 0.5nm = 1 atom)
- **Easy to produce in small quantities** (isolated in 2004 by Geim and Novoselov: graphite + scotch tape)
- **High crystal quality in 2D** (very clean system in which to experiment)
- **Exceptional properties**: stronger than steel and very stretchable, good conductor of electricity and heat (keep electronics cool), ...
- **Numerous (potential) applications** (graphene “fever” since 2010, year of the Nobel Price)
- **Fundamental studies of low-dimensional interacting systems**

Graphene as a membrane (2-brane)

Stability related to ripples (3 atoms high and 30 atoms long)
To overcome the argument of Landau & Peierls



Fundamental aspects: major early works (vast literature since then)

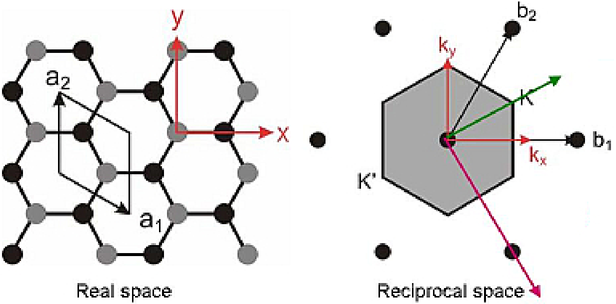
- 1947: band structure (Wallace)
- 1984: field theory approach (Semenoff)
- 1994: RG approach (González, Guinea, Vozmediano)
- 2004: experimental isolation

Band structure of graphene

P. R. Wallace, Physical Review **71** 622 (1947)

Honeycomb lattice:

- 2 inequivalent Bravais sublattices (A and B)
- 2 distinguishable points in the Brillouin zone (K and K')



Nearest-neighbour tight-binding model for π-electrons

$$H = -t \sum_{\langle i,j \rangle,\sigma} \left(a^\dagger_{i,\sigma} b_{j,\sigma} + \mathrm{h.c.} \right) - \mu \sum_{i,\sigma} n_{i,\sigma}$$

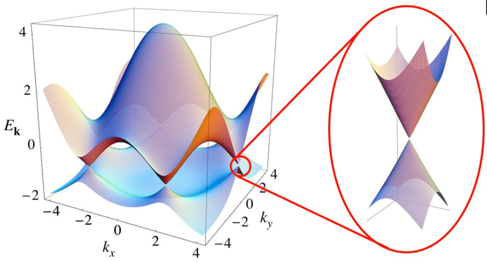
- Spinor structure (besides a real spin 1/2):
  - sublattice pseudo-spin 1/2: electron either on A or B sublattice

$$\psi_{K,\sigma}(\vec{k}) = \begin{pmatrix} a_\sigma(\vec{K}+\vec{k}) \\ b_\sigma(\vec{K}+\vec{k}) \end{pmatrix} \quad \psi_{K',\sigma}(\vec{k}) = \begin{pmatrix} b_\sigma(\vec{K}'+\vec{k}) \\ a_\sigma(\vec{K}'+\vec{k}) \end{pmatrix}$$

  - valley pseudo-spin 1/2: electron close to K or K' point

$$\Psi_\sigma(\vec{k}) = \begin{pmatrix} \psi_{K,\sigma}(\vec{k}) \\ \psi_{K',\sigma}(\vec{k}) \end{pmatrix}$$ (4 component spinor in sublattice $\otimes$ valley space).

- Spectrum: massless Dirac spectrum at low energies



Low energies ($<$ 1eV from K-points)

linear spectrum

$$E_\pm(\vec{k}) = \pm\hbar v_F|\vec{k}\,| - \mu$$

$$v_F = \frac{3lt}{2} \approx 10^6\,\mathrm{m/s}$$

Low-energy effective description

G. W. Semenoff, PRL **53** 2449 (1984)

Massless QED-like Lagrangian ($\mu = 0$, no gauge field, $N_F = 2$)

$$\mathcal{L}_0 = \sum_{\sigma=1}^{N_F} \bar{\Psi}_\sigma \left[\mathrm{i}\gamma^0 \partial_t + \mathrm{i} v_F \vec{\gamma} \cdot \vec{\nabla} \right] \Psi_\sigma$$

Dirac matrices: $\gamma^\mu = (\gamma^0, \vec{\gamma})$, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, $g^{\mu\nu} = \mathrm{diag}(+,-,-)$

Very different from Galilean invariant band metals and semi-conductors:

Intrinsic graphene is a "zero" gap semi-conductor (semi-metal)
(two-dimensional Dirac fermions)

- Fermi surface reduces to the 2 K-points
- linear spectrum and chirality (lost at high energies)
- no gap larger than 1meV (present experimental resolution)

Experimental evidence for Dirac fermions in graphene: unconventional QHE

$$\sigma_{xy} = e^2\nu/h, \qquad \nu = \pm 4 \times (n + 1/2), \qquad n = 0, 1, \ldots$$

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Remark: search for semi-metals has a long history

Early studies devoted to 3D semi-metals

- Herring, Physical Review **32** 365 (1937)
  First study - general conditions (single particle arguments)
- Halperin and Rice, Rev. Mod. Phys. **40** 755 (1968)
  Quadratic band-touching unstable to interactions (excitonic insulator)
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Systems with stable Fermi points: emergent relativity at low energies

Volovik, "The Universe in a Helium Droplet" (2009)

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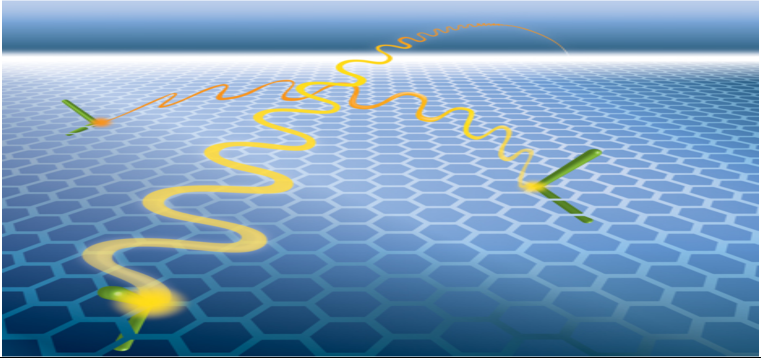
Long-range 3D electron-electron interactions

González, Guinea and Vozmediano, Nucl. Phys. **B 424** 595 (1994)

$v_F < c$: interactions break Lorentz invariance (pseudo-relativistic system)

Non-relativistic limit ($v_F/c \to 0$): instantaneous Coulomb interaction

$$H_C = \frac{1}{2}\sum_{\sigma,\sigma'}\int d^2r d^2r' \bar{\Psi}_\sigma(\vec{r})\gamma^0\Psi_\sigma(\vec{r})\,\frac{e^2}{\kappa|\vec{r}-\vec{r}'|}\,\bar{\Psi}_{\sigma'}(\vec{r}')\gamma^0\Psi_\sigma(\vec{r}')$$



Energy scales:

- Kinetic energy: $E_0 = \hbar v_F k \sim \sqrt{n}$,
- Interaction energy: $E_c = e^2/(\kappa r) \sim \sqrt{n}$,
  (κ dielectric constant and n electron density)

"Fine structure constant" of graphene

$$\alpha_g = \frac{E_c}{E_0} = \frac{e^2}{4\pi\kappa\hbar v_F} \approx \frac{2.2}{\kappa}$$

- the ratio does not depend on the density (2D),
- of the order of 1 in general (QED: $\alpha_{QED} = 1/137$),
- free standing graphene: $\kappa \approx 1$ (air), $\alpha_g \approx 2.2$,
- graphene on Boron Nitride substrate: $\kappa \approx 2.5$, $\alpha_g \approx 0.9$.

Very different from Galilean invariant band metals: $r_s = E_c/E_0 = 1/\sqrt{n}$

- High densities: $r_s \ll 1$ (Fermi liquid)
- Low densities: $r_s \gg 1$ (Wigner crystallization, ferromagnetism, ...)

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Important issue

Quantitative understanding of the effect of interactions

A priori, effects of interactions should be strong

Within the low-energy effective model:

$$S = \int \mathrm{d}^{1+2}x \left[\mathcal{L}_0 - e_0 \bar{\Psi}_\sigma \gamma^0 A_0 \Psi_\sigma + e_0 \frac{v_0}{c} \bar{\Psi}_\sigma \vec{\gamma} \cdot \vec{A} \Psi_\sigma \right] - \frac{1}{4} \int \mathrm{d}^{1+3}x \, F_{\mu\nu}^2$$

there are extensive studies related to:

review: Kotov et al., Rev. Mod. Phys. **84** 1067 (2012)

review: Gusynin et al., Int. J. Modern Phys. B **21** 4611 (2007)

- transport properties,
- spectral properties (some marginal liquid features),
- dynamical gap generation (DχSB: $U(2N_F) \to U(N_F) \times U(N_F)$),
- add disorder (random gauge fields), ripples (curved space), magnetic field...

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Optical conductivity (free fermions)

$$\vec{j}(\omega, \vec{q}\,) = \sigma(\omega, \vec{q}\,)\,\vec{E}(\omega, \vec{q}\,)$$

Optical regime: $\omega \gg v_F|\vec{q}|$

- response to a homogeneously applied electric field ($\vec{q} \to 0$)
- photon energies $\omega \approx 1\text{eV}$ (visible range of the spectrum)

Intrinsic graphene: semi-metal (no state at the Fermi points).

Does it conduct?

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Ludwig, Fisher, Shankar and Grinstein, PRB **50** 7526 (1994)

$$\sigma_0(\omega) = \frac{e^2}{4\hbar}$$

Non-zero and universal (independent of ω)

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Proof

R. R. Nair et al., Science **320** 1308 (2008)

Near the K point: $$H = v_F \hat{\sigma} \cdot \left(\vec{p} - \frac{e}{c} \vec{A}(t) \right)$$

$$\vec{E}(t) = E_0 \cos(\omega t)\hat{x} = -(1/c)\,\partial_t \vec{A}(t), \quad \vec{A}(t) = -(E_0 c/\omega)\sin(\omega t)\hat{x}$$

$$|\langle f|V(t)|i\rangle| \approx \frac{e v_F E_0}{\omega}, \qquad \rho(\hbar\omega/2) \approx \frac{\omega}{v_F^2 \hbar}$$

Transition rate from initial to final state (Fermi's Golden rule):

$$\frac{1}{\tau} = \frac{2\pi}{\hbar} |\langle f|V(t)|i\rangle|^2 \rho\,(\hbar\omega/2) \approx \frac{e^2}{\hbar^2}\frac{E_0^2}{\omega}$$

Energy absorption rate: $P_a = \hbar\omega/\tau \approx (e^2/\hbar)\; E_0^2 \quad (P_i \approx c E_0^2/(4\pi))$

Transmittance and optical conductivity

$$T_0 = 1 - \frac{P_a}{P_i} \approx 1 - \pi\alpha_{QED} \approx 97.7\% \qquad T(\omega) = [1 + 2\pi\sigma(\omega)/c]^{-2}$$

One-atom thick layer absorbs 2.3% of visible light!

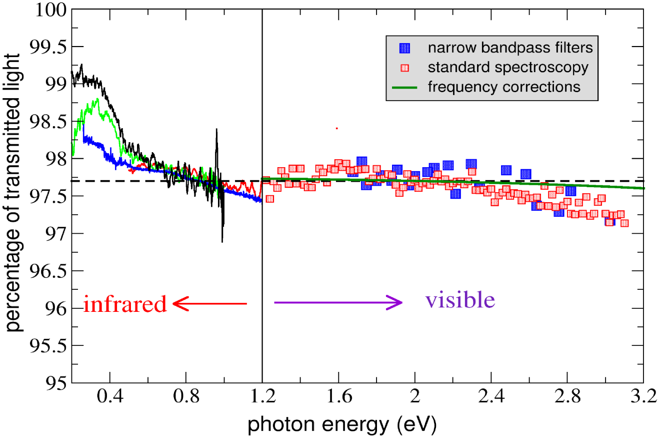
Experimentally: optical conductivity is close to ideal (!?)

$$\sigma(\omega \approx 1\text{eV}) = \sigma_0\,(1.00 \pm 0.02)$$

$\hbar\omega < 1.2$eV: Mak et al., PRL **101** 196405 (2008)

$\hbar\omega > 1.2$eV: Nair et al., Science **320** 1308 (2008)

For a review: N. M. R. Peres, RMP **82** 2673 (2010)



Theoretically: compute numbers

Optical conductivity from the polarization operator

$$\sigma(q_0) = -\lim_{\vec{q}\to 0} \frac{\mathrm{i}q_0}{|\vec{q}|^2} \Pi^{00}(q_0, \vec{q}\,), \qquad q^\mu = (q^0, v_0\vec{q})$$

$$\Pi^{00}(q) = \langle T\rho(q)\rho(-q)\rangle, \qquad \rho(q) = e_0\bar{\Psi}\gamma^0\Psi$$

Perturbative expansion: compute interaction correction coefficients

$$\sigma(q_0) = \sigma^{(0)}\left(\mathbf{1} + \mathcal{C}\alpha_r + \mathcal{C}'\alpha_r^2 + \cdots\right)$$

Notice: unrenormalized e_0, v_0 and renormalized α_r...

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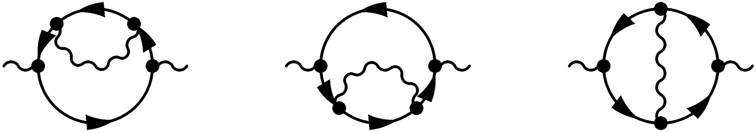
First order correction $\mathcal{C}$:



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Optical conductivity from the Kubo formula

$$\tilde{\sigma}(q_0) = \frac{1}{2\mathrm{i}q_0}\left(K^{11}(q_0,\vec{q}\to 0) + K^{22}(q_0,\vec{q}\to 0)\right)$$

$$K^{ij}(q) = \langle Tj^i(q)j^j(-q)\rangle, \qquad \vec{j}(q) = e_0 v_0\bar{\Psi}\vec{\gamma}\Psi$$

Theoretically: value of $\mathcal{C}$ is controversial (!?)

Extensive theoretical work since 2008; three different values can be found:

$$\mathcal{C}^{(1)} = \frac{25 - 6\pi}{12} \approx 0.512\,,$$ Herbut et al. 2008, hard cut-off, $\tilde{\sigma}$

$$\mathcal{C}^{(2)} = \frac{19 - 6\pi}{12} \approx 0.013\,,$$ Mishchenko 2008, hard & soft cut-off, σ, $\tilde{\sigma}$

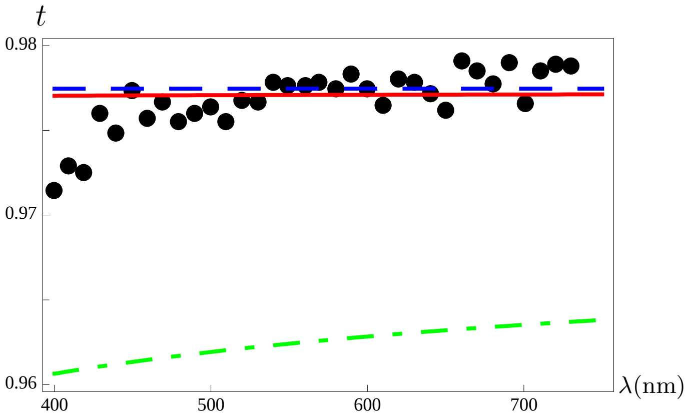
$$\mathcal{C}^{(3)} = \frac{11 - 3\pi}{6} \approx 0.263\,,$$ Juričić et al. 2010, dimensional regularization

Herbut, Juričić and Vafek, PRL **100** 046403 (2008)
Mishchenko, EPL **83** 17005 (2008)
Sheehy and Schmalian, PRB **80** 193411 (2009)
Juričić, Vafek and Herbut, PRB **82** 235402 (2010) (JVH)
Abedinpour et al., PRB **84** 045429 (2011)
Sodemann and Fogler, PRB **86** 115408 (2012)
Rosenstein, Lewkowicz and Maniv, PRL **110** 066602 (2013)
Gazzola et al., EPL **104** 27002 (2013)
Teber and Kotikov, EPL 107 57001 (2014)
Link, Orth, Sheehy and Schmalian, PRB **93** 235447 (2016) (corrects Rosenstein et al)
Boyda, Braguta, Katsnelson and Ulybyshev, arXiv:1601.05315 (Monte Carlo)

Before 2014: differences attributed to the regularization techniques used...

Tentative comparison between theory and experiments

Figure from: Sheehy and Schmalian, PRB **80** 193411 (2009) $(\alpha_g \approx 2.2)$

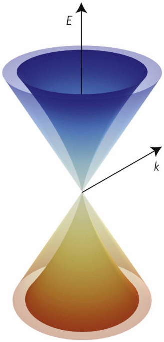


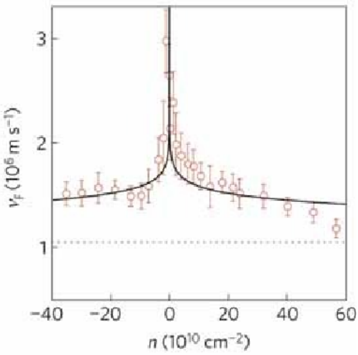
- Dashed blue line: free fermion case
- Solid red line: Mishchenko's result $\mathcal{C}^{(2)} \approx 0.013$
- Dot dashed green line: Herbut et al.'s result $\mathcal{C}^{(1)} \approx 0.512$

Experimental evidence for Fermi velocity renormalization

Band structure as a function of carrier density n ($n \to 0$ at K points)

Elias et al., Nature Physics **7** 701 (2011)





$n > 2 \times 10^{11}\text{cm}^{-2}$:
$v_F \approx 1 \times 10^6 \text{m/s}$

$n < 1 \times 10^{10}\text{cm}^{-2}$:
$v_F \approx 3 \times 10^6 \text{m/s}$

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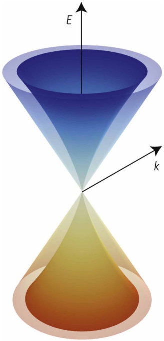

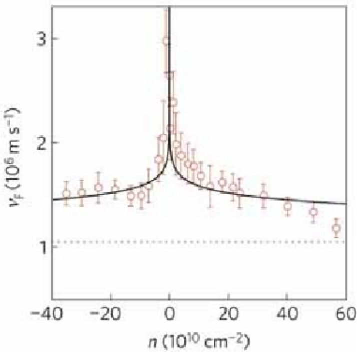

$n > 2 \times 10^{11}\mathrm{cm}^{-2}$:
$v_F \approx 1 \times 10^6 \mathrm{m/s}$

$n < 1 \times 10^{10}\mathrm{cm}^{-2}$:
$v_F \approx 3 \times 10^6 \mathrm{m/s}$

Logarithmic increase of the Fermi velocity as n decreases ($\mu \sim \sqrt{n}$):

González, Guinea and Vozmediano, Nucl. Phys. **B 424** 595 (1994)

$$v_r(\mu) = v_r(\Lambda) + \frac{e^2}{16\pi\kappa} \log \frac{\Lambda}{\mu}, \qquad v_r(\Lambda) = v_F$$

Experimental evidence for Fermi velocity renormalization

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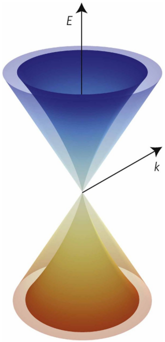

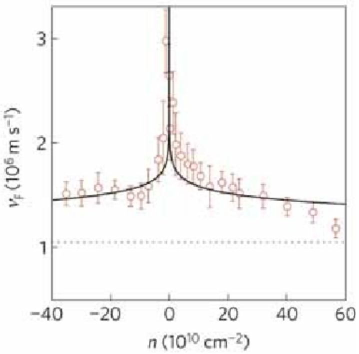

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Corresponding Fermi velocity beta-function:

González, Guinea and Vozmediano, Nucl. Phys. **B 424** 595 (1994)

$$\beta_v = \frac{\mathrm{d}\log v_r(\mu)}{\mathrm{d}\log\mu} = -\frac{\alpha_r(\mu)}{4} + \mathrm{O}(\alpha_r^2)\,, \qquad \alpha_r(\mu) = \frac{e^2}{4\pi\kappa v_r(\mu)}$$

Lorentz-invariant infra-red fixed point

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$$v(\mu) = v_0 + \frac{e^2}{16\pi\kappa} \ln\frac{\Lambda}{\mu} \quad \xrightarrow[\mu\to 0]{} \quad c$$

$$\alpha(\mu) = \frac{e^2}{4\pi\kappa v(\mu)} = \left(\frac{1}{\alpha_0} + \frac{1}{4}\ln\frac{\Lambda}{\mu}\right)^{-1} \quad \xrightarrow[\mu\to 0]{} \quad \alpha_{QED}$$

Existence of such fixed point is generic to systems with Fermi points

emergent relativity at low energies   Volovik, "The Universe in a Helium Droplet" (2009)

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de Juan, Grushin and Vozmediano, PRB **82** 125409 (2010)

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Outline

Reduced or Pseudo Quantum Electrodynamics

Terminology from: Gorbar, Gusynin and Miransky PRD **64** 105028 (2001)
Marino, Nucl. Phys. **B408** 551 (1993)

Basics of massless reduced QED$_{d_\gamma,d_e}$ $(d_e < d_\gamma)$

Fermion field in d_e-dimensions (mem-brane) $\Rightarrow d_e = 4 - 2\varepsilon_e - 2\varepsilon_\gamma$
Photon field in d_γ-dimensions (bulk gauge field) $\Rightarrow d_\gamma = 4 - 2\varepsilon_\gamma$

$$\mathcal{L} = \bar{\Psi}(x)\mathrm{i}\gamma^{\mu_e} D_{\mu_e}\Psi(x)\,\delta^{(d_\gamma - d_e)}(x) - \frac{1}{4}\,F_{\mu_\gamma\nu_\gamma}F^{\mu_\gamma\nu_\gamma} - \frac{1}{2a}\left(\partial_{\mu_\gamma}A^{\mu_\gamma}\right)^2$$

- case $d_\gamma = d_e$ $(\varepsilon_e = 0)$: usual QEDs
  - QED$_4$ (renormalizable),
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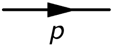
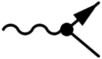
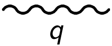
Reduced QED Feynman rules (photon propagator has a branch cut):
(natural units are used: $\hbar = c = 1$)



$$= \frac{i}{\not{p}}$$



$$= -ie\gamma^{\mu}$$



$$= \frac{i\Gamma(1-\varepsilon_e)}{(4\pi)^{\varepsilon_e}(-q^2)^{1-\varepsilon_e}} \left(g^{\mu\nu} - \tilde{\xi}\,\frac{q^{\mu}q^{\nu}}{q^2} \right) \quad (\tilde{\xi} = \frac{\xi}{2} = \frac{1-a}{2})$$

Case of reduced $QED_{4,3}$: $\varepsilon_e = 1/2$ and $\varepsilon_\gamma \to 0$ ($d_\gamma = 4$, $d_e = 3$)

graphene at the IR fixed point

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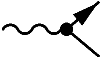
μ  $\quad = -ie\gamma^\mu$

μ  ν, q

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Local interactions but free effective gauge-field action is non-local

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- square-root branch cut in the photon propagator: $\propto (-q^2)^{-1/2}$
- feynman diagrams with non-integer indices

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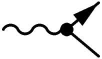

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Dimensional Reg. and Renormalization

Bare fields and parameters should be expressed in terms of renormalized ones with the help of renormalization constants:

$$\Psi_n = Z_\psi^{1/2}(\mu)\,\Psi_{nr}(\mu), \quad A^\mu = Z_A^{1/2}(\mu) A_r^\mu(\mu), \quad \Gamma^\mu = Z_\Gamma(\mu)\,\Gamma_r^\mu\,,$$
$$e_0^2 = Z_\alpha(\mu)\,e^2(\mu)\,\left(\mu^2 \frac{e^{\gamma_E}}{4\pi}\right)^{\varepsilon_\gamma}$$

Note: we work in $\overline{\mathrm{MS}}$-scheme where $\mu^2 \to \mu^2\, e^{\gamma_E}/(4\pi)$ and

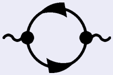
$$Z = \sum_{n=0}^{\infty} \frac{z_n}{\varepsilon_\gamma^n} = 1 + \frac{z_1}{\varepsilon_\gamma} + \frac{z_2}{\varepsilon_\gamma^2} + \ldots$$

Anomalous dimensions of fields and beta-functions of parameters, *e.g.*,

$$S(p) = Z_\psi(\mu)\,S_r(p;\mu), \qquad \gamma_\psi(\mu) = \frac{\mathrm{d}\log Z_\psi(\mu)}{\mathrm{d}\log\mu}\,,$$
$$\beta(\alpha) = \frac{\mathrm{d}\log\alpha(\mu)}{\mathrm{d}\log\mu} \qquad (\alpha = \frac{e^2}{4\pi})$$

Reduced $QED_{4,3}$: renormalizable & scale-invariant QFT
(naive power counting: $[e] = \varepsilon_\gamma, \ \forall \varepsilon_e$)

Photon self-energy free of UV divergences: no charge renormalization



finite : $\quad Z_\alpha = Z_A^{-1} = 1, \quad \beta(\alpha) = 0 \quad$ (counterterms only local)

Fermion self-energy is UV singular: wave-function renormalization

divergent : $Z_\psi = Z_\Gamma^{-1} = 1 - \frac{3a-1}{3}\frac{\alpha}{4\pi\varepsilon_\gamma} + \mathrm{O}(\alpha^2)$

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Kotikov and ST, PRD **89**065038 (2014)

$$\gamma_\psi = 2\,\frac{3a-1}{3}\frac{\alpha}{4\pi} + 16\left(\zeta_2 N_F + \frac{4}{27}\right)\left(\frac{\alpha}{4\pi}\right)^2 + \mathrm{O}(\alpha^3)$$

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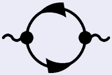

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Curious QFT (Tomonaga-Luttinger like):
finite (photon self-energy) 1PI graphs with divergent subraphgs

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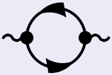

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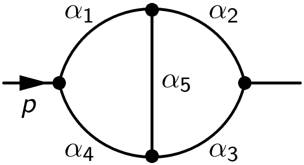
Massless propagator type 2-loop diagram

Basic building block of multi-loop calculations:

$$J(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = \int\int \frac{\mathrm{d}^D k_1\, \mathrm{d}^D k_2}{k_1^{2\alpha_1}\, k_2^{2\alpha_2}\,(k_2-p)^{2\alpha_3}\,(k_1-p)^{2\alpha_4}\,(k_2-k_1)^{2\alpha_5}}$$

Arbitrary indices α_i and external momentum p in Euclidean space (D)



Coefficient function (dimensionless):

$$I(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = \frac{(p^2)^{\sum_{i=1}^5 \alpha_i - D}}{\pi^D}\, J(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$$

Goal of multi-loop computation:

in $D = n - 2\varepsilon$ ($n \in \mathbb{N}$), compute $I(\{\alpha_i\})$ as a Laurent series in $\varepsilon \to 0$

Long history of the massless 2-loop diagram (basic building block):

for a review, see: Grozin, Int. J. Mod. Phys. **A27** 1230018 (2012)

- all indices integers: well-known and easy to compute, *e.g.* IBP

  Vasil'ev, Pismak and Khonkonen, TMF **47** 291 (1981)

  Tkachov, Phys. Lett. **B 100** 65 (1981)

  Chetyrkin and Tkachov, Nucl. Phys. **B 192** 159 (1981)

- all indices arbitrary: highly non-trivial (combination of 2-fold series)

  Bierenbaum and Weinzierl, Eur. Phys. J. **C 32** 67 (2003)

- particular cases: simpler forms can be reached, see, for example

  Vasil'ev, Pismak and Khonkonen, TMF **47** 291 (1981)

  Kazakov, TMF **62** 127 (1985)

  Gracey, Phys. Lett. **B 277** 249 (1992)

  Kivel, Stepenenko and Vasil'ev, Nucl. Phys. **B 424** 619 (1994)

  Vasiliev, Derkachov, Kivel, and Stepanenko, TMF **94** 179 (1993)

  Kotikov, Phys. Lett. **B 375** 240 (1996)

  Broadhurst, Gracey and Kreimer, Z. Phys. **C 75** 559 (1997)

  Broadhurst and Kotikov, Phys. Lett. **B 441** 345 (1998)

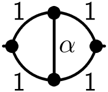
Within reduced QED

Optical conductivity (general case): $\alpha = 1 - \varepsilon_e = \lambda + \varepsilon_\gamma$

ST, PRD **86** 025005 (2012)

Kotikov, Phys. Lett. **B 375** 240 (1996)

$$I(1,1,1,1,\alpha) = C_D \left[\right.$$



$$\left. \right] = -\frac{2}{\pi^{d_e}} \Gamma(\lambda)\Gamma(\lambda-\alpha)\Gamma(1-2\lambda+\alpha) \times$$

$$\left[\frac{\Gamma(\lambda)}{\Gamma(2\lambda)\Gamma(3\lambda-\alpha-1)} \sum_{n=0}^{\infty} \frac{\Gamma(n+2\lambda)\Gamma(n+1)}{n!\,\Gamma(n+1+\alpha)} \frac{1}{n+1-\lambda+\alpha} + \frac{\pi \cot \pi(2\lambda-\alpha)}{\Gamma(2\lambda)} \right]$$

Optical conductivity (particular case): $\varepsilon_\gamma = 0$ and $\lambda = \frac{d_e}{2} - 1 \to 1/2$

Kotikov and ST, PRD **87** 087701 (2013)

Vasil'ev, Pismak and Khonkonen, TMF **47** 291 (1981)

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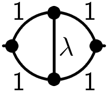


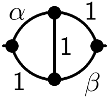
$$\left. \right] = 3\, \frac{\Gamma(\lambda)\Gamma(1-\lambda)}{\Gamma(2\lambda)} \left[\psi'(\lambda) - \psi'(1) \right]$$

(method of uniqueness in momentum space: Kotikov and ST, PRD **87** 087701 (2013))

For fermion self-energy (general case): $\tilde{\alpha} = \frac{D}{2} - \alpha$, $\lambda = D/2 - 1$

Kotikov, Phys. Lett. **B 375** 240 (1996)

Kotikov and ST, PRD **89**065038 (2014)

$I(\alpha, 1, \beta, 1, 1) = C_D$ []

$$= \frac{1}{\pi^D} \frac{1}{\tilde{\alpha} - 1} \frac{1}{1 - \tilde{\beta}} \times$$

$$\times \quad \frac{\Gamma(\tilde{\alpha})\Gamma(\tilde{\beta})\Gamma(3 - \tilde{\alpha} - \tilde{\beta})}{\Gamma(\alpha)\Gamma(\lambda - 2 + \tilde{\alpha} + \tilde{\beta})} \frac{\Gamma(\lambda)}{\Gamma(2\lambda)} I(\tilde{\alpha}, \tilde{\beta})$$

$$I(\tilde{\alpha}, \tilde{\beta}) = \frac{\Gamma(1 + \lambda - \tilde{\alpha})}{\Gamma(3 - \tilde{\alpha} - \tilde{\beta})} \frac{\pi \sin[\pi\tilde{\alpha}]}{\sin[\pi(\lambda - 1 + \tilde{\beta})] \sin[\pi(\tilde{\alpha} + \tilde{\beta} + \lambda - 1)]}$$

$$+ \sum_{n=0}^{\infty} \frac{\Gamma(n + 2\lambda)}{n!} \left(\frac{1}{n + \lambda + \tilde{\alpha} - 1} \frac{\Gamma(n + 1)}{\Gamma(n + 2 + \lambda - \tilde{\beta})} + \frac{1}{n + \lambda + 1 - \tilde{\alpha}} \times \right.$$

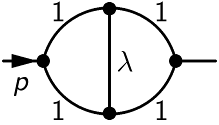
$$\left. \times \frac{\Gamma(n + 2 - \tilde{\alpha})\Gamma(2 - \tilde{\beta})\Gamma(\lambda)}{\Gamma(n + 3 + \lambda - \tilde{\alpha} - \tilde{\beta})\Gamma(3 - \tilde{\alpha} - \tilde{\beta})\Gamma(\lambda + \tilde{\alpha} - 1)} \frac{\sin[\pi(\tilde{\beta} + \lambda - 1)]}{\sin[\pi(\tilde{\alpha} + \tilde{\beta} + \lambda - 1)]} \right)$$

Application: reduced $QED_{3,2}$ ST, PRD **89** 067702 (2014)

Example of a computation

Consider the simplest but important case of

$$J(1,1,1,1,\lambda) \; = $$



$$= \frac{\pi^D}{p^{2(2-\lambda)}}\, I(\lambda), \quad \lambda = \frac{D}{2} - 1$$

Vasil'ev, Pismak and Khonkonen, TMF **47** 291 (1981)

Kivel, Stepenenko and Vasil'ev, Nucl. Phys. **B 424** 619 (1994)

Vasiliev, Derkachov, Kivel, and Stepanenko, TMF **94** 179 (1993)

Within reduced QED:

- simpler derivation via the method of uniqueness in momentum space
- application to an odd-dimensional QFT (reduced QED $D = 3 - 2\varepsilon$)
- interaction correction to the conductivity at the IR fixed point

The Method of Uniqueness

Also known as the star-triangle or Yang-Baxter relation

Origins:

- first appeared in theories with conformal symmetry

  Polyakov, JETP Lett. **12** 381 (1970)

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Idea of the method (algebraic, no explicit integration):

compute complicated Feynman diagrams
with the help of a sequence of simple transformations

(finding such sequence is generally highly non trivial)

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Some Transformations (in momentum space)

- Plain line with an arbitrary index α:

$$\overset{\alpha}{\overline{\hspace{2em}}} \iff \frac{1}{k^{2\alpha}}$$

- Chains reduce to the product of propagators:

$$\overset{\alpha \quad \beta}{\overline{\hspace{1em}\bullet\hspace{1em}}} = \overset{\alpha+\beta}{\overline{\hspace{2em}}}$$

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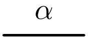

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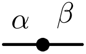

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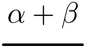
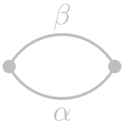


- Simple loops involve an integration:



$$= \pi^{D/2} G(\alpha, \beta)$$

$$G(\alpha, \beta) = \frac{a(\alpha)a(\beta)}{a(\alpha+\beta-D/2)}, \quad a(\alpha) = \frac{\Gamma(D/2-\alpha)}{\Gamma(\alpha)}$$

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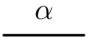

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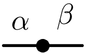

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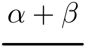

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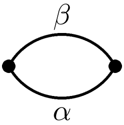

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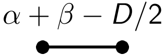
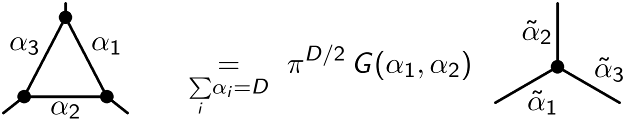


$$G(\alpha, \beta) = \frac{a(\alpha)a(\beta)}{a(\alpha+\beta-D/2)}, \quad a(\alpha) = \frac{\Gamma(D/2-\alpha)}{\Gamma(\alpha)}$$

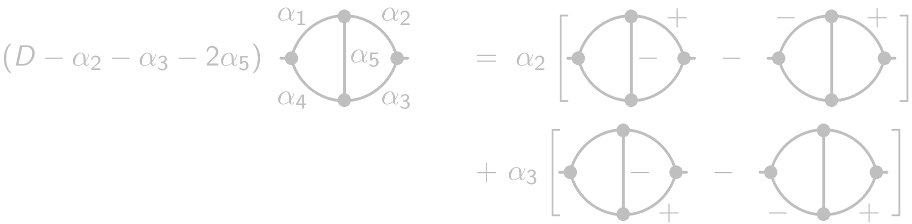
Some Transformations (in momentum space)

- Uniqueness relation ($\tilde{\alpha} = D/2 - \alpha$):



(Note: unique triangle has index $\sum_i \alpha_i = 2\lambda + 2 = D$)

- Integration by parts (IBP):



(Note: $\pm$ correponds to add or subtract 1 to index α_i)

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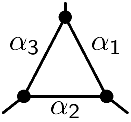
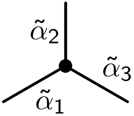


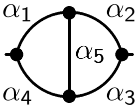
$\underset{\sum_i \alpha_i = D}{=} \quad \pi^{D/2}\, G(\alpha_1, \alpha_2)$

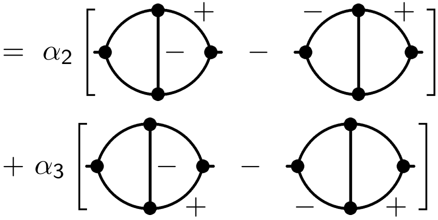


(Note: unique triangle has index $\sum_i \alpha_i = 2\lambda + 2 = D$)

- Integration by parts (IBP):

$(D - \alpha_2 - \alpha_3 - 2\alpha_5)$





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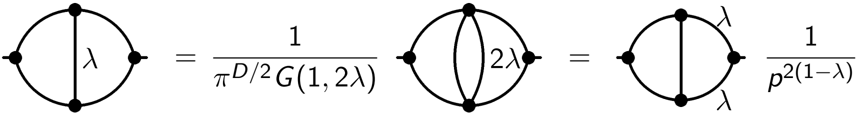
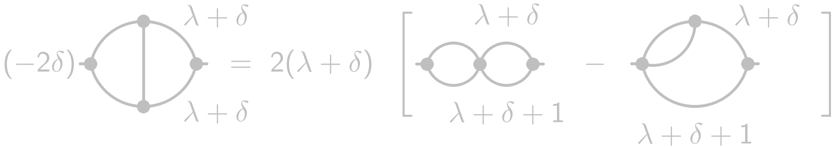
Application to $J(1,1,1,1,\lambda)$

- Replace line by loop to make right triangle unique (index $2+2\lambda = D$):



- Apply IBP to reduce the diagram to simple chains and loops:



$$= \frac{\pi^D 2(\lambda+\delta)}{p^{2(1+2\delta)}} G(1,1) \left[G(\lambda+\delta+1, \lambda+\delta) - G(\lambda+\delta+1, 1+\delta) \right]$$

(Note: $\delta \to 0$ additional regularization parameter)

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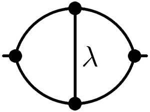
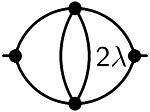
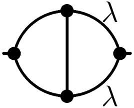
- Replace line by loop to make right triangle unique (index $2+2\lambda = D$):



$$= \frac{1}{\pi^{D/2} G(1,2\lambda)}$$



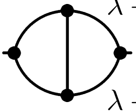
$$=$$



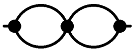
$$\frac{1}{p^{2(1-\lambda)}}$$

- Apply IBP to reduce the diagram to simple chains and loops:

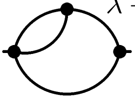
$$(-2\delta)$$



$$= 2(\lambda+\delta) \Bigg[$$



$$-$$



$$\Bigg]$$

$$= \frac{\pi^D 2(\lambda+\delta)}{p^{2(1+2\delta)}} G(1,1) \left[G(\lambda+\delta+1, \lambda+\delta) - G(\lambda+\delta+1, 1+\delta) \right]$$

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Final result

Vasil'ev, Pismak and Khonkonen, TMF **47** 291 (1981)

$$I(\lambda) = 3\,\frac{\Gamma(\lambda)\Gamma(1-\lambda)}{\Gamma(2\lambda)}\left[\psi'(\lambda) - \psi'(1)\right]$$

($\psi'(x)$ is the trigamma function)

- Even-dimensional QFT ($\lambda \to 1$ or $D \to 4$), well-known result:

$$I(1) = 6\,\zeta(3)$$

- Odd-dimensional QFT ($\lambda \to 1/2$ or $D \to 3$):

Kivel, Stepenenko and Vasil'ev, Nucl. Phys. **B 424** 619 (1994)

Vasiliev, Derkachov, Kivel, and Stepanenko, TMF **94** 179 (1993)

$$I(1/2) = 6\pi\,\zeta(2)$$

(odd-dimensional case is transcendentally more complex: $\zeta(2) = \pi^2/6$)

Interaction correction coefficient at the IR fixed point

$$\sigma(q_0) = \sigma^{(0)} \left(1 + \mathcal{C}\alpha_r + \mathrm{O}(\alpha_r^2) \right)$$

At 2-loops, using the expression of $I(1,1,1,1,\lambda)$:

$$\mathcal{C}(\lambda) = -\frac{1}{2\pi} \left(3\left[\psi'(\lambda+2) - \psi'(1) \right] + \frac{4}{1+\lambda} + \frac{1}{(1+\lambda)^2} \right)$$

In reduced $QED_{4,3}$, the interaction correction coefficient is small

ST, PRD **86** 025005 (2012)

Kotikov and ST, PRD **87** 087701 (2013)

$$\mathcal{C}^* = \mathcal{C}(1/2) = \frac{92 - 9\pi^2}{18\pi} \approx 0.056$$

At the Lorentz-invariant IR fixed point
interactions (up to 2 loops) have negligible effects on the conductivity

Outline

Graphene field theory

Feynman rules (instantaneous Coulomb interaction)

- unrenormalized free fermion propagator ($v_0 = 1$ not a "natural" unit)

$$S_0(p) = \frac{i\not{p}}{p^2}, \qquad \not{p} = \gamma^\mu p_\mu = \gamma^0 p_0 - v_0 \vec{\gamma} \cdot \vec{p},$$

- unrenormalized free photon propagator

$$V_0(\vec{q}) = \frac{\mathrm{i}}{2(|\vec{q}|^2)^{1/2}},$$

- unrenormalized free vertex: $\Gamma_0^0 = -\mathrm{i} e_0 \gamma^0$.

Immediate consequences: González, Guinea and Vozmediano (1994)

- the photon self-energy is finite (no UV singularity): $Z_e = Z_A^{-1/2} = 1$
- the one-loop fermion self-energy does not depend on frequency
  - no wave function renormalization: $Z_\psi = Z_\Gamma^{-1} = 1 + \mathrm{O}(\alpha^2)$
  - Fermi velocity renormalization: $Z_v = 1 - \frac{\alpha(\mu)}{8\varepsilon_\gamma} + \mathrm{O}(\alpha^2)$, $\quad \alpha = \frac{e^2}{4\pi\kappa v}$

One-loop fermion self-energy ($D_e = 2 - 2\varepsilon_\gamma$)

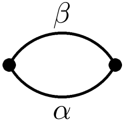
$$-i\Sigma_1(k) =$$  $$= \int [\mathrm{d}^{1+D_e} q]\, (-\mathrm{i} e_0 \gamma^0)\, S_0(k+q)\, (-\mathrm{i} e_0 \gamma^0)\, V_0(q)$$

Integrating over frequency and using the parametrization:

$$\Sigma_1(\vec{k}\,) = v_0 \vec{\gamma} \cdot \vec{k}\, \Sigma_{k1}(|\vec{k}\,|^2)\,, \quad \Sigma_{k1}(|\vec{k}\,|^2) = -\frac{\mathrm{Tr}[\vec{\gamma}\cdot\vec{k}\,\Sigma_1(\vec{k}\,)]}{4 N_F v_0 |\vec{k}\,|^2}\,,$$

yields: $$\Sigma_{k1}(|\vec{k}\,|^2) = \frac{e_0^2}{4\, v_0\, |\vec{k}\,|^2} \int [\mathrm{d}^{D_e} q]\, \frac{\vec{k}\cdot(\vec{k}+\vec{q}\,)}{|\vec{k}+\vec{q}\,|\,|\vec{q}\,|}\,.$$

Note: massless one-loop propagator-type master integral



$$= \int \frac{[\mathrm{d}^D q]}{[q^2]^\alpha\, [(q-k)^2]^\beta} = \frac{(k^2)^{D/2-\alpha-\beta}}{(4\pi)^{D/2}}\, G(\alpha,\beta)$$

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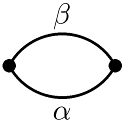
$G(\alpha,\beta)$: coefficient function of the one-loop p-type massless integral



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After ε_γ-expansion in the $\overline{\mathrm{MS}}$ scheme ($L_k = \log(|\vec{k}\,|^2/\mu^2)$):

$$\Sigma_{k1}(|\vec{k}\,|^2) = \frac{\alpha(\mu)}{8} \left(\frac{1}{\varepsilon_\gamma} - L_k + 4 \log 2 + \mathrm{O}(\varepsilon_\gamma) \right)$$

Computation of Z_v: similar to mass renormalization in QED

$$S(k) = \frac{S_0(k)}{1 + \mathrm{i}\Sigma(k)\, S_0(k)}, \quad \Sigma(k) = \gamma^0 k_0 \Sigma_\omega(k^2) + v_0 \vec{\gamma} \cdot \vec{k}\, \Sigma_k(k^2)$$

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$$S(k) = \frac{1}{1 - \Sigma_\omega} \frac{1}{\gamma^0 k_0 - v_0 \vec{\gamma} \cdot \vec{k}\, \frac{1+\Sigma_k}{1-\Sigma_\omega}} = Z_\psi(\mu)\, S_r(k;\mu), \quad v_0 = Z_v(\mu) v(\mu)$$

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Constraints (in $\overline{\mathrm{MS}}$-scheme):

$$(1 - \Sigma_\omega)\, Z_\psi = 1, \qquad \frac{1 + \Sigma_k}{1 - \Sigma_\omega}\, Z_v = 1$$

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$$(1 - \Sigma_\omega)\, Z_\psi = 1, \qquad \frac{1 + \Sigma_k}{1 - \Sigma_\omega}\, Z_v = 1$$

Our case: $\Sigma_\omega = 0 + \mathrm{O}(\alpha^2)$, so:

$$Z_\psi = Z_\Gamma^{-1} = 1 + \mathrm{O}(\alpha^2), \quad Z_v = 1 - \frac{\alpha(\mu)}{8\varepsilon_\gamma} + \mathrm{O}(\alpha^2), \qquad \alpha(\mu) = \frac{e^2(\mu)}{\kappa v(\mu)}$$

Computation of Z_v: similar to mass renormalization in QED

$$S(k) = \frac{S_0(k)}{1 + \mathrm{i}\Sigma(k)\, S_0(k)}, \quad \Sigma(k) = \gamma^0 k_0 \Sigma_\omega(k^2) + v_0 \vec{\gamma} \cdot \vec{k}\, \Sigma_k(k^2)$$

$$S(k) = \frac{1}{1 - \Sigma_\omega} \frac{1}{\gamma^0 k_0 - v_0 \vec{\gamma} \cdot \vec{k}\, \frac{1+\Sigma_k}{1-\Sigma_\omega}} = Z_\psi(\mu)\, S_r(k;\mu), \quad v_0 = Z_v(\mu) v(\mu)$$

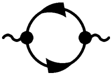
Constraints (in $\overline{\mathrm{MS}}$-scheme):

$$(1 - \Sigma_\omega)\, Z_\psi = 1, \qquad \frac{1 + \Sigma_k}{1 - \Sigma_\omega}\, Z_v = 1$$

Our case: $\Sigma_\omega = 0 + \mathrm{O}(\alpha^2)$, so:

$$Z_\psi = Z_\Gamma^{-1} = 1 + \mathrm{O}(\alpha^2), \quad Z_v = 1 - \frac{\alpha(\mu)}{8\varepsilon_\gamma} + \mathrm{O}(\alpha^2), \qquad \alpha(\mu) = \frac{e^2(\mu)}{\kappa v(\mu)}$$

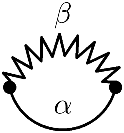
One-loop photon self-energy

$$\mathrm{i}\Pi_1^{\mu\nu}(q) =$$  $$= -\int [\mathrm{d}^{1+D_e}k]\,\mathrm{Tr}\left[(-\mathrm{i}e_0\gamma^\mu)\,S_0(k+q)\,(-\mathrm{i}e_0\gamma^\nu)\,S_0(k)\right]$$

Focusing on Π^{00} and after frequency integration $(q_0 = \mathrm{i}q_{E0})$:

$$\Pi_1^{00}(q_{E0}, \vec{q} \to 0) = \frac{N_F}{2v_0}\, e_0^2\, |\vec{q}\,|^2\, \frac{D_e - 1}{D_e} \int \frac{[\mathrm{d}^{D_e}k]}{|\vec{k}\,|\,[|\vec{k}\,|^2 + m_0^2]}\,, \quad m_0 = \frac{q_{E0}}{2v_0}$$

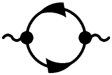
Note: master integral is of the semi-massive tadpole type

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{[k^2]^\alpha [k^2 + m^2]^\beta} =$$ 

$$= \frac{(m^2)^{D/2-\alpha-\beta}}{(4\pi)^{D/2}}\, B(\beta, \alpha)\,,$$

$$B(\beta, \alpha) = \frac{\Gamma(D/2-\alpha)\,\Gamma(\alpha+\beta-D/2)}{\Gamma(D/2)\,\Gamma(\beta)}\,.$$

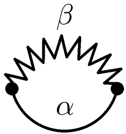
One-loop photon self-energy

$$\mathrm{i}\Pi_1^{\mu\nu}(q) = $$  $$= -\int [\mathrm{d}^{1+D_e} k]\, \mathrm{Tr}\, [(-\mathrm{i}e_0\gamma^\mu)\, S_0(k+q)\, (-\mathrm{i}e_0\gamma^\nu)\, S_0(k)]$$

Focusing on Π^{00} and after frequency integration ($q_0 = \mathrm{i}q_{E0}$) :

$$\Pi_1^{00}(q_{E0}, \vec{q} \to 0) = N_F \, \frac{|\vec{q}\,|^2}{q_{E0}} \, \frac{e_0^2 \, (m_0^2)^{-\varepsilon_\gamma}}{(4\pi)^{D_e/2}} \, \frac{D_e - 1}{D_e} \, B(1, 1/2) \, , \quad m_0 = \frac{q_{E0}}{2v_0}$$

$B(\beta, \alpha)$: coefficient function of the the semi-massive tadpole diagram

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{[k^2]^\alpha [k^2 + m^2]^\beta} = $$ 

$$= \frac{(m^2)^{D/2-\alpha-\beta}}{(4\pi)^{D/2}} \, B(\beta, \alpha) \, ,$$

$$B(\beta, \alpha) = \frac{\Gamma(D/2 - \alpha)\, \Gamma(\alpha + \beta - D/2)}{\Gamma(D/2)\, \Gamma(\beta)} \, .$$

One-loop photon self-energy

$\mathrm{i}\Pi_1^{\mu\nu}(q) =$  $= -\int [\mathrm{d}^{1+D_e} k] \,\mathrm{Tr}\left[(-\mathrm{i}e_0\gamma^\mu)\, S_0(k+q)\,(-\mathrm{i}e_0\gamma^\nu)\, S_0(k)\right]$

Focusing on Π^{00} and after frequency integration $(q_0 = \mathrm{i}q_{E0})$:

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Renormalization (simple substitution):

$$(m_0^2)^{-\varepsilon_\gamma} = (v_0)^{2\varepsilon_\gamma} = (Z_v\, v)^{2\varepsilon_\gamma} = 1 - \frac{\alpha}{4} + \mathrm{O}(\alpha^2)$$

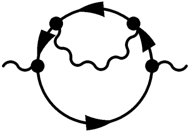
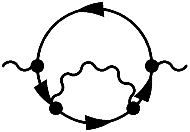
With 2-loop accuracy:

$$\Pi_1^{00}(q_0, \vec{q} \to 0) = -\frac{N_F\, e^2}{8} \, \frac{|\vec{q}\,|^2}{\mathrm{i}q_0} \, \left(1 - \frac{\alpha}{4}\right), \quad \sigma_1(q_0) = \sigma_0 \, \left(1 - \frac{\alpha}{4} + \mathrm{O}(\alpha^2)\right)$$

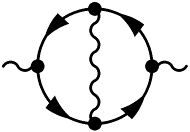
Two-loop photon self-energy

$$\Pi_2^{\mu\nu}(q) = 2\Pi_{2a}^{\mu\nu}(q) + \Pi_{2b}^{\mu\nu}(q)$$

The first contribution is primitively one-loop:

$$2\Pi_{2a}^{\mu\nu}(q) =$$  $+$ 

The second contribution is truly two-loop:

$$\Pi_{2b}^{\mu\nu}(q) =$$ 

Two-loop photon self-energy (a)

After integration over frequencies, Wick rotation and expansion in $\vec{q} \to 0$:

$$\Pi^{00}_{2a}(q_{E0}, \vec{q} \to 0) = -\frac{N_F}{32}\,|\vec{q}\,|^2\,\frac{e_0^4\,(m^2)^{D_e/2-3/2-\varepsilon_\gamma}}{v^2\,(4\pi)^{D_e}}$$

$$\times\frac{(D_e-1)\,(D_e-2-2\varepsilon_\gamma)}{D_e}\,G(1/2,1/2)\,B(1,1/2+\varepsilon_\gamma)$$

Note: the diagram is finite but has a divergent fermion self-energy subraph!

$$2\,\Pi^{00}_{2a}(q_0, \vec{q} \to 0) = -\frac{N_F\,e^2}{8}\,\frac{\alpha}{2}\,\frac{|\vec{q}\,|^2}{\mathrm{i}q_0}$$

Contribution to the conductivity:

$$\sigma_{2a}(q_0) = \sigma_0\,\frac{\alpha}{2} + \mathrm{O}(\alpha^2)$$

Agreement with JVH but not with Mishchenko ($\alpha/4$ instead of $\alpha/2$)

Two-loop photon self-energy (b)

After integration over frequencies, Wick rotation and expansion in $\vec{q} \to 0$:

$$\Pi_{2b}^{00}(m, \vec{q} \to 0\,) = \frac{N_F\, e_0^4}{8\, v^2}\, \frac{|\vec{q}\,|^2}{D_e} \times \Big\{ (D_e - 1)\, I_1(1/2) \\ -m^2\, I_2(3/2) - m^2 (D_e - 2)\, I_0(1/2) \Big\}$$

where $I_n(\alpha)$ are semi-massive 2-loop tadpole master integrals

$$\begin{aligned} I_n(\alpha) &= \int [\mathrm{d}^{D_e} k_1][\mathrm{d}^{D_e} k_2]\, \frac{(\vec{k}_1 \cdot \vec{k}_2\,)^n [|\vec{k}_1 - \vec{k}_2\,|^2]^{-1/2}}{[|\vec{k}_1\,|^2]^\alpha\, [|\vec{k}_1\,|^2 + m^2]\, [|\vec{k}_2\,|^2]^\alpha\, [|\vec{k}_2\,|^2 + m^2]} \\ &= \frac{(m^2)^{D_e + n - 2\alpha - 5/2}}{(4\pi)^{D_e}}\, \tilde{I}_n(\alpha)\,. \end{aligned}$$

Note: in $\Pi_{2b}^{00}(q)$, $I_0(1/2)$, $I_1(1/2)$ and $I_2(3/2)$ are UV finite.

To compute the master integrals, use a combination of transformations:

- simple identities to related diagrams with different α values:

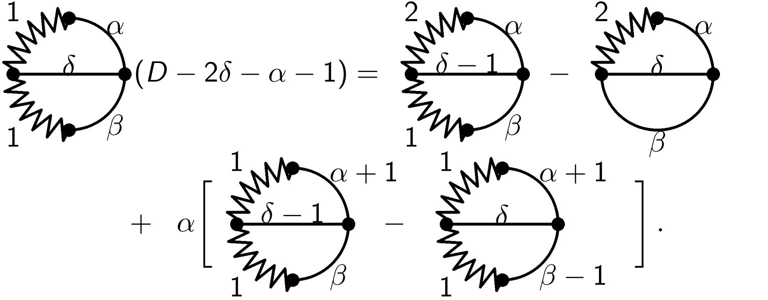
$$\frac{1}{k^{2\alpha}\,(k^2+m^2)} = \frac{1}{m^2}\,\left(\frac{1}{k^{2\alpha}} - \frac{1}{k^{2(\alpha-1)}\,(k^2+m^2)}\right)$$

- Mellin-Barnes transformation: Boos and Davydychev, TMP **89**, 1052 (1991)

$$\frac{1}{k^2+m^2} = \frac{1}{2\mathrm{i}\pi}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}s\,\Gamma(-s)\Gamma(1+s)\frac{(m^2)^s}{(k^2)^{1+s}}$$

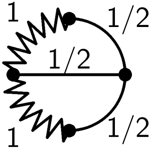
- integration by parts for a 2-loop diagram with massive lines

Kotikov, Mod. Phys. Lett. A, **06** 677 (1991)



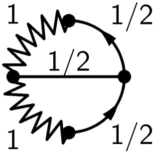
Two-loop photon self-energy (b)

$I_0(1/2) =$



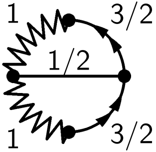
$$= \frac{(m^2)^{D_e-7/2}}{(4\pi)^{D_e}}\,\pi^2$$

$I_1(1/2) =$



$$= \frac{(m^2)^{D_e-5/2}}{(4\pi)^{D_e}}\,\pi\,(4-\pi)\,,$$

$I_2(3/2) =$



$$= \frac{(m^2)^{D_e-7/2}}{(4\pi)^{D_e}}\,\pi\,\left(\pi-\frac{4}{3}\right)$$

Two-loop photon self-energy (b)

$$\Pi_{2b}^{00}(m, \vec{q} \to 0\,) = \frac{N_F\, e_0^4}{8\, v^2}\, \frac{|\vec{q}\,|^2}{D_e}\, \times \Big\{ (D_e - 1)\, I_1(1/2)$$
$$- m^2\, I_2(3/2) - m^2 (D_e - 2)\, I_0(1/2) \Big\}$$

Then:

$$\Pi_{2b}^{00}(q_0, \vec{q} \to 0) = -\frac{N_F\, e^2}{8}\, \alpha\, \frac{8 - 3\pi}{6}\, \frac{|\vec{q}\,|^2}{\mathrm{i} q_0}\, .$$

Contribution to the conductivity:

$$\sigma_{2b}(q_0) = \sigma_0\, \frac{8 - 3\pi}{6}\, \alpha + \mathrm{O}(\alpha^2)$$

Agreement with JVH and Mishchenko.

Optical conductivity up to 2 loops

- One-loop contribution computed with 2-loop accuracy:

$$\sigma_1(q_0) = \sigma_0 + \sigma_{2a'}(q_0), \qquad \sigma_{2a'}(q_0) = -\sigma_0 \frac{\alpha}{4}$$

- contribution of the fermion self-energy correction

$$\sigma_{2a}(q_0) = \sigma_0 \frac{\alpha}{2}$$

- contribution of the vertex correction

$$\sigma_{2b}(q_0) = \sigma_0 \frac{8-3\pi}{6} \alpha$$

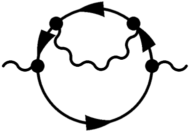
Total conductivity up to 2-loops

$$\sigma(q_0) = \sigma_0(q_0) + \sigma_{2a}(q_0) + \sigma_{2a'}(q_0) + \sigma_{2b}(q_0) = \sigma_0 \left(1 + \mathcal{C}^{(2)} \alpha + \mathrm{O}(\alpha^2)\right)$$

We recover Mishchenko's result (2008): $\mathcal{C}^{(2)} = (19 - 6\pi)/12 \approx 0.013$

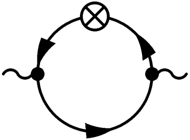
Method of Counterterms

Finite diagram with a divergent fermion self-energy subgraph:

$$\Pi^{00}_{2a}(q_{E0}, \vec{q} \to 0) =$$  $$= -\frac{N_F}{32}\, |\vec{q}\,|^2\, \frac{e_0^4\,(m^2)^{D_e/2-3/2-\varepsilon_\gamma}}{v^2\,(4\pi)^{D_e}}$$

$$\times \frac{(D_e-1)\,(D_e-2-2\varepsilon_\gamma)}{D_e}\, G(1/2,1/2)\, B(1, 1/2+\varepsilon_\gamma)$$

Add the corresponding (local) counter-term:

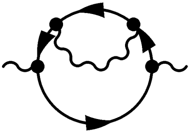
$$\Pi^{00}_{2a'}(q_{E0}, \vec{q} \to 0) =$$  $$= \frac{N_F\, e^2}{32}\, |\vec{q}\,|^2\, \frac{e_0^2\,(m^2)^{D_e/2-3/2}}{v^2\,(4\pi)^{D_e/2}}$$

$$\times \frac{(D_e-1)\,(D_e-2)}{4\pi\varepsilon_\gamma\, D_e}\, B(1, 1/2)$$

Method of Counterterms

Finite diagram with a divergent fermion self-energy subgraph:

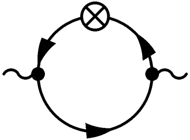
$$2\Pi^{00}_{2a}(q_{E0}, \vec{q} \to 0) = 2$$



$$= -\frac{N_F\, e^2}{8}\, \frac{\alpha}{2}\, \frac{|\vec{q}\,|^2}{\mathrm{i} q_0}$$

Add the corresponding (local) counter-term:

$$2\Pi^{00}_{2a'}(q_{E0}, \vec{q} \to 0) = 2$$



$$= -\frac{N_F\, e^2}{8}\, \left(-\frac{\alpha}{4}\right)\, \frac{|\vec{q}\,|^2}{\mathrm{i} q_0}$$

Hence, in agreement with the simple substitution, we recover:

$$\sigma_{2a'}(q_0) = -\sigma_0\, \alpha/4$$

(besides subtracting the subdivergence
the counterterm graph has a finite contribution to the final result)

Summary of density-density correlation function approach

Total conductivity up to 2-loops

$$\sigma(q_0) = - \lim_{\vec{q} \to 0} \frac{\mathrm{i} q_0}{|\vec{q}|^2} \Pi^{00}(q_0, \vec{q}\,), \qquad q^\mu = (q^0, v_0 \vec{q})$$

$$\Pi^{00}(q) = \langle T \rho(q) \rho(-q) \rangle, \qquad \rho(q) = e_0 \bar{\Psi} \gamma^0 \Psi$$

$$\sigma(q_0) = \sigma_0 \left(1 + \mathcal{C}^{(2)} \alpha + \mathrm{O}(\alpha^2) \right)$$

We recover Mishchenko's result (2008): $\mathcal{C}^{(2)} = (19 - 6\pi)/12 \approx 0.013$

Crucial distinction between regularization and renormalization

- Dimensional regularization works as well as the hard cut-off approach.
- Renormalization of the Fermi velocity ($v_0 = 1$ not a "natural" unit)

Clarifies the origin of (half of) the controversy

ST and Kotikov, EPL **107** 57001 (2014)

A word on the Kubo formula approach

$$\tilde{\sigma}(q_0) = \frac{1}{2\mathrm{i}q_0}\left(K^{11}(q_0, \vec{q} \to 0) + K^{22}(q_0, \vec{q} \to 0)\right)$$

$$K^{ij}(q) = \langle T j^i(q) j^j(-q) \rangle, \qquad \vec{j}(q) = e_0 v_0 \bar{\Psi}\vec{\gamma}\Psi$$

To better exploit the $O(2)$ symmetry, attempt parametrization (as $\vec{q} \to 0$):

$$\Pi^{\mu\nu}(q) = (g^{\mu\nu}q^2 - q^\mu q^\nu)\,\Pi(q^2), \quad \Pi(q^2) = \frac{-\Pi^{\mu}{}_{\mu}(q)}{D_e(-q^2)}$$

(encodes transversality $q_\mu \Pi^{\mu\nu}(q) = 0$ or current conservation)

Kubo formula

$$\tilde{\sigma}(q_0) = \mathrm{i}q_0\, K(q_0), \qquad K(q_0) = v_0^2\,\Pi(q_0^2, \vec{q} \to 0)$$

According to Mishchenko, there is a “Coulomb anomaly” (2008):

- with a hard cut-off: $\tilde{\sigma}(q_0) = \sigma_0\left(1 + \mathcal{C}^{(1)}\alpha\right) \neq \sigma(q_0)$
- soft cut-off must be used: $\tilde{\sigma}(q_0) = \sigma(q_0) = \sigma_0\left(1 + \mathcal{C}^{(2)}\alpha\right)$

A word on the Kubo formula approach

$$\tilde{\sigma}(q_0) = \frac{1}{2\mathrm{i}q_0}\,\left(K^{11}(q_0, \vec{q} \to 0) + K^{22}(q_0, \vec{q} \to 0)\right)$$

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- soft cut-off must be used: $\tilde{\sigma}(q_0) = \sigma(q_0) = \sigma_0\,\left(1 + \mathcal{C}^{(2)}\,\alpha\right)$

One-loop case (finite)

$$K_1(q_{E0}) = \frac{N_F}{2 v_0 m_0} \, \frac{e_0^2 \, (m_0^2)^{-\varepsilon_\gamma}}{(4\pi)^{D_e/2}} \, \frac{D_e - 1}{D_e} \, B(1, -1/2)$$

$B(\beta, \alpha)$: coefficient function of the the semi-massive tadpole diagram

Expressing all bare parameters in terms of renormalized ones and performing the ε_γ-expansion yields, with two-loop accuracy:

$$K_1(q_0) = \frac{N_F \, e^2}{8 \, \mathrm{i} q_0} \left(1 - \frac{\alpha}{4} \right)$$

Contribution to the conductivity with two-loop accuracy:

$$\tilde{\sigma}_1(q_0) = \sigma_0 \left(1 - \frac{\alpha}{4} + \mathrm{O}(\alpha^2)\right) = \sigma_1(q_0)$$

$$\text{Notation:} \quad \tilde{\sigma}_{2a'}(q_0) = \sigma_{2a'}(q_0) = -\frac{\alpha}{4} \, \sigma_0$$

Two-loop case (diagrams are individually divergent)

$$K_2(q_0) = 2K_{2a}(q_0) + K_{2b}(q_0)$$

Local singularities (simple poles):

$$2K_{2a}(q_0) = \frac{N_F\, e^2}{8\mathrm{i}q_0}\,\frac{\alpha}{4}\left(-\frac{1}{\varepsilon_\gamma} + 2L_q + 3 - 4\log 2 + \mathrm{O}(\varepsilon_\gamma)\right)$$

$$K_{2b}(q_0) = -2K_{2a}(q_0) + \frac{N_F\, e^2}{8\,\mathrm{i}q_0}\,\alpha\,\frac{11-3\pi}{6}$$

Using the simple substitution:

$$\frac{\tilde{\sigma}_2}{\sigma_0} = \frac{\tilde{\sigma}_{2a} + \tilde{\sigma}_{2b} + \tilde{\sigma}_{2a'}}{\sigma_0} = \mathcal{C}^{(2)}\,\alpha = \frac{\sigma_2}{\sigma_0}\,.$$

$$\tilde{\sigma}_{2a'} = \sigma_{2a'} = -\frac{\alpha}{4}\,\sigma_0, \quad \tilde{\sigma}_{2a} + \tilde{\sigma}_{2b} = \sigma_0\,\frac{11-3\pi}{6}\,\alpha\,.$$

All approaches yield the same result: $\mathcal{C}^{(2)} = (19-6\pi)/12 \approx 0.013$.

Outline

Conclusion

Rich interplay between:

- condensed matter physics motivations
- high-energy physics algebraic multi-loop techniques

Interaction correction to the optical conductivity

- in the non-relativistic limit ($v/c \to 0$):

$$\mathcal{C}^{(2)} = \frac{19 - 6\pi}{12} = \frac{19}{12} - \frac{\pi}{2} \approx 0.013$$

  - consistent with present experimental results ($\alpha \approx 2.2$: $\alpha\mathcal{C}^{(2)} \approx 2.9\%$)

- in the ultra-relativistic limit ($v/c \to 1$, stable IR fixed point):

$$\mathcal{C}^{*} = \frac{92 - 9\pi^2}{18\pi} = \frac{46}{9\pi} - \frac{\pi}{2} \approx 0.056$$

  - same order of magnitude as in the non-relativistic limit
  - same structure as in the non-relativistic limit
  - universality (quantitative)? future: case of arbitrary v/c

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Spasibo Bolchoi!