Divergences in Maximal SYM Theories in Diverse Dimensions

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JHEP 1404 (2014) 121, arXiv:1402.1024 [hep-th]
Phys.Lett. B 734 (2014) 111, arXiv:1404.6998 [hep-th]
JHEP (2015), arXiv:1508.05570 [hep-th]

Motivation

Maximal SYM

D=4 N=4
D=6 N=2
D=8 N=1
D=10 N=1

- **Partial or total cancellation of UV divergences (all bubble and triangle diagrams cancel)**
- **First UV divergent diagrams at D=4+6/L**
- **Conformal or dual conformal symmetry**
- **Common structure of the integrands**

Bern, Dixon &Co 10
Drummond, Henn, Korchemsky, Sokatchev 10
Arkani-Hammed 12

Object: Helicity Amplitudes on mass shell with arbitrary number of legs and loops

The case: Planar limit $N_c \to \infty,\ g^2_{YM} \to 0$ and $g^2_{YM} N_c$ - fixed

The aim: to get all loop (exact) result

UV & IR Divergences

D=4 N=4

- **No UV divergences in all loops**
- **IR & Collinear Divs on shell**

BDS conjecture *Bern, Dixon, Smirnov 05*

$$\mathcal{M}_n \equiv \frac{A_n}{A_n^{tree}} = 1 + \sum_{L=1}^{\infty} \left(\frac{g^2 N_c}{16\pi^2}\right)^L M_n^{(L)}(\epsilon) = \exp\left[\sum_{l=1}^{\infty} \left(\frac{g^2 N_c}{16\pi^2}\right)^l \Big(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)\Big)\right]$$

$$\mathcal{M}_n(\epsilon) \quad = \quad \exp\left[-\frac{1}{8}\sum_{l=1}^{\infty}\left(\frac{g^2 N_c}{16\pi^2}\right)^l \left(\frac{\gamma_{cusp}^{(l)}}{(l\epsilon)^2} + \frac{2G_0^{(l)}}{l\epsilon}\right)\sum_{i=1}^{n}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{l\epsilon} + \frac{1}{4}\sum_{l=1}^{\infty}\left(\frac{g^2 N_c}{16\pi^2}\right)^l \gamma_{cusp}^{(l)} F_n^{(1)}(0) + C(g)\right]$$

IR & Collinear Divs in dimensional regularization **Cusp anom dim**

$$M_4^{(1-loop)}(\epsilon) = A_4^{(1-loop)}/A_4^{(tree)} \quad = \frac{\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}\left[\frac{1}{\epsilon^2}\left(\left(\frac{\mu^2}{s}\right)^{\epsilon} + \left(\frac{\mu^2}{-t}\right)^{\epsilon}\right) - \frac{1}{2}\log^2\left(\frac{s}{-t}\right) - \frac{\pi^2}{3}\right] + \mathcal{O}(\epsilon)$$

UV & IR Divergences

D=6 N=2

N=(1,1)

$$[g^2] \sim \frac{1}{M^2}$$

- **No IR & Collinear divergences in all loops**
- **UV Divs starting from L=6/(D-4)=3 loops**

Toy model for gravity

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D=10 N=1

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Compactification on a torus of higher dim maximal SYM theories gives lower dimensional maximal SYM theories

Colour decomposition

Colour ordered amplitude

$$\mathcal{A}_n^{a_1\ldots a_n}(p_1^{\lambda_1}\ldots p_n^{\lambda_n}) = \sum_{\sigma\in S_n/Z_n} Tr[\sigma(T^{a_1}\ldots T^{a_n})]A_n(\sigma(p_1^{\lambda_1}\ldots p_n^{\lambda_n})) + \mathcal{O}(1/N_c)$$

Planar Limit $N_c \to \infty,\ g^2_{YM} \to 0$ and $g^2_{YM}N_c$ - fixed

This is what we calculate

Four-point amplitude

$$A_4^{(1),phys.}(1,2,3,4) = T^1 A_4^{(0)}(1,2,3,4) M^{(1)}(s,t) + T^2 A_4^{(0)}(1,2,4,3) M^{(1)}(s,u) + T^3 A_4^{(0)}(1,4,2,3) M^{(1)}(t,u).$$

$T^1 = Tr(T^{a1}T^{a2}T^{a3}T^{a4}) + Tr(T^{a1}T^{a4}T^{a3}T^{a2}),$

$T^2 = Tr(T^{a1}T^{a2}T^{a4}T^{a3}) + Tr(T^{a1}T^{a3}T^{a4}T^{a2}),$

$T^3 = Tr(T^{a1}T^{a4}T^{a2}T^{a3}) + Tr(T^{a1}T^{a3}T^{a2}T^{a4})$

Tree level amplitude usually has a simple universal form proportional to the delta function (conservation of momenta), in SUSY case - conservation of supercharge in on shell momentum superspace

Spinor helicity formalism

Spinor helicity formalism in D=4 and in D=6

D=4

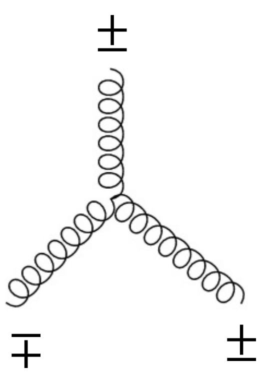
Momentum $p^\mu, p^2 = 0, \mu = 0, ..., 3$

$$p_\mu^{(i)} \to (\sigma^\mu)_{\alpha\dot{\alpha}} p_\mu^{(i)} = \lambda_\alpha^{(i)} \tilde{\lambda}_{\dot{\alpha}}^{(i)} \qquad \lambda_\alpha \in SL(2, \mathbb{C})$$

$$\epsilon^{\alpha\beta} \lambda_\alpha^{(i)} \lambda_\beta^{(j)} \equiv \langle ij \rangle = \sqrt{(p_i + p_j)^2} e^{i\phi_{ij}} = \sqrt{s_{ij}} e^{i\phi_{ij}}$$

$$\phi_{ij} \in \mathbb{R}$$

$$(\langle ij \rangle)^* \equiv [ij]$$



$$A_3(g_1^- g_1^- g_3^+) \sim \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}$$

$p_i \in \mathbb{C}$

$$A_n^{(0)}(g_1^+ ... g_k^- ... g_j^- ... g_n^+) = \frac{\langle kj \rangle^4}{\langle 12 \rangle ... \rangle \langle n1 \rangle}$$

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Cheung, O'Connell 09, Bern&Co 10

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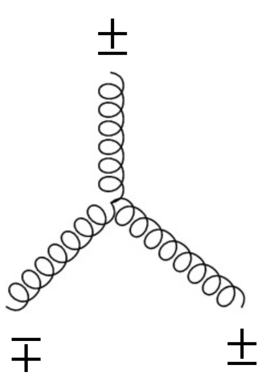

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D=6

$p^\mu, \ p^2 = 0, \ \mu = 0, ..., 5$

$$p_{AB} = p_\mu (\sigma^\mu)_{AB}, \ p^{AB} = p^\mu (\bar\sigma_\mu)^{AB}$$

$$p^{AB} = \lambda^{Aa} \lambda_a^B, \ p_{AB} = \tilde\lambda_A^{\dot a} \tilde\lambda_{B\dot a}$$

$SU^*(4) \to \lambda^{Aa}$ ← $SU(2)$; $SO(5,1)$

$SO(4) \simeq SU(2) \times SU(2)$

Helicity is no longer conserved in D=6!

$$\lambda(i)^{Aa} \tilde\lambda(j)_A^{\dot a} \doteq \langle i_a | j_{\dot a}] = [j_{\dot a} | i_a \rangle$$

$$\mathcal{A}_4^{(0)}(1_{a\dot a} 2_{b\dot b} 3_{c\dot c} 4_{d\dot d}) = -ig_{YM}^2 \frac{\langle 1_a 2_b 3_c 4_d \rangle [1_{\dot a} 2_{\dot b} 3_{\dot c} 4_{\dot d}]}{st}$$

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Cheung, O'Connell 09, Bern&Co 10

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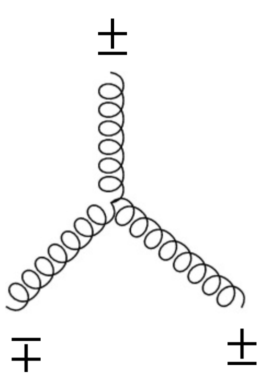

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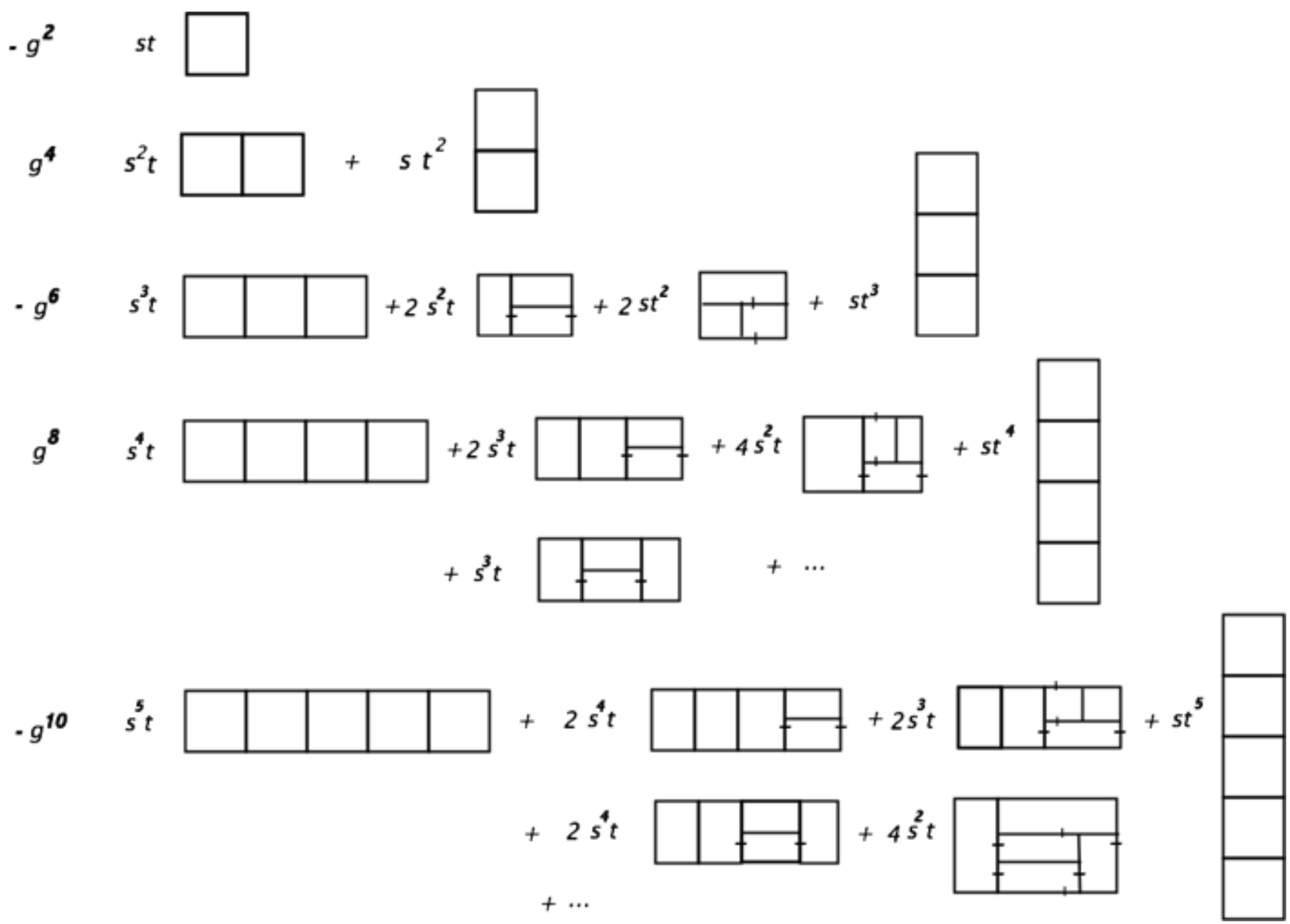
Similar but more complicated in D=8 and D=10

R.H.Boles D O'Connell 12
S.Caron-Huot D.O'Connell 10

Perturbation Expansion for the Amplitudes for any D

A_4/A_4^{tree}



1

2

4

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No bubbles
No Triangles

First UV div at
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IR finite

T. Dennen Yu-yin Huang 10 ,
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Universal expansion for any D in maximal SYM due to Dual conformal invariance

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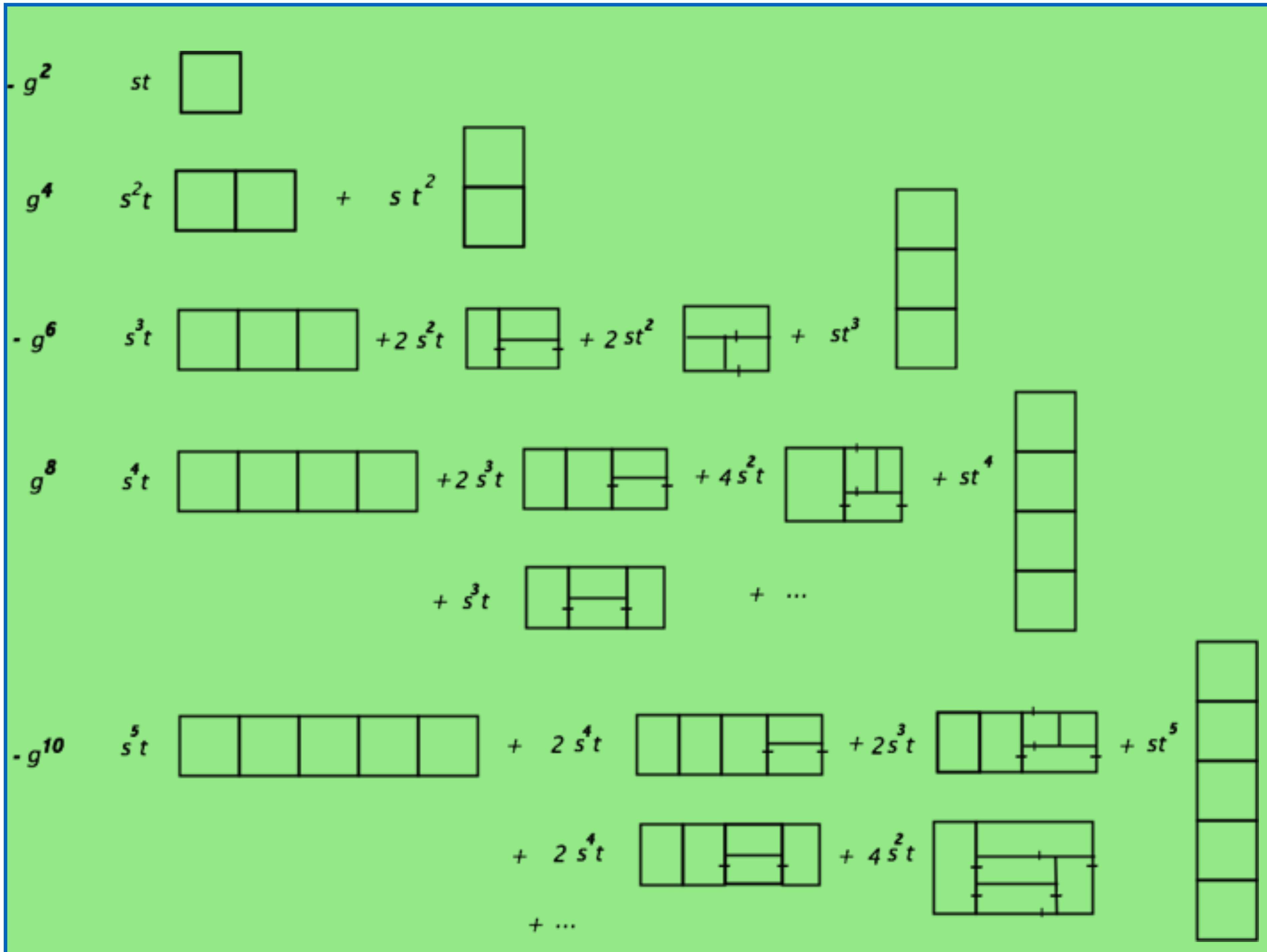

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Perturbation Expansion for the Amplitudes

Leading Divergences

The master integrals with leading divergences up to four loops

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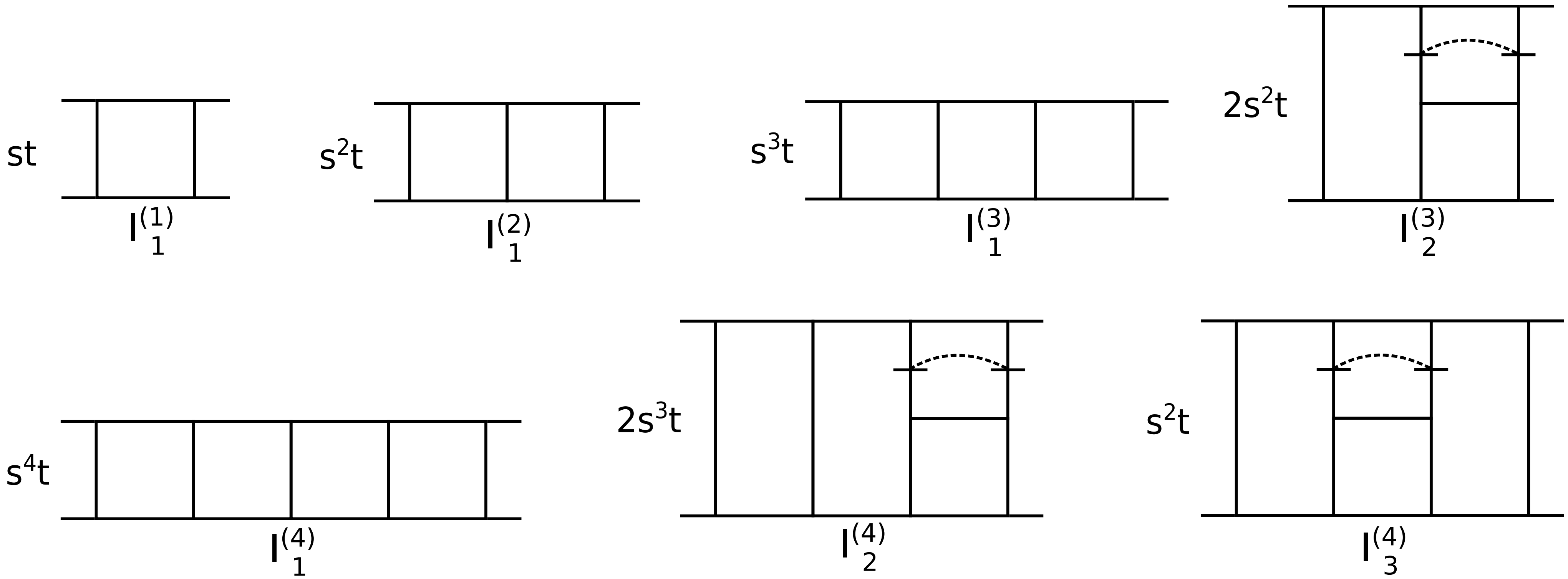

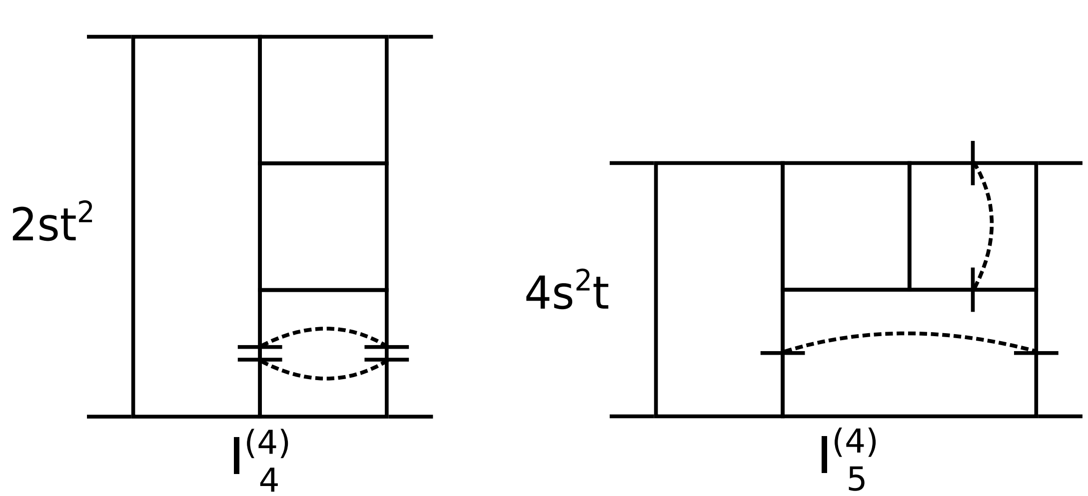

The diagrams with the substitution s<->t are not shown

Everything was checked also numerically!

Leading Divergences from Generalized «Renormalization Group»

- **In renormalizable theories the leading divergences can be found from the 1-loop term due to the renormalization group, in particular, for a single coupling theory the coefficient of** $1/\epsilon^n$ **in n loops in given by**

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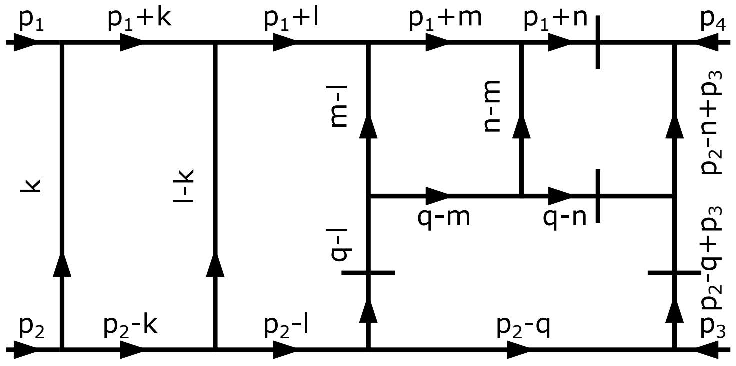
$$A_n = (-1)^{n-1}\frac{A_1}{n}$$

Leading pole **Coeff of 1 loop graph**

$\mathbf{R}'$- operation and Leading Divergences



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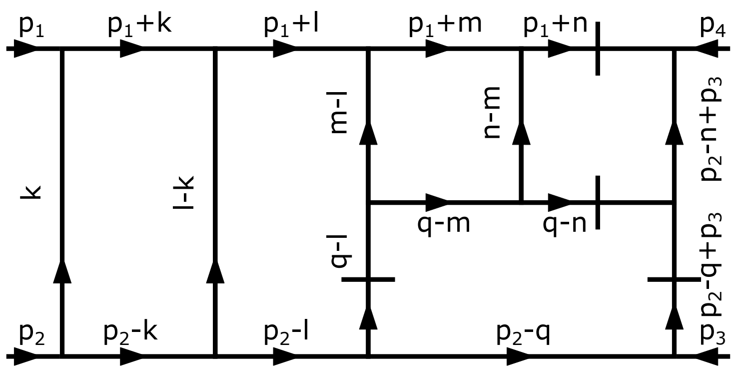

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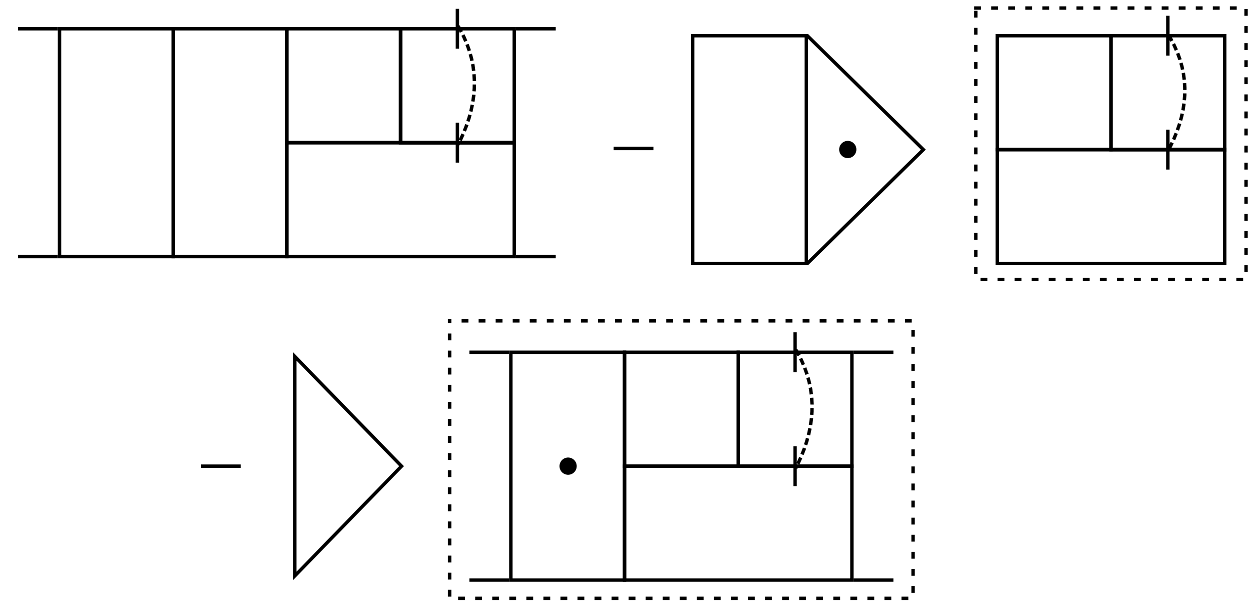

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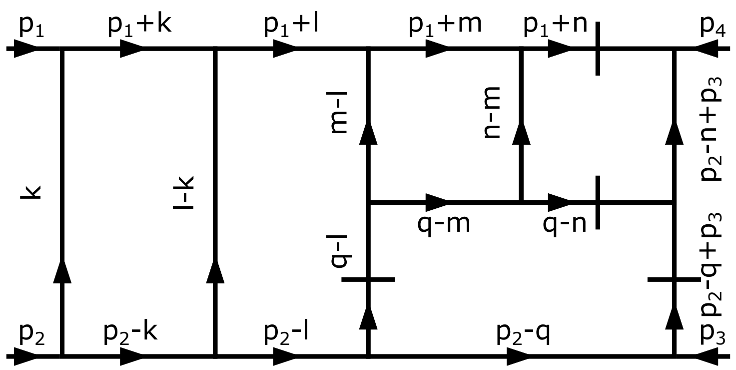

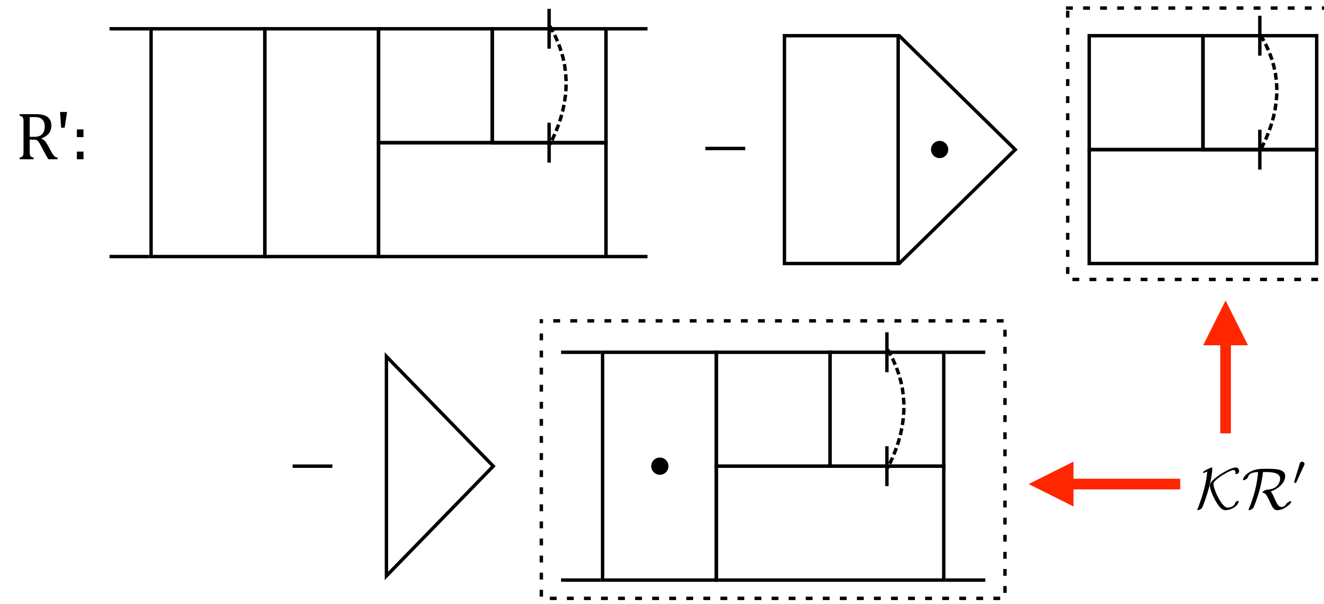

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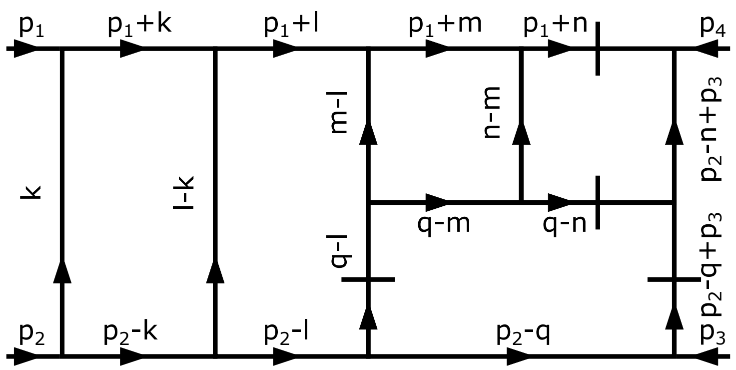

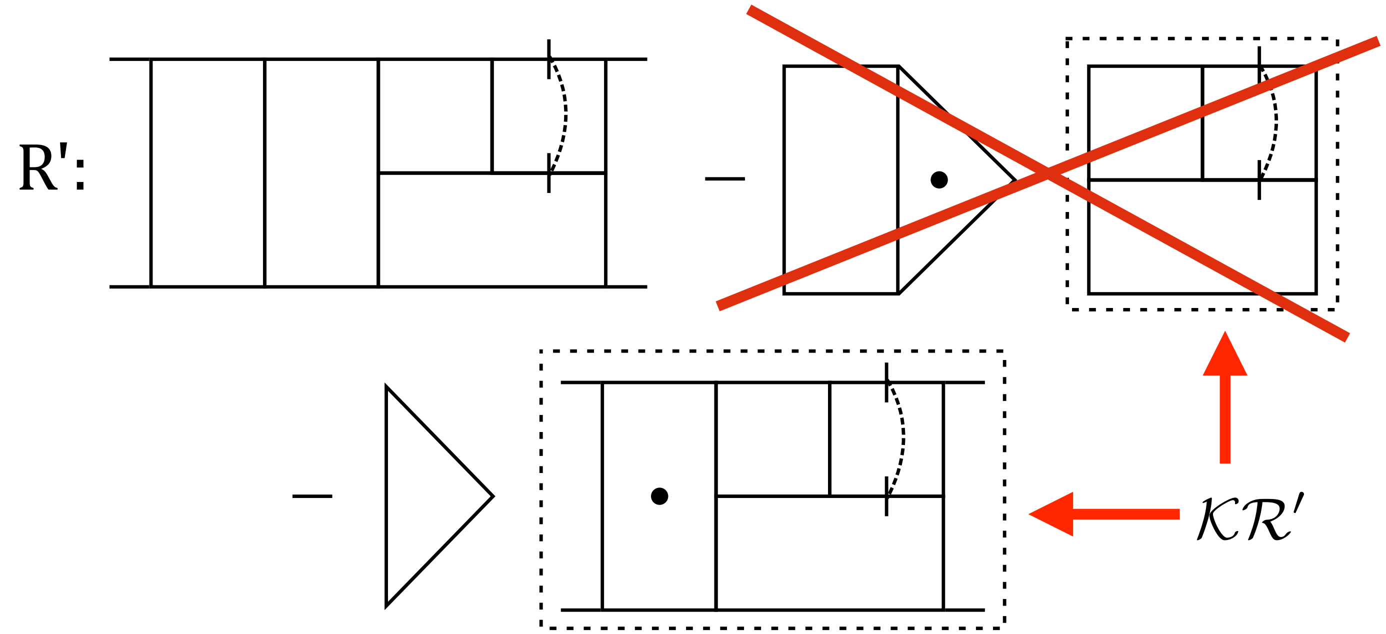

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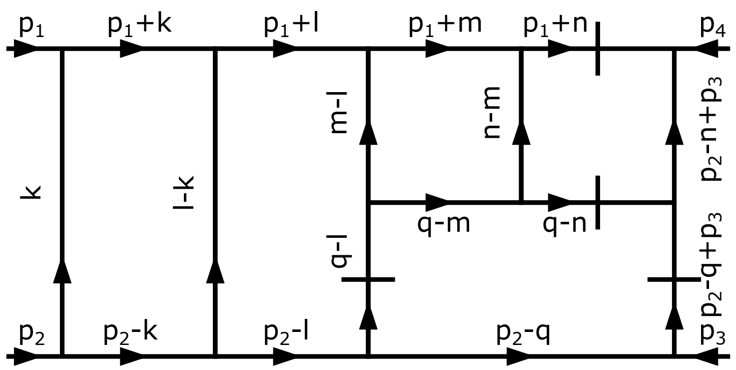

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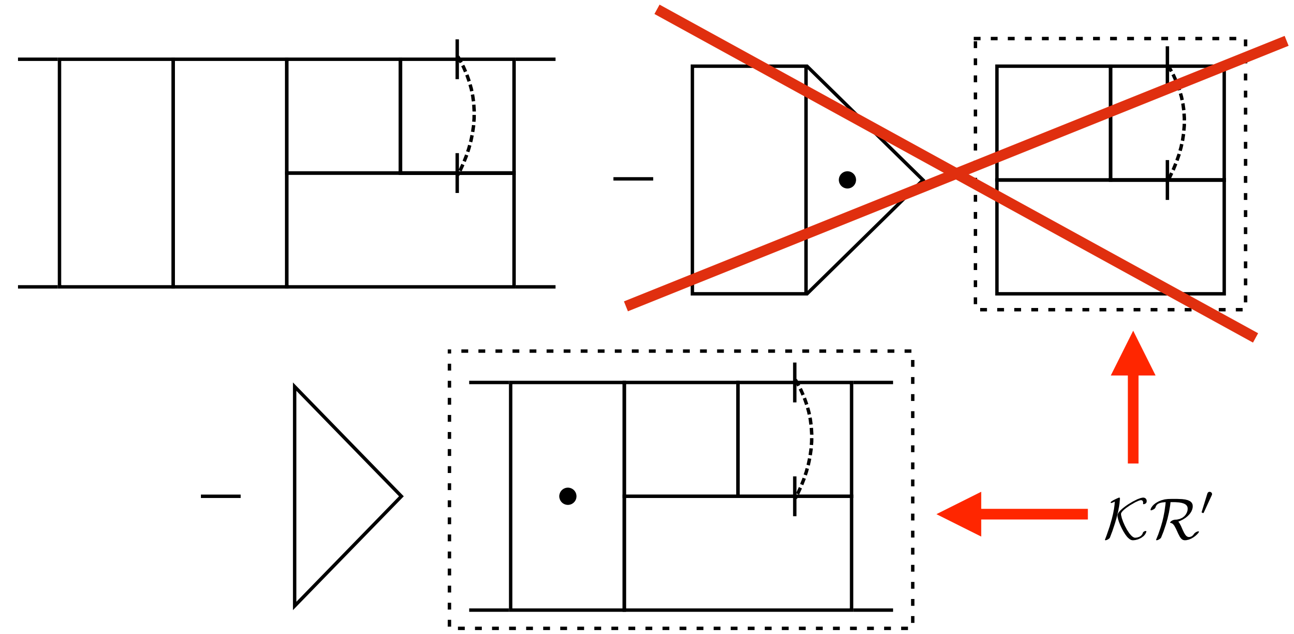

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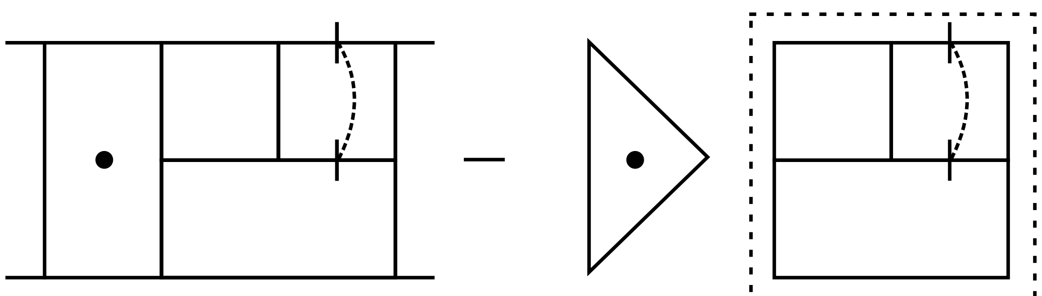

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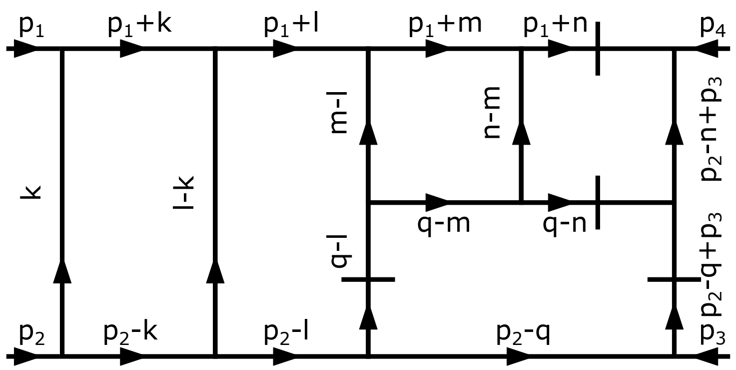

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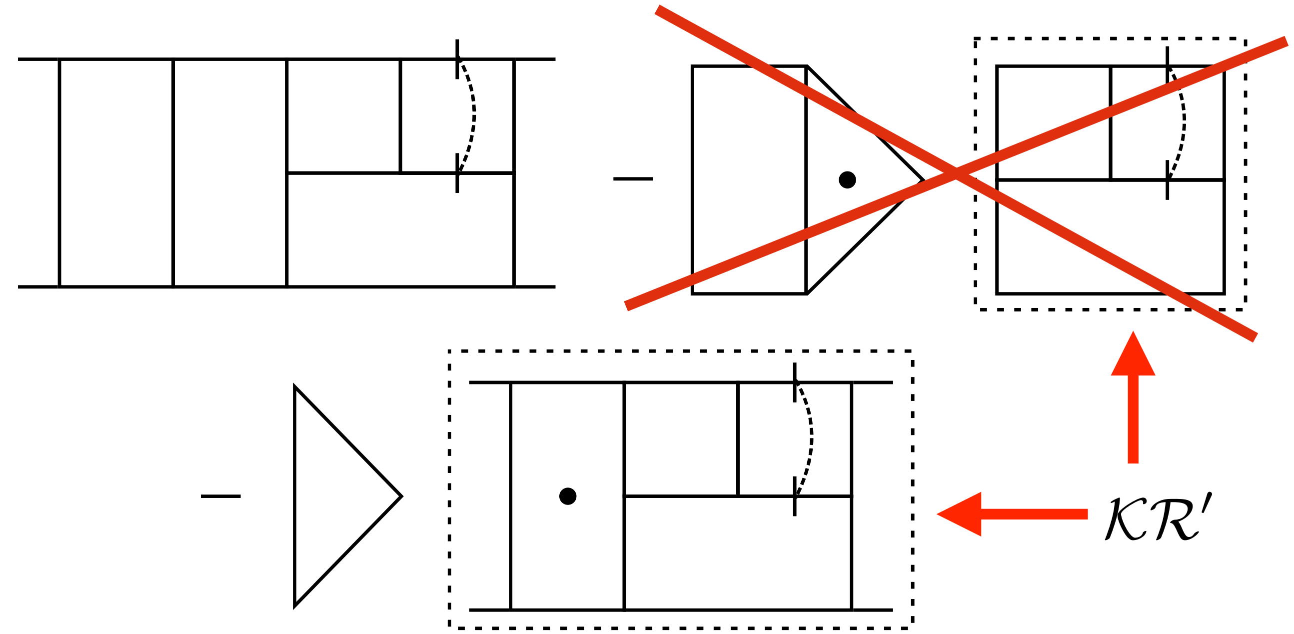

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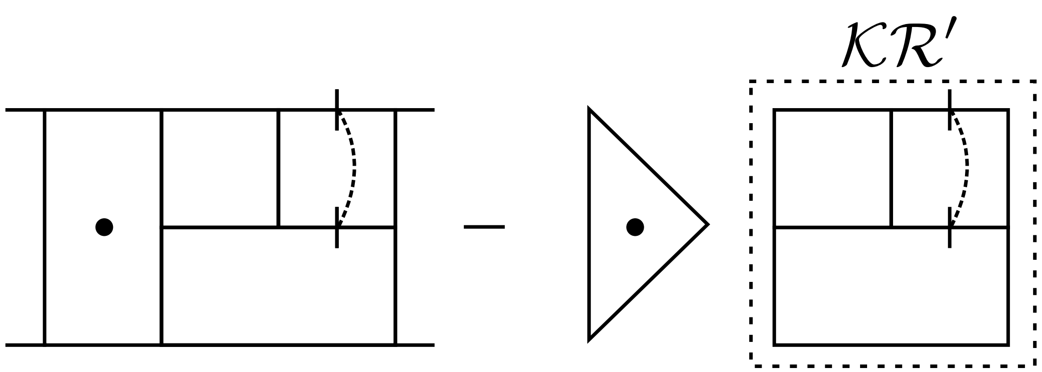

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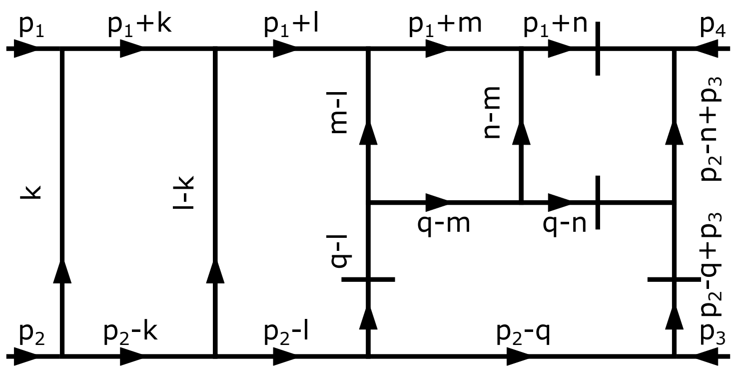

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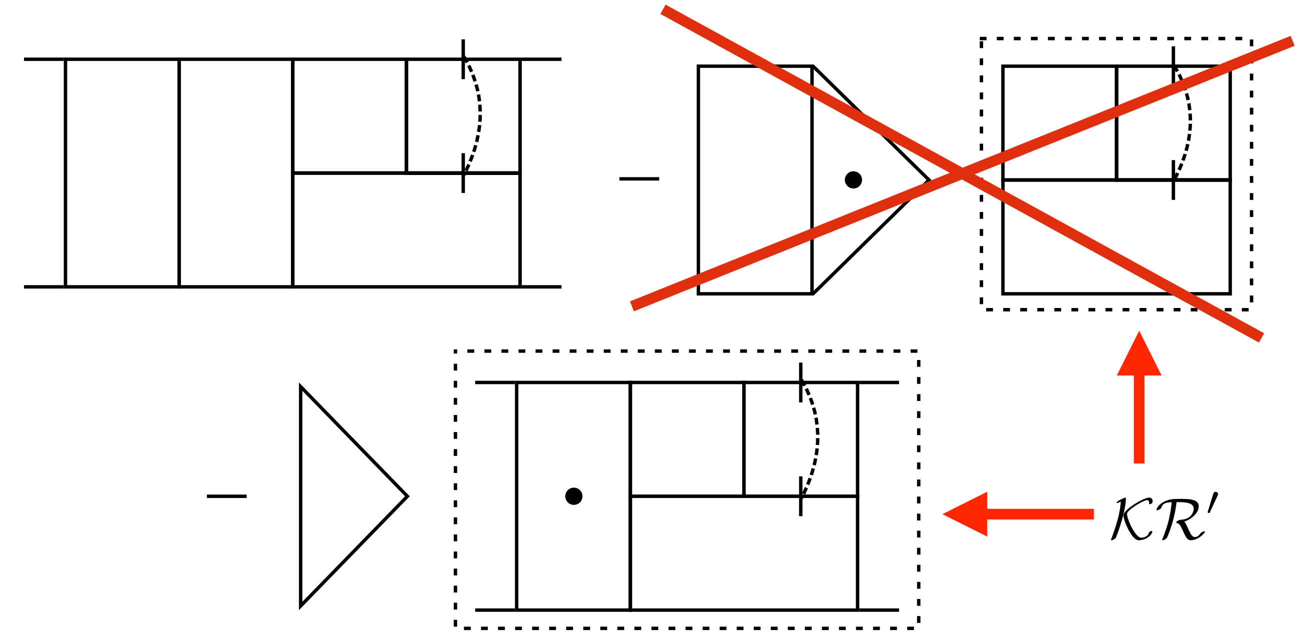

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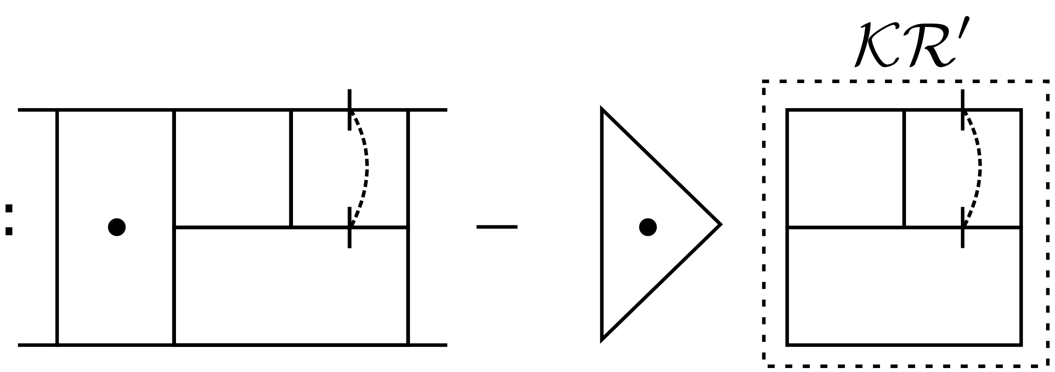

$$\mathcal{R}' : A_4 \frac{\mu^{4\epsilon}}{\epsilon^2} - \left(-\frac{1}{6\epsilon}\right)\left(-\frac{\mu^{\epsilon}}{6\epsilon} 2p_3(2p_2 - k + p_1)\right)$$

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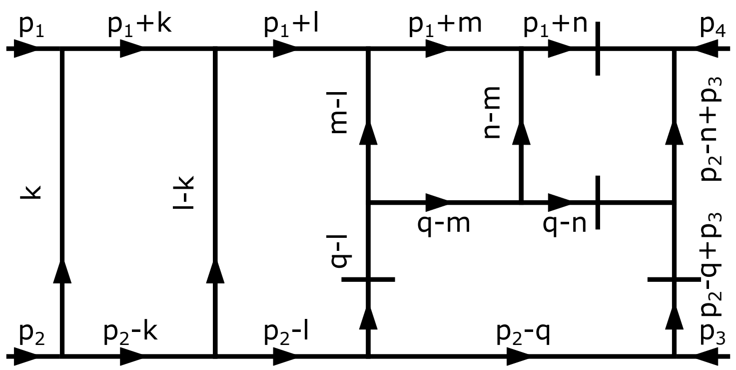

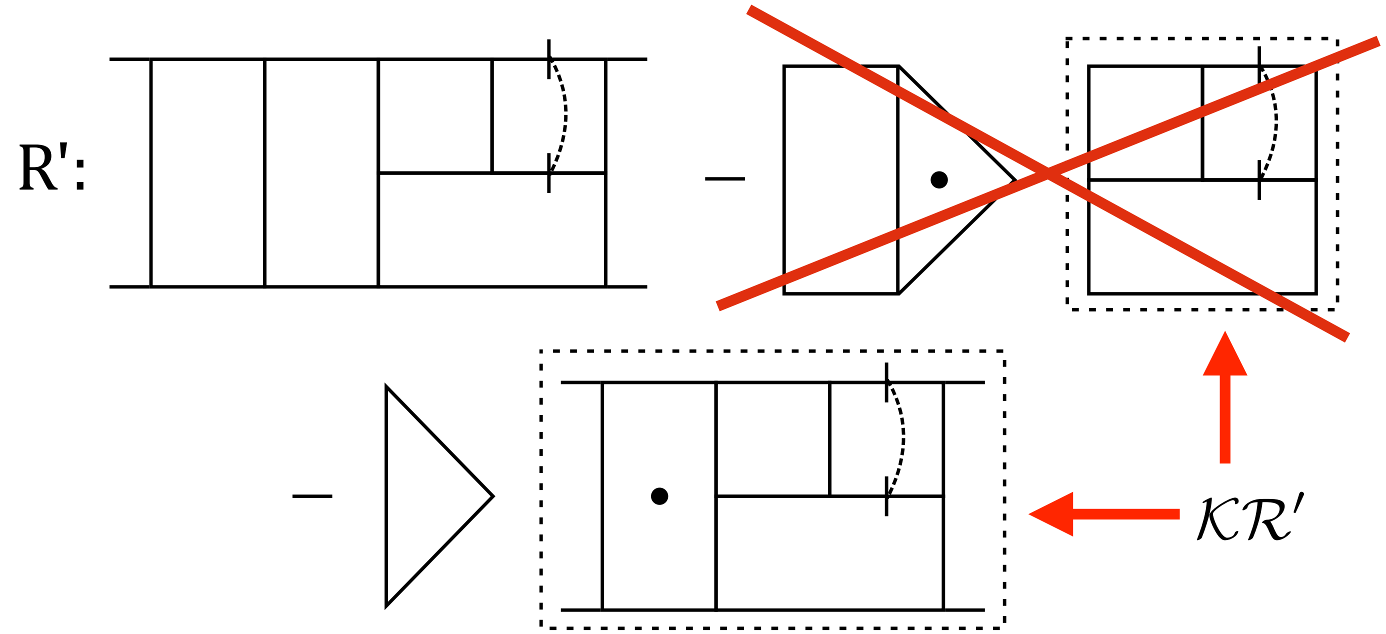

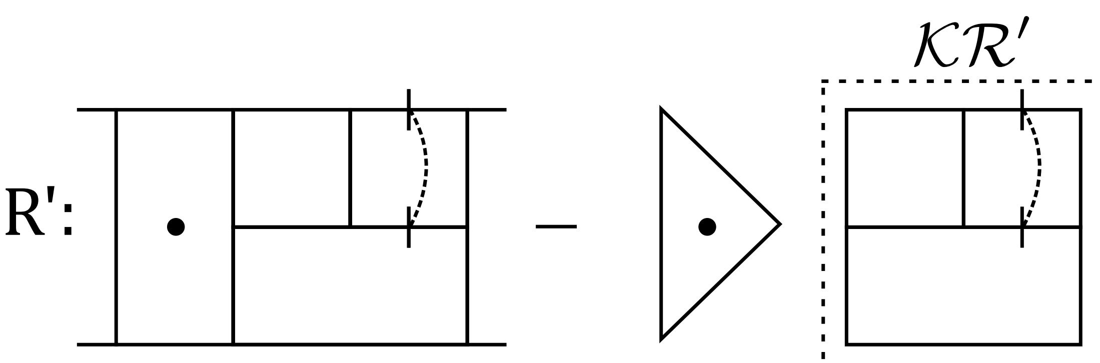

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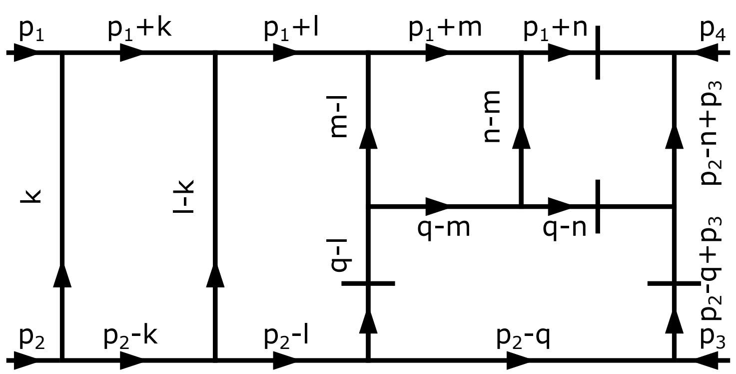
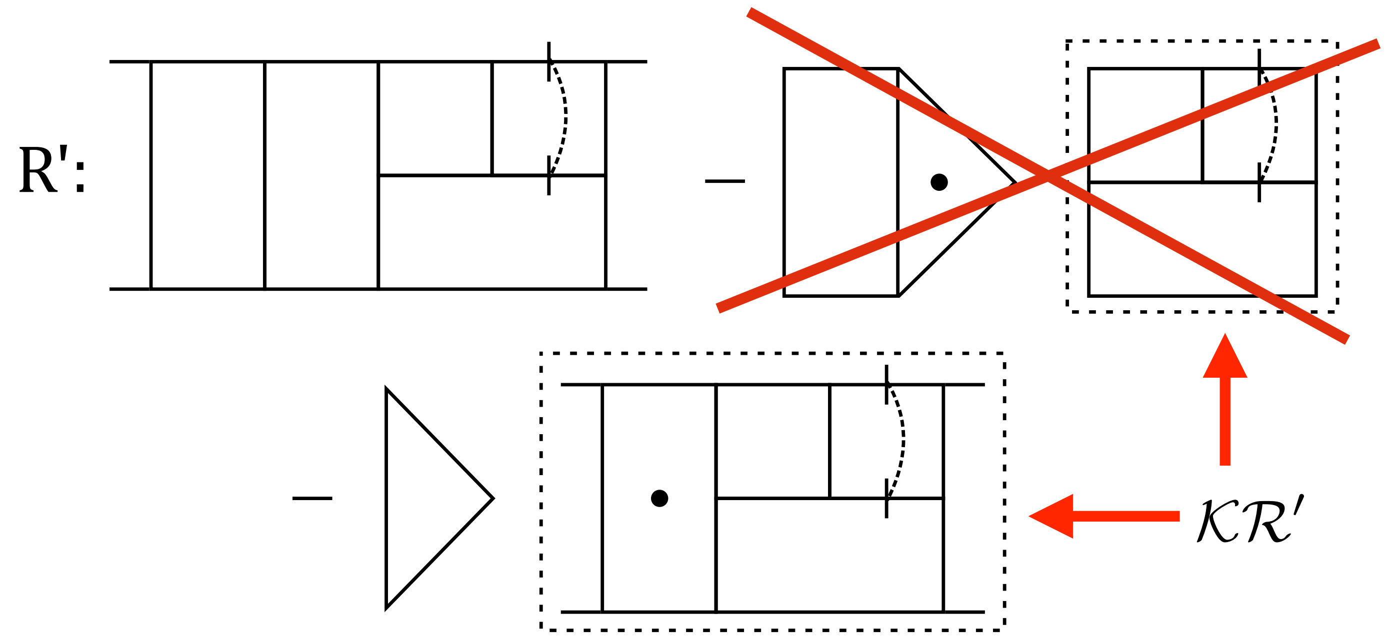
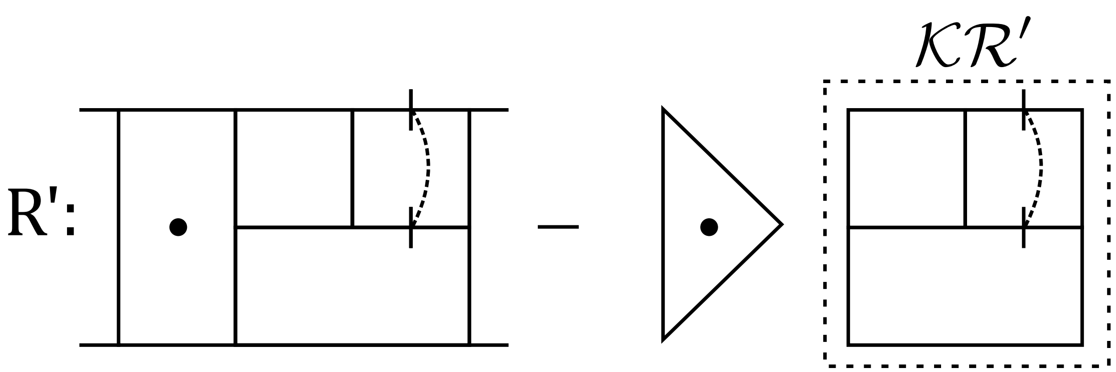
$$A_4 = \frac{2p_3(2p_2 - k + p_1)}{4 \cdot 36}$$

$\mathbf{R}'$- operation and Leading Divergences





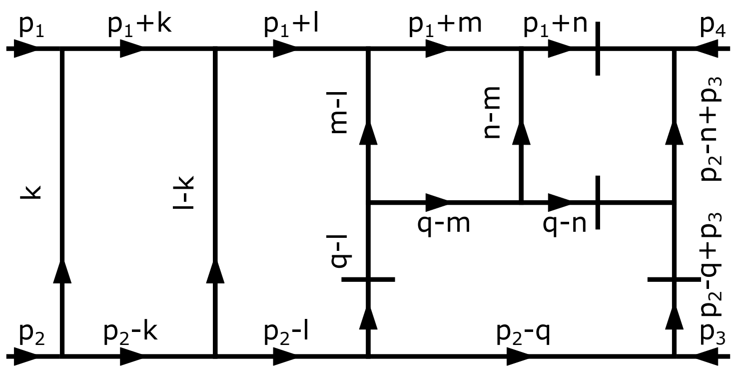


$$\mathcal{R}' : A_4 \frac{\mu^{4\epsilon}}{\epsilon^2} - \left(-\frac{1}{6\epsilon}\right)\left(-\frac{\mu^{\epsilon}}{6\epsilon} 2p_3(2p_2 - k + p_1)\right)$$

$$A_4 = \frac{2p_3(2p_2 - k + p_1)}{4 \cdot 36}$$

$$\mathcal{KR}' = \frac{2p_3(2p_2 - k + p_1)}{4 \cdot 36\epsilon^2}\mu^{4\epsilon} - \frac{2p_3(2p_2 - k + p_1)}{36\epsilon^2}\mu^{\epsilon} = -3\frac{2p_3(2p_2 - k + p_1)}{4 \cdot 36\epsilon^2}$$

$\mathbf{R}'$- operation and Leading Divergences



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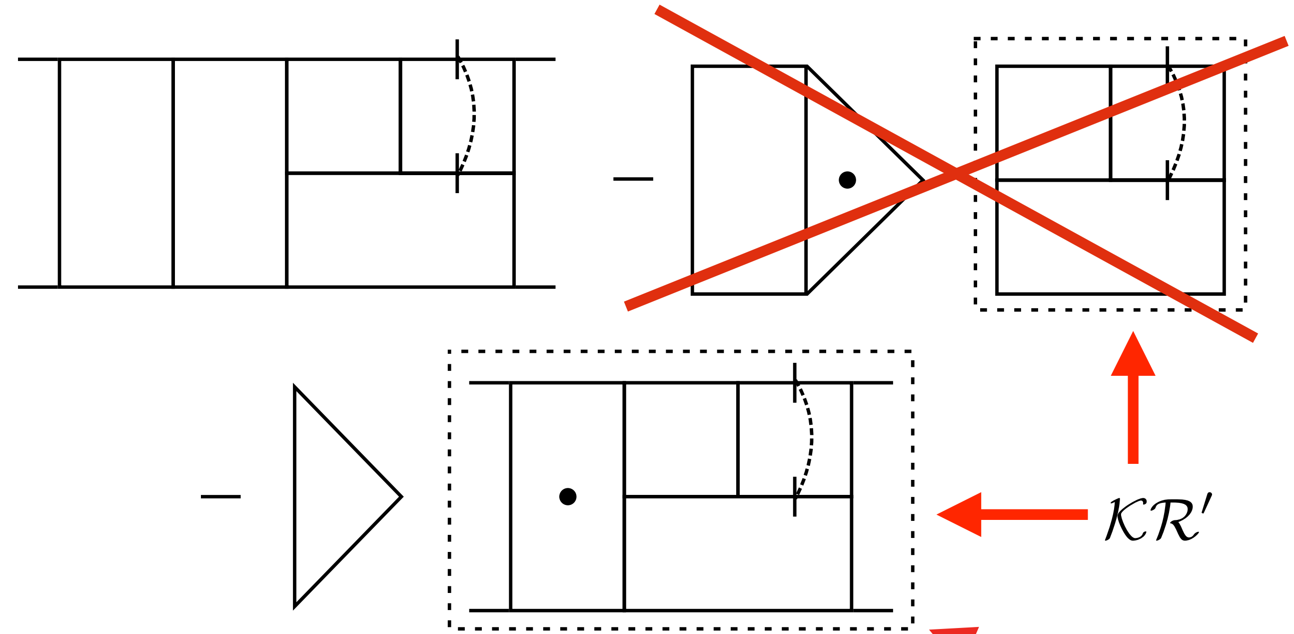
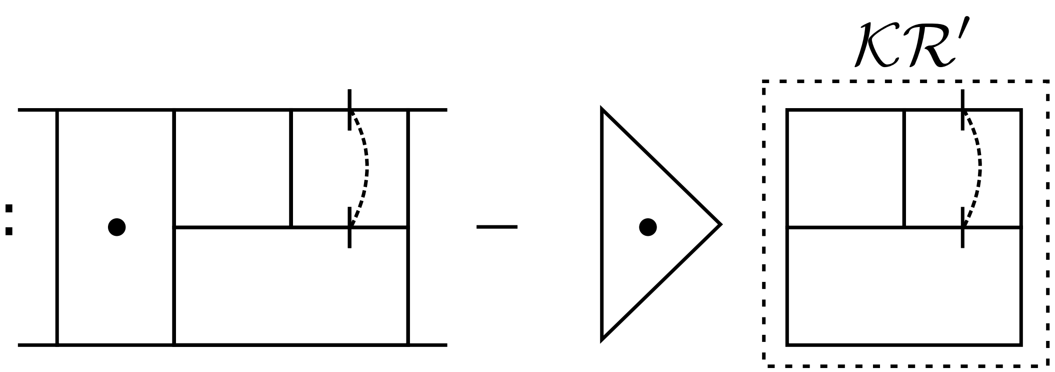


R':



$$\mathcal{R}' : A_4 \frac{\mu^{4\epsilon}}{\epsilon^2} - \left(-\frac{1}{6\epsilon}\right)\left(-\frac{\mu^{\epsilon}}{6\epsilon} 2p_3(2p_2 - k + p_1)\right)$$

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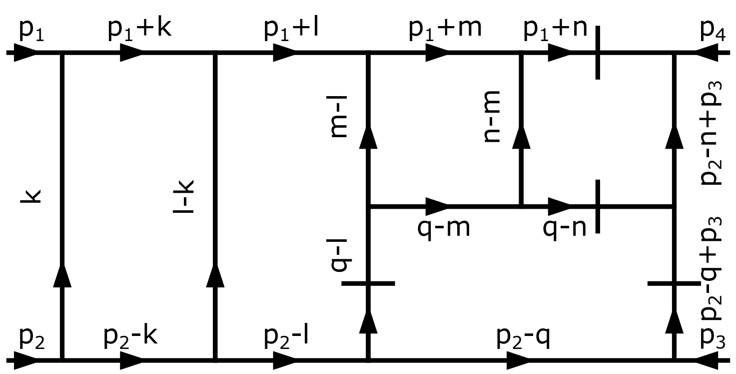
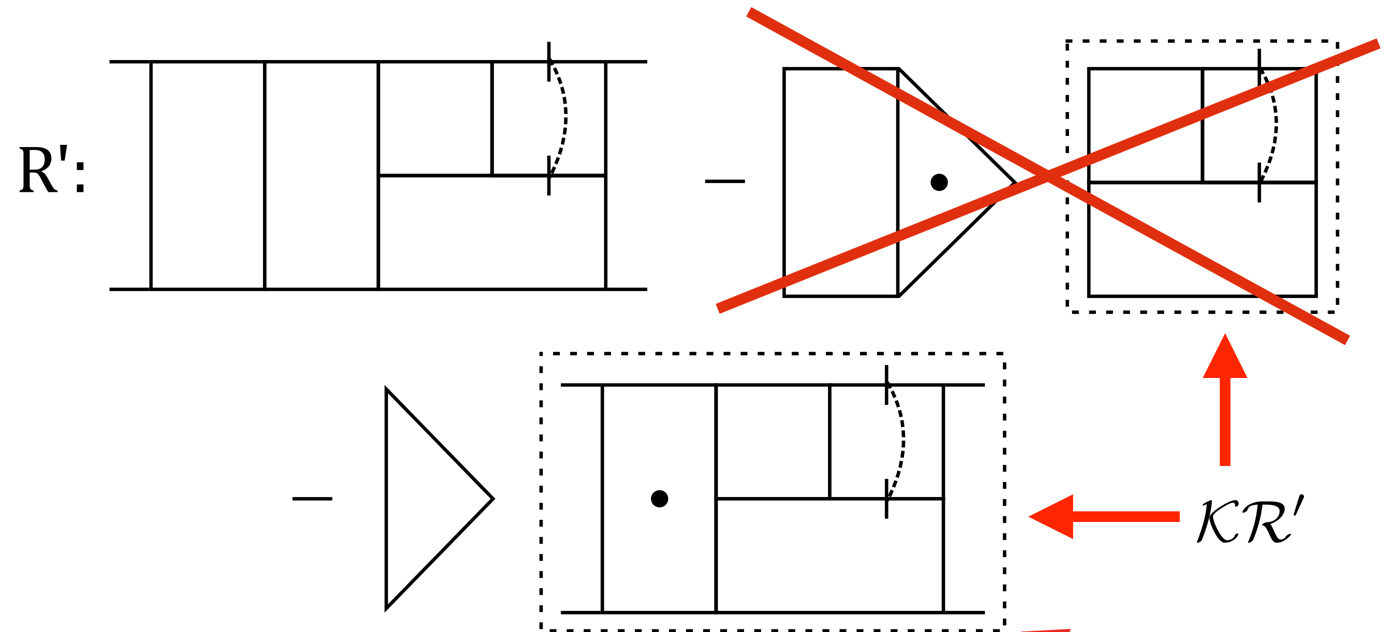
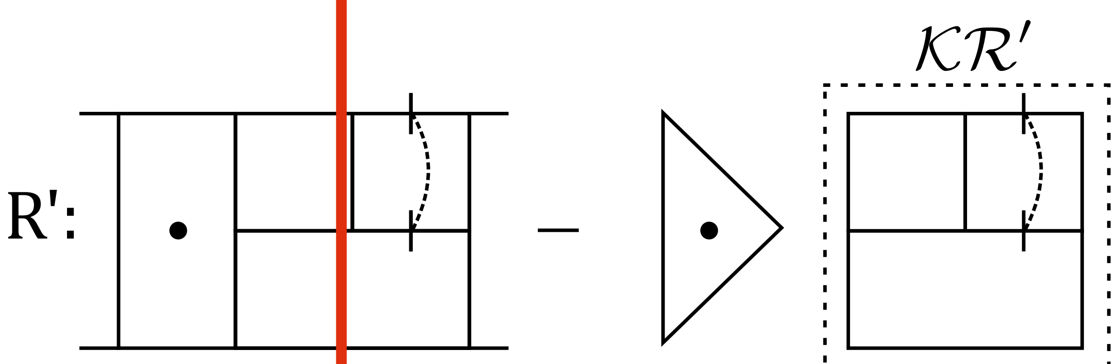
$\mathbf{R}'$- operation and Leading Divergences







$$\mathcal{R}' : A_4 \frac{\mu^{4\epsilon}}{\epsilon^2} - \left(-\frac{1}{6\epsilon}\right)\left(-\frac{\mu^{\epsilon}}{6\epsilon} 2p_3(2p_2 - k + p_1)\right)$$

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$$\mathcal{KR}' = \frac{2p_3(2p_2 - k + p_1)}{4 \cdot 36\epsilon^2}\mu^{4\epsilon} - \frac{2p_3(2p_2 - k + p_1)}{36\epsilon^2}\mu^{\epsilon} = -3\frac{2p_3(2p_2 - k + p_1)}{4 \cdot 36\epsilon^2}$$

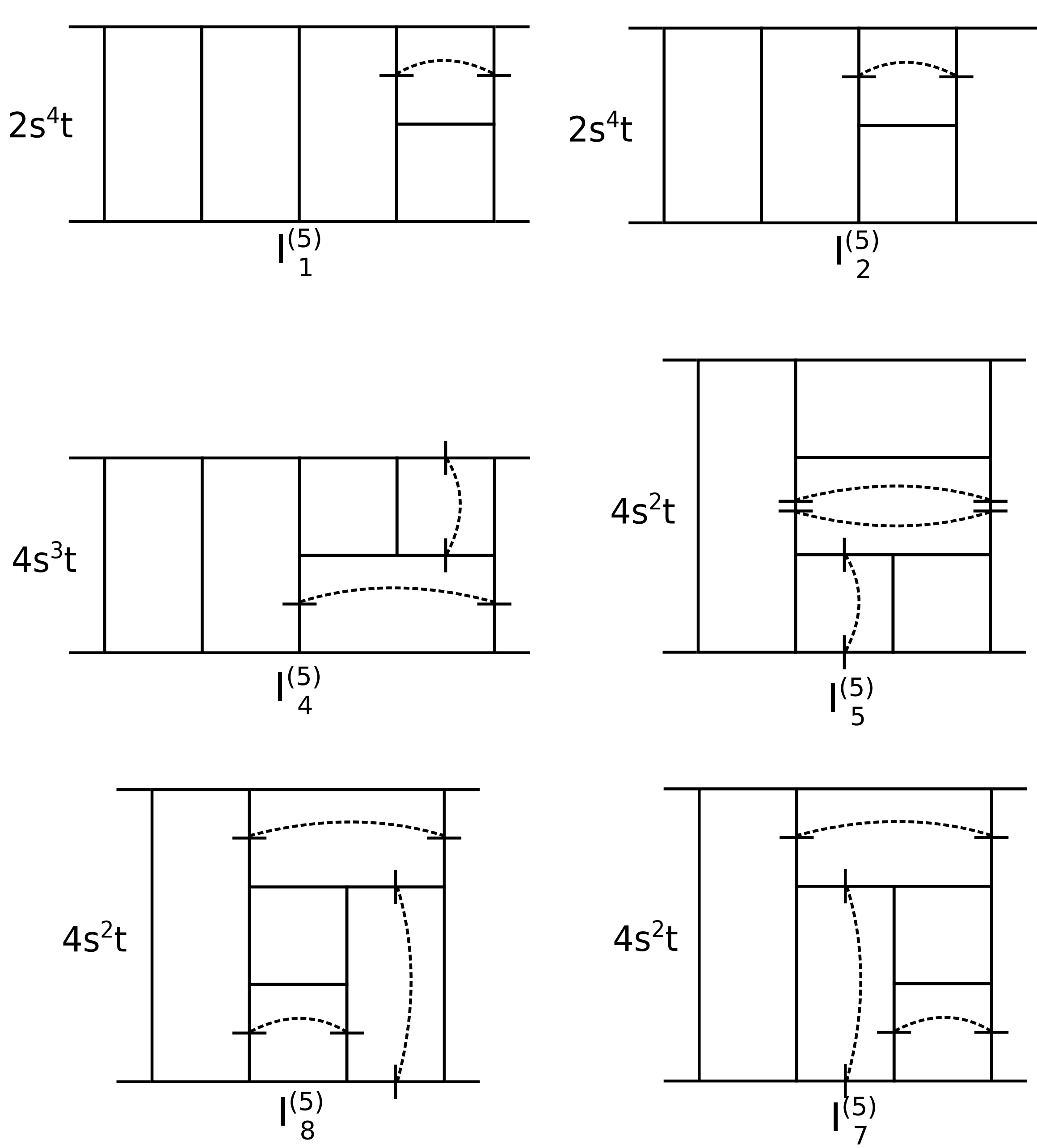
$$L.P. = \frac{s - t/4}{30 \cdot 36 \cdot \epsilon^3}$$

The leading Divergences

MI	Comb	$D=6$	$D=8$	$D=10$
$I_1^{(1)}$	st	conv	$\frac{1}{3!\epsilon}$	$\frac{s+t}{5!\epsilon}$
$I_1^{(2)}$	s^2t	conv	$-\frac{s}{3!4!\epsilon^2}$	$\frac{-s^2(8s+2t)}{5!7!\epsilon^2}$
$I_1^{(3)}$	s^3t	conv	$\frac{s^2}{4!5!\epsilon^3}$	$\frac{-2s^4(135s+11t)}{5!7!7!3\epsilon^3}$
$I_2^{(3)}$	$2s^2t$	$-\frac{1}{6\epsilon}$	$\frac{s(3s^2-2st+t^2)}{3!4!5!9\epsilon^3}$	$\frac{-s^2\left(14s^4-10s^3t+\frac{33}{5}s^2t^2-\frac{19}{5}st^3+\frac{8}{5}t^4\right)}{5!7!7!9\epsilon^3}$
$I_1^{(4)}$	s^4t	conv	$-\frac{210s^3}{3!4!5!6!\epsilon^4}$	$\frac{-32s^6(99s+2t)}{5!7!7!7!3\epsilon^4}$
$I_2^{(4)}$	$2s^3t$	$\frac{1}{48\epsilon^2}$	$\frac{s^2\left(-\frac{430}{21}s^2+\frac{4}{9}st-\frac{1}{18}t^2\right)}{3!4!5!6!\epsilon^4}$	$\frac{-2s^4\left(\frac{1502144}{33}s^4-\frac{1085791}{33}s^3t+\frac{2044}{5}s^2t^2-\frac{1001}{15}st^3+\frac{112}{15}t^4\right)}{5!7!7!7!7!\epsilon^4}$
$I_3^{(4)}$	s^3t	$\frac{1}{24\epsilon^2}$	$\frac{s^2\left(-\frac{20}{3}s^2+\frac{8}{9}st-\frac{1}{9}t^2\right)}{3!4!5!6!\epsilon^4}$	$\frac{-28s^4\left(8512s^4-1043s^3t+\frac{876}{5}s^2t^2-\frac{143}{5}st^3+\frac{16}{5}t^4\right)}{5!7!7!7!7!3\epsilon^4}$
$I_4^{(4)}$	$2s^2t$	$\sim\frac{1}{\epsilon}$	$\frac{s\left(-\frac{45}{14}s^4+\frac{18}{7}s^3t-\frac{27}{14}s^2t^2+\frac{9}{7}st^3-\frac{9}{14}t^4\right)}{3!4!5!6!\epsilon^4}$	$\frac{-s^2\left(-\frac{7504}{1287}s^7+\frac{7819}{1716}s^6t-\frac{1475}{429}s^5t^2+\frac{12745}{5148}s^4t^3-\frac{716}{429}s^3t^4+\frac{1747}{1716}s^2t^5-\frac{673}{1287}st^6+\frac{105}{572}t^7\right)}{5!7!7!7!\epsilon^4}$
$I_5^{(4)}$	$4s^2t$	$\frac{t-s}{3\cdot 48\epsilon^2}$	$\frac{s\left(-\frac{15}{28}s^4+\frac{25}{63}s^3t-\frac{65}{252}s^2t^2+\frac{5}{42}st^3-\frac{1}{28}t^4\right)}{3!4!5!6!\epsilon^4}$	$\frac{-4s^2\left(-\frac{95200}{143}s^7+\frac{67634}{143}s^6t-\frac{225008}{715}s^5t^2+\frac{136514}{715}s^4t^3-\frac{6608}{65}s^3t^4+\frac{6706}{143}s^2t^5-\frac{7420}{429}st^6+\frac{1715}{429}t^7\right)}{5!7!7!7!7!\epsilon^4}$

Perturbation Expansion for the Amplitudes

D=6 N=2 **<u>Leading Divergences</u>** **5 loops**



The diagrams with the substitution s<->t are not shown

MI	$I_1^{(5)}$	$I_2^{(5)}$	$I_3^{(5)}$	$I_4^{(5)}$
Comb	$2s^4t$	$2s^4t$	$4s^3t$	$2s^3t$
Int	$-\frac{1}{\epsilon^3}\frac{3}{36\cdot 40}$	$-\frac{1}{\epsilon^3}\frac{9}{36\cdot 40}$	$\frac{1}{\epsilon^3}\frac{s-t/4}{36\cdot 15}$	$\frac{1}{\epsilon^3}\frac{s-t/4}{36\cdot 30}$
MI	$I_5^{(5)}$	$I_6^{(5)}$	$I_7^{(5)}$	$I_8^{(5)}$
comb	$4s^2t$	$2s^2t$	$4s^2t$	$4s^2t$
Int	$-\frac{1}{\epsilon^3}\frac{s^2-st+t^2}{36\cdot 80}$	$-\frac{1}{\epsilon^3}\frac{s^2-st+t^2}{36\cdot 40}$	$\frac{1}{\epsilon^3}\frac{s^2-st+t^2/3}{36\cdot 80}$	$\frac{1}{\epsilon^3}\frac{s^2-st+t^2/3}{36\cdot 80}$

Numerical evaluation of Integrals

D=6 N=2 **Leading Divergences** α **-representation**

$$I(s,t,m_i)=\frac{(\pi)^{DL/2}}{\prod\limits_{i=1}^{n}\Gamma(\lambda_i)}\left(\left(\prod_{i=n+1}^{n+k}(-\partial_{\alpha_i})^{\kappa_i}\right)\int_0^\infty\frac{d\alpha_1...d\alpha_n}{U^{d/2}}e^{-V/U-\sum\limits_{j=1}^{n}m_j\alpha_j}\right)\Big|_{\alpha_{n+1}=...=\alpha_{n+k}=0}$$

Numerical evaluation of Integrals

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s=t=0, m $\neq 0$

Numerical evaluation of Integrals

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s=t=0, m $\neq 0$

$$\tilde{G}^{(D=8)}_{i,3-i}(s=0,t=0,m_i) = (\pi)^{3D/2} \left((-\partial_{\alpha_{11}}) \int_0^\infty \frac{d\alpha_1 ... d\alpha_{10} (-P_s)^i (-P_t)^{3-i}}{U^{d/2+3}} e^{-\sum\limits_{j=1}^{10} m_j \alpha_j} \right) \Big|_{\alpha_{11}=0}$$

Numerical evaluation of Integrals

D=6 N=2 **Leading Divergences** α **-representation**

$$I(s,t,m_i) = \frac{(\pi)^{DL/2}}{\prod\limits_{i=1}^{n}\Gamma(\lambda_i)}\left(\left(\prod_{i=n+1}^{n+k}(-\partial_{\alpha_i})^{\kappa_i}\right)\int_0^\infty \frac{d\alpha_1 \ldots d\alpha_n}{U^{d/2}} e^{-V/U-\sum\limits_{j=1}^{n} m_j\alpha_j}\right)\Big|_{\alpha_{n+1}=\ldots=\alpha_{n+k}=0}$$

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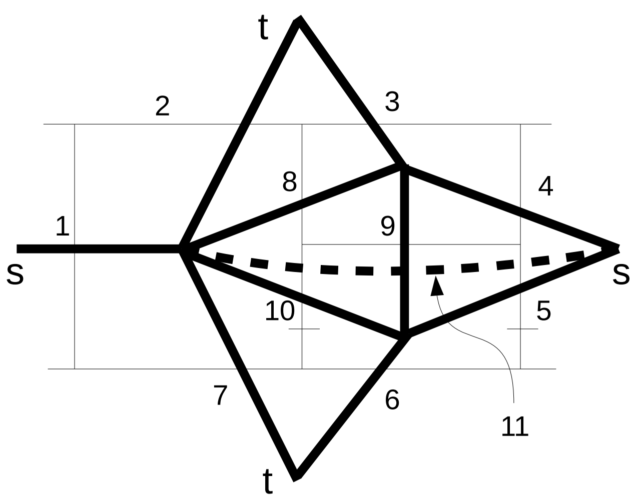

Dual graph

Numerical evaluation of Integrals

D=6 N=2 **Leading Divergences** α **-representation**

$$I(s,t,m_i) = \frac{(\pi)^{DL/2}}{\prod\limits_{i=1}^{n}\Gamma(\lambda_i)}\left(\left(\prod_{i=n+1}^{n+k}(-\partial_{\alpha_i})^{\kappa_i}\right)\int_0^\infty \frac{d\alpha_1 \ldots d\alpha_n}{U^{d/2}} e^{-V/U-\sum\limits_{j=1}^{n} m_j\alpha_j}\right)\Big|_{\alpha_{n+1}=\ldots=\alpha_{n+k}=0}$$

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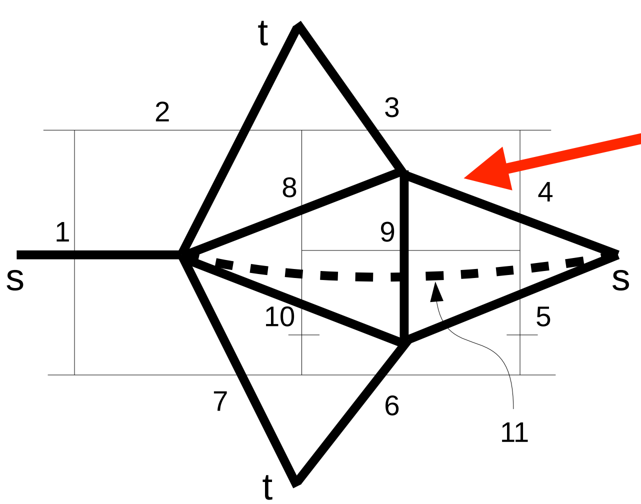
Numerator



Dual graph

Numerical evaluation of Integrals

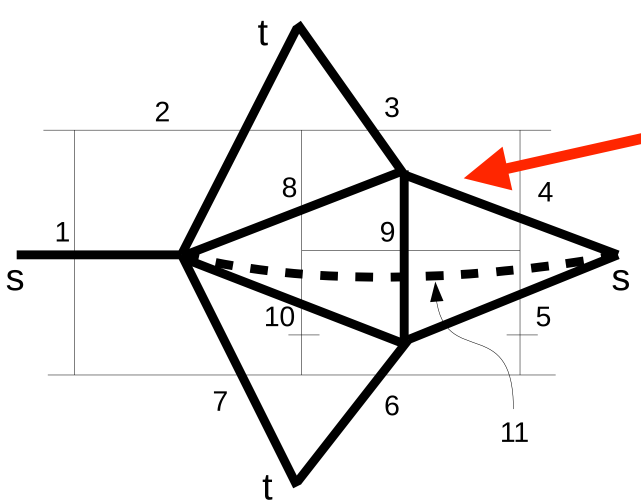
D=6 N=2 **Leading Divergences** α **-representation**

$$I(s,t,m_i) = \frac{(\pi)^{DL/2}}{\prod_{i=1}^{n}\Gamma(\lambda_i)} \left(\left(\prod_{i=n+1}^{n+k} (-\partial_{\alpha_i})^{\kappa_i} \right) \int_0^\infty \frac{d\alpha_1 ... d\alpha_n}{U^{d/2}} e^{-V/U - \sum_{j=1}^{n} m_j \alpha_j} \right) \Big|_{\alpha_{n+1}=...=\alpha_{n+k}=0}$$

s=t=0, m $\neq 0$

$$\tilde{G}^{(D=8)}_{i,3-i}(s=0,t=0,m_i) = (\pi)^{3D/2} \left((-\partial_{\alpha_{11}}) \int_0^\infty \frac{d\alpha_1 ... d\alpha_{10} (-P_s)^i (-P_t)^{3-i}}{U^{d/2+3}} e^{-\sum_{j=1}^{10} m_j \alpha_j} \right) \Big|_{\alpha_{11}=0}$$

Numerator



Dual graph

graph	term	numerical	exact
$I_1^{(4)}$	$s^0\,t^0$	0	0
$I_2^{(4)}$	$s^0\,t^0$	0.0416652(17)	1/24
$I_3^{(4)}$	$s^0\,t^0$	0.0208328(7)	1/48

graph	term	numerical	exact
$I_1^{(4)}$	s^3	$-209.997(5)$	-210
$I_2^{(4)}$	s^4	-6.6661(10)	-20/3
	s^3t	0.888900(24)	8/9
	s^2t^2	-0.1111105(7)	-1/9
$I_3^{(4)}$	s^4	-20.4765(8)	-430/21
	s^3t	0.444420(25)	4/9
	s^2t^2	-0.0555541(10)	-1/18

Comparison with analytical evaluation

Perturbation Expansion for the Amplitudes

D=6 N=2

Leading Divergences

$$L.P. = 2stg^4 \left[g^2 \frac{s+t}{6\epsilon} + g^4 \frac{s^2+st+t^2}{36\epsilon^2} + g^6 \frac{s^3+\frac{2}{5}s^2t+\frac{2}{5}st^2+t^3}{216\epsilon^3} \right]$$

Perturbation Expansion for the Amplitudes

D=6 N=2

Result up to 5 loops

Leading Divergences

$$L.P. = 2stg^4\left[g^2\frac{s+t}{6\epsilon} + g^4\frac{s^2+st+t^2}{36\epsilon^2} + g^6\frac{s^3+\frac{2}{5}s^2t+\frac{2}{5}st^2+t^3}{216\epsilon^3}\right]$$

Perturbation Expansion for the Amplitudes

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Geom progression !?

Perturbation Expansion for the Amplitudes

D=6 N=2

Result up to 5 loops

Leading Divergences

$$L.P. = 2stg^4\left[g^2\frac{s+t}{6\epsilon} + g^4\frac{s^2+st+t^2}{36\epsilon^2} + g^6\frac{s^3+\frac{2}{5}s^2t+\frac{2}{5}st^2+t^3}{216\epsilon^3}\right]$$

Geom progression !?

Leading powers of s > 0

$$\sum_{n=1}^{\infty}\left(\frac{g^2 s}{6\epsilon}\right)^n = \frac{\frac{g^2 s}{6\epsilon}}{1-\frac{g^2 s}{6\epsilon}}$$

Pole!

$\epsilon \to +0$

Perturbation Expansion for the Amplitudes

D=6 N=2

Result up to 5 loops

Leading Divergences

$$L.P. = 2stg^4\left[g^2\frac{s+t}{6\epsilon} + g^4\frac{s^2+st+t^2}{36\epsilon^2} + g^6\frac{s^3+\frac{2}{5}s^2t+\frac{2}{5}st^2+t^3}{216\epsilon^3}\right]$$

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Pole!

$\epsilon \to +0$

Leading powers of t < 0

$$\sum_{n=1}^{\infty}\left(\frac{g^2t}{6\epsilon}\right)^n = \frac{\frac{g^2t}{6\epsilon}}{1-\frac{g^2t}{6\epsilon}}$$

-1

$\epsilon \to +0$

Perturbation Expansion for the Amplitudes

D=6 N=2

Result up to 5 loops

Leading Divergences

$$L.P. = 2stg^4\left[g^2\frac{s+t}{6\epsilon} + g^4\frac{s^2+st+t^2}{36\epsilon^2} + g^6\frac{s^3+\frac{2}{5}s^2t+\frac{2}{5}st^2+t^3}{216\epsilon^3}\right]$$

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Pole!

$\epsilon \to +0$

Leading powers of t < 0

$$\sum_{n=1}^{\infty}\left(\frac{g^2 t}{6\epsilon}\right)^n = \frac{\frac{g^2 t}{6\epsilon}}{1-\frac{g^2 t}{6\epsilon}}$$

-1

$\epsilon \to +0$

Compare D=4 YM

$$g^2 = \frac{g_B^2}{1-\frac{11C_2}{3}\frac{g_B^2}{\epsilon}}$$

Perturbation Expansion for the Amplitudes

D=6 N=2

Result up to 5 loops

Leading Divergences

$$L.P. = 2stg^4 \left[g^2 \frac{s+t}{6\epsilon} + g^4 \frac{s^2+st+t^2}{36\epsilon^2} + g^6 \frac{s^3 + \frac{2}{5}s^2t + \frac{2}{5}st^2 + t^3}{216\epsilon^3} \right]$$

Geom progression !?

Leading powers of s > 0

$$\sum_{n=1}^{\infty} \left(\frac{g^2 s}{6\epsilon} \right)^n = \frac{\frac{g^2 s}{6\epsilon}}{1 - \frac{g^2 s}{6\epsilon}}$$

Pole!

$\epsilon \to +0$

Leading powers of t < 0

$$\sum_{n=1}^{\infty} \left(\frac{g^2 t}{6\epsilon} \right)^n = \frac{\frac{g^2 t}{6\epsilon}}{1 - \frac{g^2 t}{6\epsilon}}$$

-1

$\epsilon \to +0$

Compare D=4 YM

$$g^2 = \frac{g_B^2}{1 - \frac{11C_2}{3}\frac{g_B^2}{\epsilon}}$$

General case will be given below

Perturbation Expansion for the Amplitudes

D=8 N=1

Leading Divergences

Result up to 4 loops

$$
\begin{aligned}
L.P. \quad &= \quad -st\left[g^2\frac{1}{3!\epsilon}+g^4\frac{s^2+t^2}{3!4!\epsilon^2}+g^6\frac{4}{3}\frac{15s^4-s^3t+s^2t^2-st^3+15t^4}{3!4!5!\epsilon^3}\right. \\
&+ \quad \left. g^8\frac{1}{63}\frac{16770s^6-536s^5t+412s^4t^2-384s^3t^3+412s^2t^4-536st^5+16770t^6}{3!4!5!6!\epsilon^4}\right].
\end{aligned}
$$

Perturbation Expansion for the Amplitudes

D=8 N=1

Leading Divergences

Result up to 4 loops

$$L.P. = -st\left[g^2\frac{1}{3!\epsilon} + g^4\frac{s^2+t^2}{3!4!\epsilon^2} + g^6\frac{4}{3}\frac{15s^4 - s^3t + s^2t^2 - st^3 + 15t^4}{3!4!5!\epsilon^3}\right.$$
$$\left. + \quad g^8\frac{1}{63}\frac{16770s^6 - 536s^5t + 412s^4t^2 - 384s^3t^3 + 412s^2t^4 - 536st^5 + 16770t^6}{3!4!5!6!\epsilon^4}\right].$$

D=10 N=1

Leading Divergences

Result up to 4 loops

$$L.P. = -st\left[g^2\frac{s+t}{5!\epsilon} + g^4\frac{8s^4 + 2s^3t + 2st^3 + 8t^4}{5!7!\epsilon^2}\right.$$
$$+ \quad g^6\frac{2(2095s^7 + 115s^6t + 33s^5t^2 - 11s^4t^3 - 11s^3t^4 + 33s^2t^5 + 115st^6 + 2095t^7)}{5!7!7!45\epsilon^3}$$
$$+ \quad g^8\frac{32(211218880s^{10} + 753490s^9t - 1395096s^8t^2 + 1125763s^7t^3 - 916916s^6t^4}{13!7!7!5!5\epsilon^4}$$
$$\left.\frac{+843630s^5t^5 - 916916s^4t^6 + 1125763s^3t^7 - 1395096s^2t^8 + 753490st^9 + 211218880t^{10})}{13!7!7!5!5\epsilon^4}\right].$$

Perturbation Expansion for the Amplitudes

D=8 N=1

Leading Divergences

Result up to 4 loops

$$L.P. = -st\left[g^2\frac{1}{3!\epsilon}+g^4\frac{s^2+t^2}{3!4!\epsilon^2}+g^6\frac{4}{3}\frac{15s^4-s^3t+s^2t^2-st^3+15t^4}{3!4!5!\epsilon^3}\right.$$
$$\left.+\ g^8\frac{1}{63}\frac{16770s^6-536s^5t+412s^4t^2-384s^3t^3+412s^2t^4-536st^5+16770t^6}{3!4!5!6!\epsilon^4}\right].$$

D=10 N=1

Leading Divergences

Result up to 4 loops

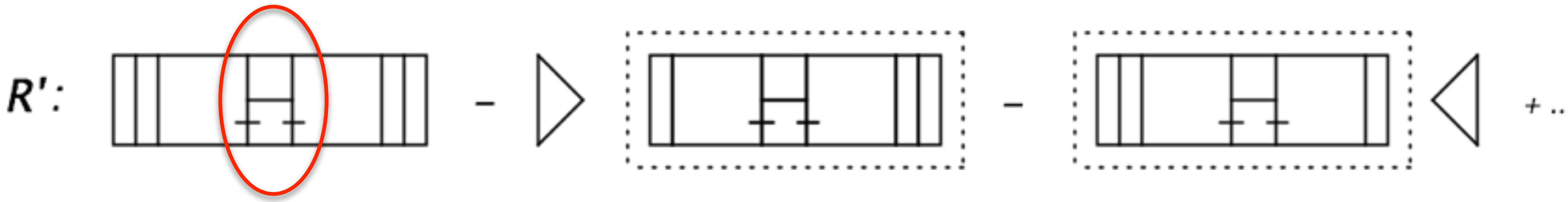
$$L.P. = -st\left[g^2\frac{s+t}{5!\epsilon}+g^4\frac{8s^4+2s^3t+2st^3+8t^4}{5!7!\epsilon^2}\right.$$
$$+\ g^6\frac{2(2095s^7+115s^6t+33s^5t^2-11s^4t^3-11s^3t^4+33s^2t^5+115st^6+2095t^7)}{5!7!7!45\epsilon^3}$$
$$+\ g^8\frac{32(211218880s^{10}+753490s^9t-1395096s^8t^2+1125763s^7t^3-916916s^6t^4}{13!7!7!5!5\epsilon^4}$$
$$\left.\frac{+843630s^5t^5-916916s^4t^6+1125763s^3t^7-1395096s^2t^8+753490st^9+211218880t^{10})}{13!7!7!5!5\epsilon^4}\right].$$

Doesn't look like Geom progression anymore, however, coefficients grow slowly

R-operation and Recurrence Relation

D=6 N=2

Horizontal boxes + tennis court



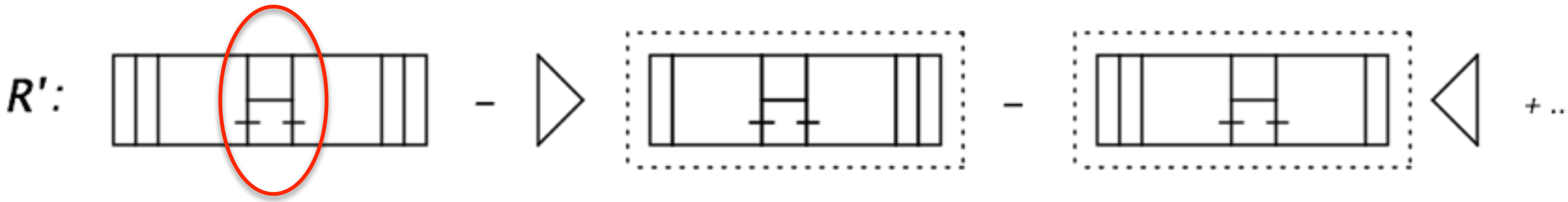
A_n $\quad$ A_{n-1}

$$nA_n = -A_{n-1} \quad \longrightarrow \quad A_n = (-1)^n \frac{2}{n!} \qquad (-g^2 s)^n$$

R-operation and Recurrence Relation

D=6 N=2

Horizontal boxes + tennis court



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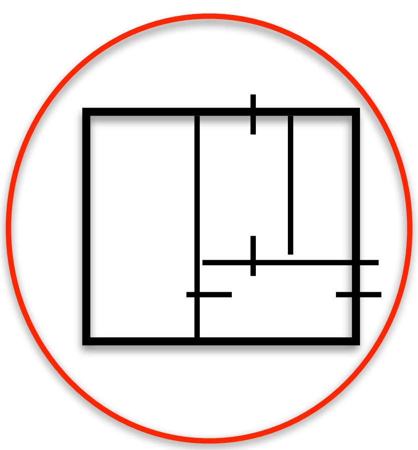
Horizontal boxes + double tennis court

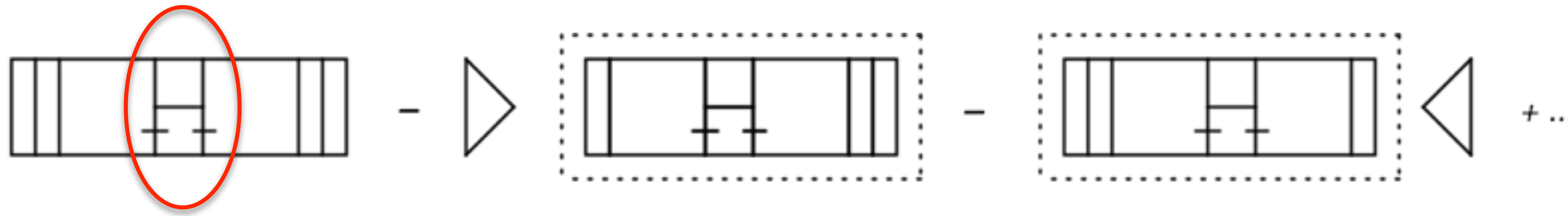


R-operation and Recurrence Relation

D=6 N=2

Horizontal boxes + tennis court

R':



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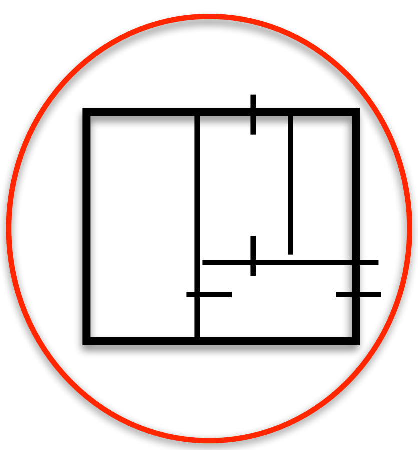


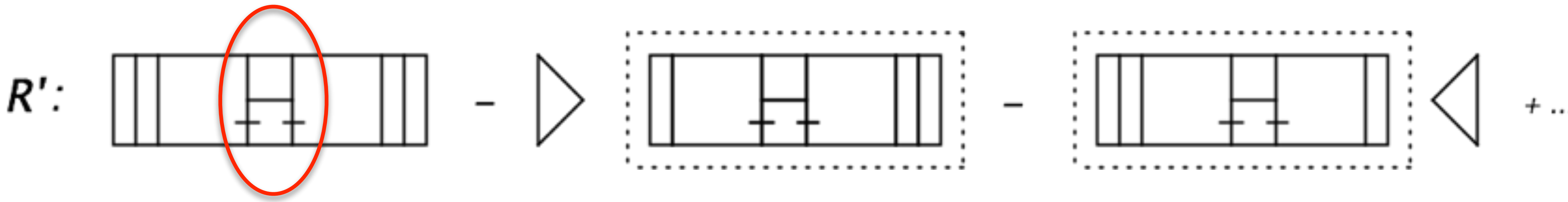
Horizontal boxes + double tennis court

$$nA_n^t = -\frac{1}{3} A_{n-1}^t, \qquad nA_n^s = -A_{n-1}^s + \frac{1}{3} A_{n-1}^t$$

R-operation and Recurrence Relation

D=6 N=2

Horizontal boxes + tennis court



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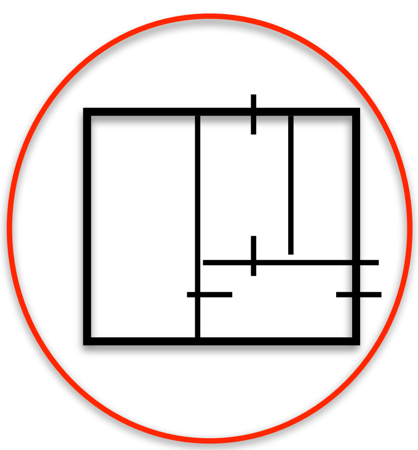
Horizontal boxes + double tennis court



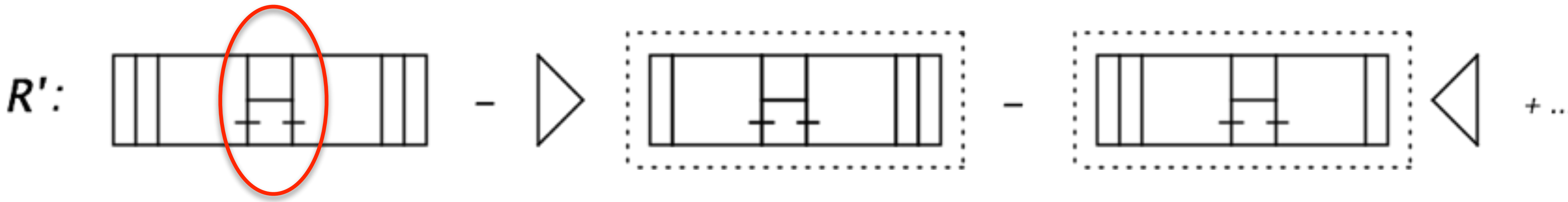
$$nA^t_n = -\frac{1}{3} A^t_{n-1}, \qquad nA^s_n = -A^s_{n-1} + \frac{1}{3} A^t_{n-1}$$

$$A^t_n = \frac{(-1)^n}{3^{n-3}} \frac{1}{n!}, \qquad A^s_n = \frac{1}{2} \frac{(-1)^n}{3^{n-3}} \frac{1}{n!} - \frac{1}{2} (-1)^n \frac{1}{n!}$$

R-operation and Recurrence Relation

D=6 N=2

Horizontal boxes + tennis court



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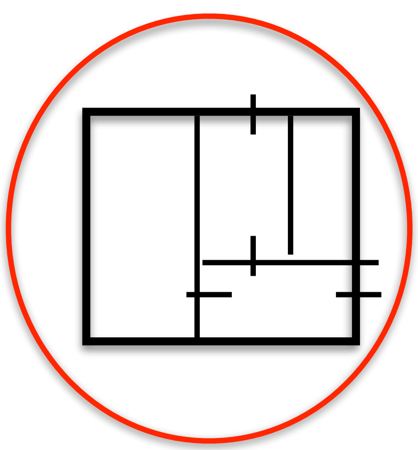


Horizontal boxes + double tennis court

$$nA_n^t = -\frac{1}{3} A_{n-1}^t, \qquad nA_n^s = -A_{n-1}^s + \frac{1}{3} A_{n-1}^t$$

$$\downarrow$$

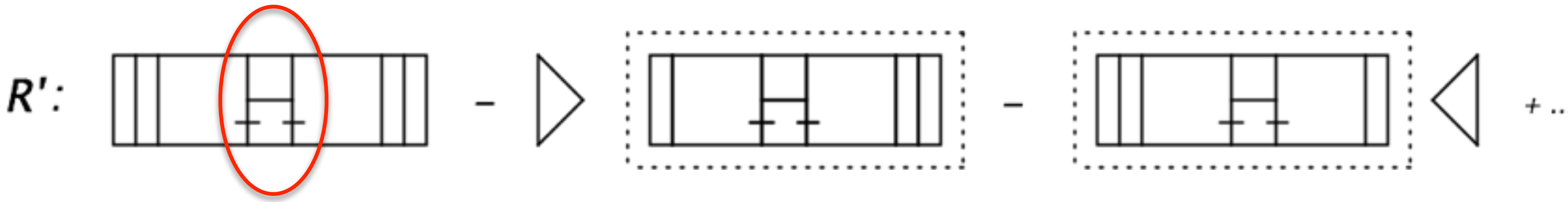
$$A_n^t = \frac{(-1)^n}{3^{n-3}} \frac{1}{n!}, \qquad A_n^s = \frac{1}{2} \frac{(-1)^n}{3^{n-3}} \frac{1}{n!} - \frac{1}{2} (-1)^n \frac{1}{n!}$$

$$(-g^2 s)^{n-1} (-g^2 t)$$

R-operation and Recurrence Relation

D=6 N=2

Horizontal boxes + tennis court



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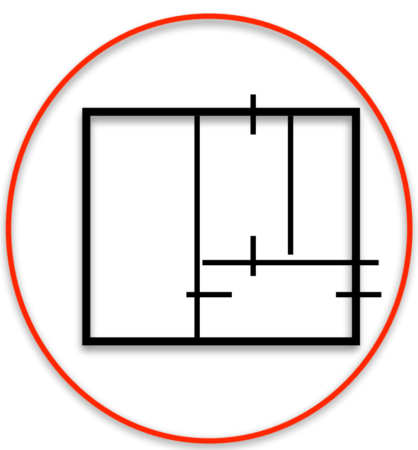
Horizontal boxes + double tennis court



$$nA_n^t = -\frac{1}{3} A_{n-1}^t, \qquad nA_n^s = -A_{n-1}^s + \frac{1}{3} A_{n-1}^t$$

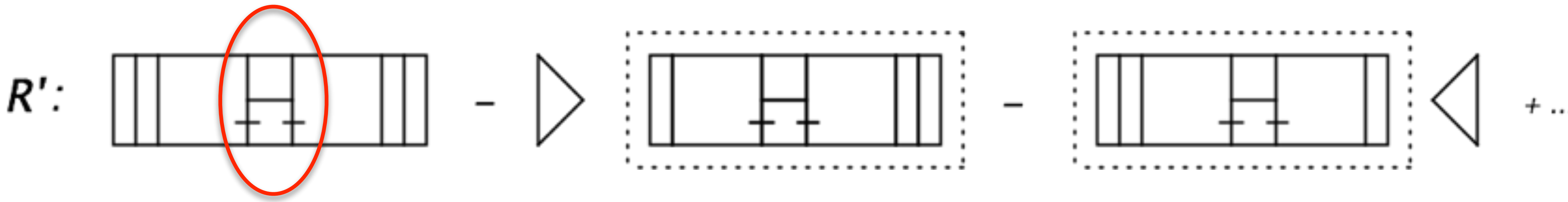
$$A_n^t = \frac{(-1)^n}{3^{n-3}} \frac{1}{n!}, \qquad A_n^s = \frac{1}{2} \frac{(-1)^n}{3^{n-3}} \frac{1}{n!} - \frac{1}{2} (-1)^n \frac{1}{n!}$$

$$(-g^2 s)^{n-1} (-g^2 t) \qquad (-g^2 s)^n$$

R-operation and Recurrence Relation

D=6 N=2

Horizontal boxes + tennis court



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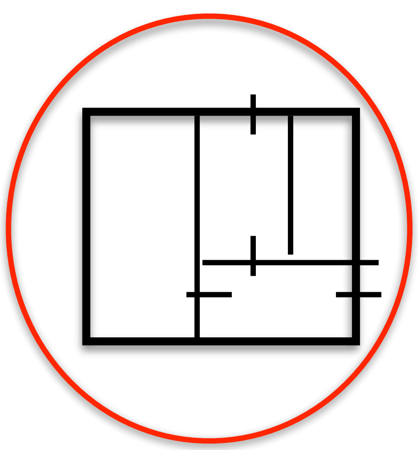
Horizontal boxes + double tennis court



$$nA_n^t = -\frac{1}{3}A_{n-1}^t, \qquad nA_n^s = -A_{n-1}^s + \frac{1}{3}A_{n-1}^t$$

$$\downarrow$$

$$A_n^t = \frac{(-1)^n}{3^{n-3}}\frac{1}{n!}, \qquad A_n^s = \frac{1}{2}\frac{(-1)^n}{3^{n-3}}\frac{1}{n!} - \frac{1}{2}(-1)^n\frac{1}{n!}$$

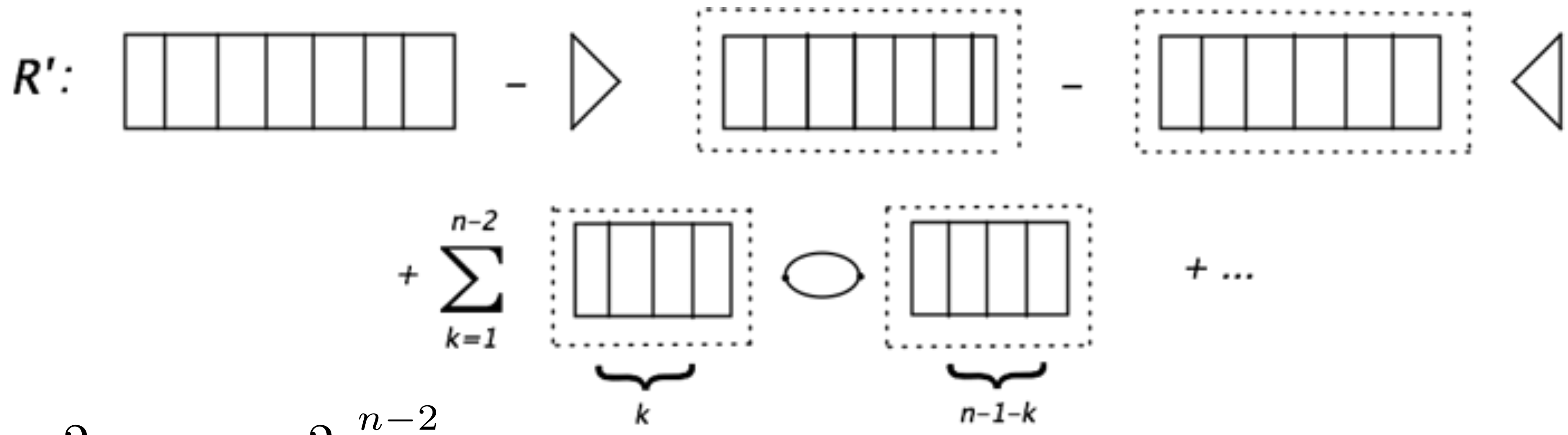
$$(-g^2 s)^{n-1}(-g^2 t) \qquad (-g^2 s)^n$$

- **Similar relations one can get for all other series**
- **All of them have 1/n! behavior**
- **Number of these series group as n!**

R-operation and Recurrence Relation

D=8 N=1

Horizontal boxes

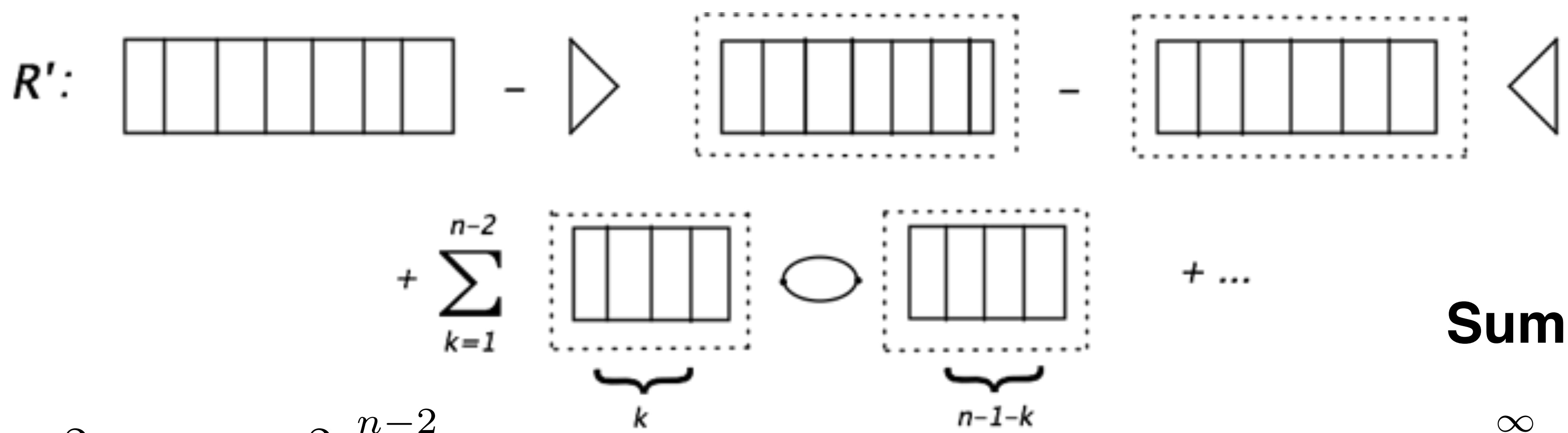


$$nA_n = -\frac{2}{4!}A_{n-1} + \frac{2}{5!}\sum_{k=1}^{n-2} A_k A_{n-1-k}, \quad n \geq 3$$

R-operation and Recurrence Relation

D=8 N=1

Horizontal boxes



Summation

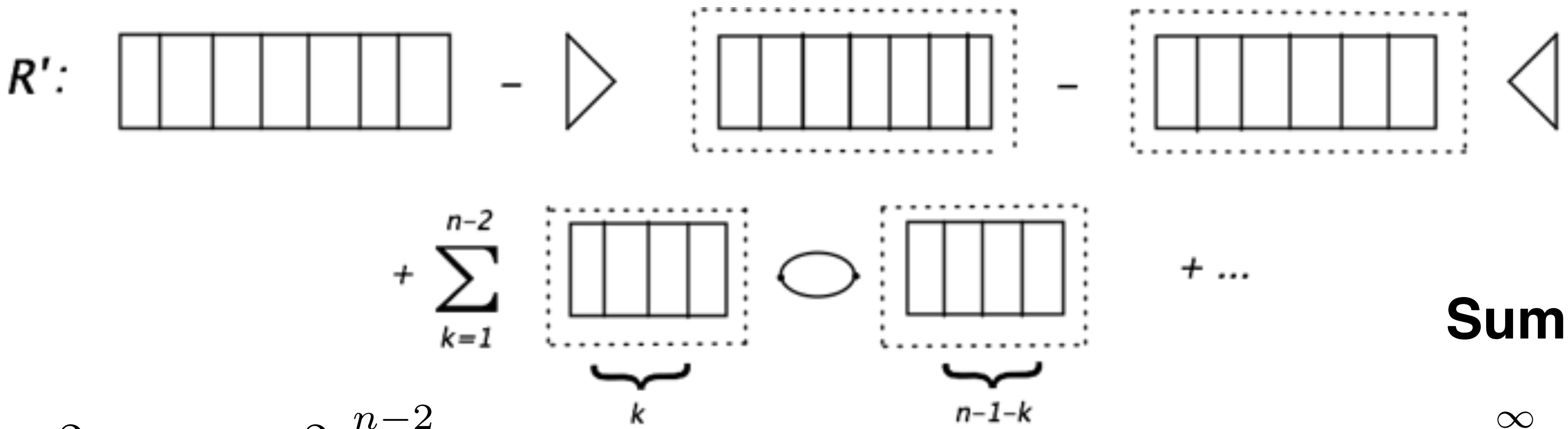
$$nA_n = -\frac{2}{4!}A_{n-1} + \frac{2}{5!}\sum_{k=1}^{n-2} A_k A_{n-1-k}, \quad n \geq 3$$

$$\Sigma_m(z) = \sum_{n=m}^{\infty} A_n(-z)^n$$

R-operation and Recurrence Relation

D=8 N=1

Horizontal boxes



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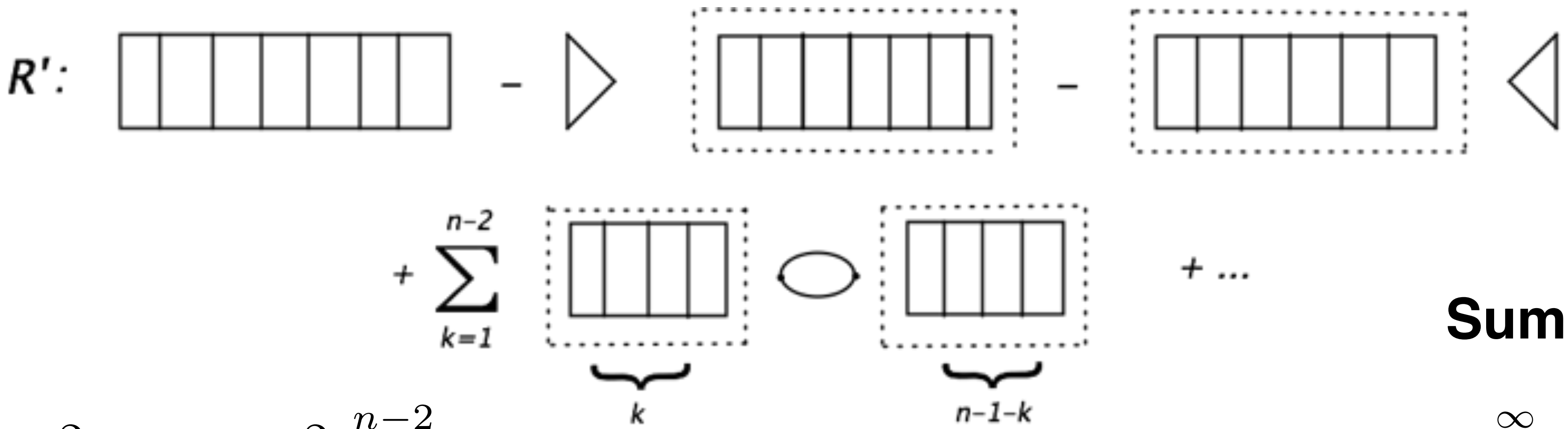
$$\Sigma_m(z) = \sum_{n=m}^{\infty} A_n(-z)^n$$

$$-\frac{d}{dz}\Sigma_3 = -\frac{2}{4!}\Sigma_2 + \frac{2}{5!}\Sigma_1\Sigma_1.$$

R-operation and Recurrence Relation

D=8 N=1

Horizontal boxes



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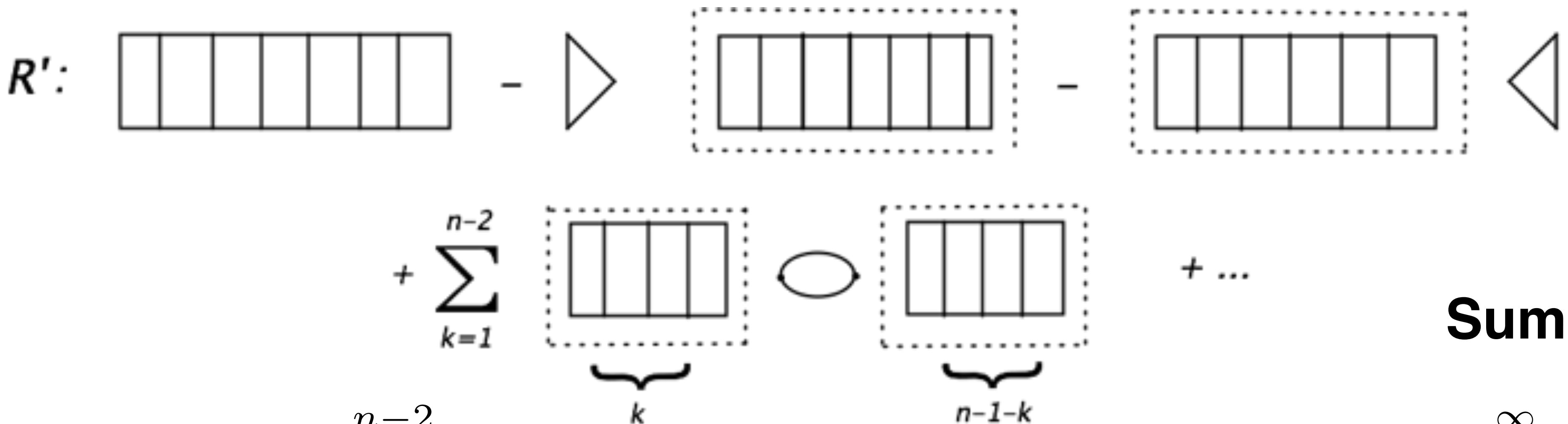
$$-\frac{d}{dz}\Sigma_3 = -\frac{2}{4!}\Sigma_2 + \frac{2}{5!}\Sigma_1\Sigma_1. \quad \Sigma_3 = \Sigma_1 + A_1 z - A_2 z^2, \quad \Sigma_2 = \Sigma_1 + A_1 z, \quad A_1 = \frac{1}{3!}, \; A_2 = -\frac{1}{3!4!}$$

R-operation and Recurrence Relation

D=8 N=1

Horizontal boxes



Summation

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$$\Sigma \equiv \Sigma_1$$

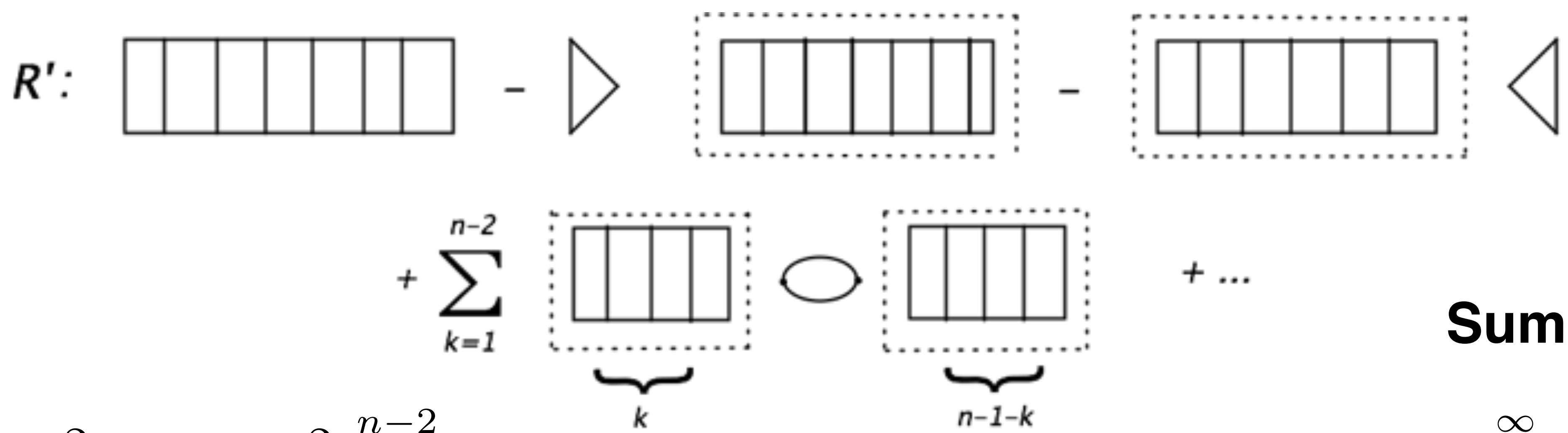
Diff eqn

$$\Sigma' = -\frac{1}{3!} + \frac{2}{4!}\Sigma - \frac{2}{5!}\Sigma^2$$

R-operation and Recurrence Relation

D=8 N=1

Horizontal boxes



Summation

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$$\Sigma \equiv \Sigma_1$$

Diff eqn

$$\Sigma' = -\frac{1}{3!} + \frac{2}{4!}\Sigma - \frac{2}{5!}\Sigma^2$$

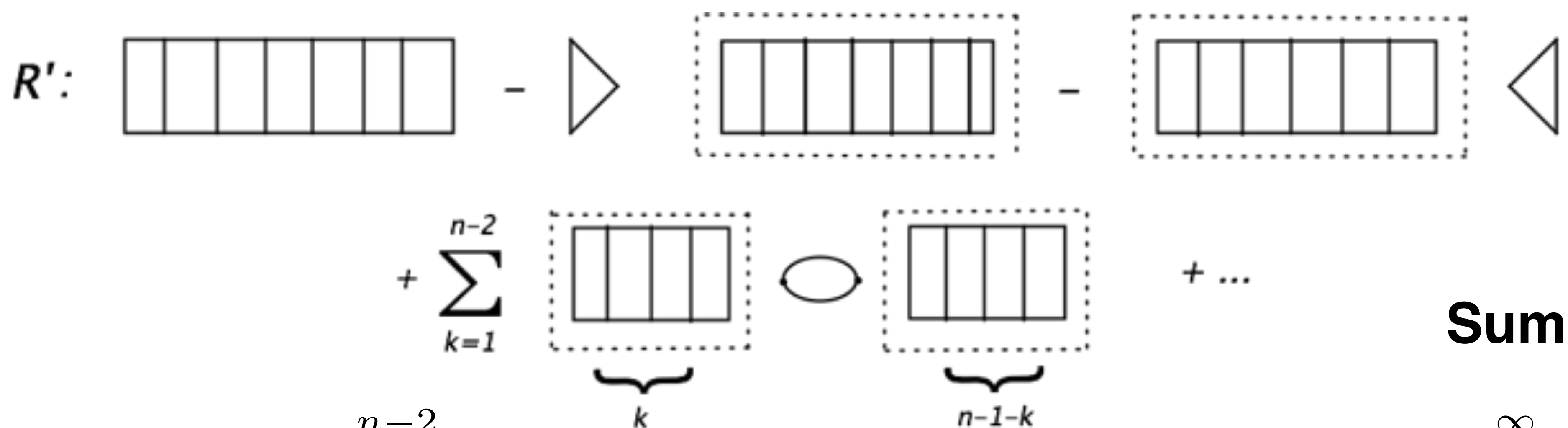
$$\Sigma(z) = -\sqrt{5/3}\,\frac{4\tan[z/(8\sqrt{15})]}{1 - \tan[z/(8\sqrt{15})]\sqrt{5/3}}$$

$$z = g^2 s^2/\epsilon$$

R-operation and Recurrence Relation

D=8 N=1

Horizontal boxes



Summation

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$$\Sigma(z) = -\sqrt{5/3}\,\frac{4\tan[z/(8\sqrt{15})]}{1 - \tan[z/(8\sqrt{15})]\sqrt{5/3}}$$

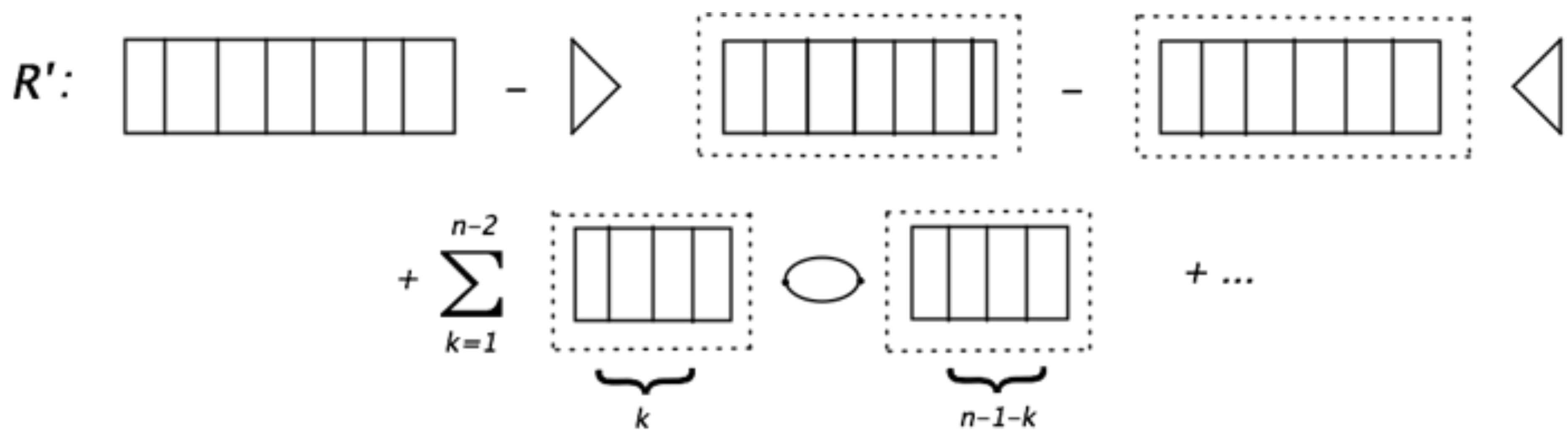
$$z = g^2 s^2/\epsilon$$

$$\Sigma(z) = -(z/6 + z^2/144 + z^3/2880 + 7z^4/414720 + \dots)$$

R-operation and Recurrence Relation

D=10 N=1

Horizontal boxes



$$nA_n^t = -2\frac{2}{7!}A_{n-1}^t + \frac{1}{3\cdot 7!}\sum_{k=1}^{n-2} A_k^t A_{n-1-k}^t,$$

$$nA_n^s = -2\left[\frac{1}{3\cdot 5!}A_{n-1}^s - \frac{6}{7!}A_{n-1}^t\right]$$

$$+ \frac{3}{7!}\sum_{k=1}^{n-2}\left(2A_k^s A_{n-1-k}^s - A_k^s A_{n-1-k}^t - A_k^t A_{n-1-k}^s + \frac{5}{9}A_k^t A_{n-1-k}^t\right)$$

$A_1^s = A_1^t = 1/5!$

All loop Exact Recurrence Relation

D=6 N=2

s-channel term $S_n(s,t)$ **t-channel term** $T_n(s,t)$ $T_n(s,t) = S_n(t,s)$

Exact relation for ALL diagrams

$$nS_n(s,t) = -2s \int_0^1 dx \int_0^x dy \; (S_{n-1}(s,t') + T_{n-1}(s,t'))$$

$$n \geq 4$$

$$t' = t(x-y) - sy$$

$$S_3 = -s/3, \; T_3 = -t/3$$

All loop Exact Recurrence Relation

D=6 N=2

s-channel term $S_n(s,t)$ **t-channel term** $T_n(s,t)$ $T_n(s,t) = S_n(t,s)$

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$$n \geq 4$$

$$t' = t(x-y) - sy$$

$$S_3 = -s/3, \; T_3 = -t/3$$

Summation

$$\Sigma_k(s,t,z) = \sum_{n=k}^{\infty} (-z)^n S_n(s,t)$$

All loop Exact Recurrence Relation

D=6 N=2

s-channel term $S_n(s,t)$ **t-channel term** $T_n(s,t)$ $T_n(s,t) = S_n(t,s)$

Exact relation for ALL diagrams

$$nS_n(s,t) = -2s \int_0^1 dx \int_0^x dy \, \left(S_{n-1}(s,t') + T_{n-1}(s,t')\right)$$

$n \geq 4$

$t' = t(x-y) - sy$

$$S_3 = -s/3, \; T_3 = -t/3$$

Summation $$\Sigma_k(s,t,z) = \sum_{n=k}^{\infty} (-z)^n S_n(s,t)$$

Diff eqn $$\frac{d}{dz}\Sigma_4(s,t,z) = 2s \int_0^1 dx \int_0^x dy \, \left(\Sigma_3(s,t',z) + \Sigma_3(t',s,z)\right)|_{t'=xt+yu}$$

All loop Exact Recurrence Relation

D=6 N=2

s-channel term $S_n(s,t)$ **t-channel term** $T_n(s,t)$ $T_n(s,t) = S_n(t,s)$

Exact relation for ALL diagrams

$$nS_n(s,t) = -2s \int_0^1 dx \int_0^x dy \; (S_{n-1}(s,t') + T_{n-1}(s,t'))$$

$n \geq 4$

$t' = t(x-y) - sy$

$$S_3 = -s/3, \; T_3 = -t/3$$

Summation $$\Sigma_k(s,t,z) = \sum_{n=k}^{\infty} (-z)^n S_n(s,t)$$

Diff eqn $$\frac{d}{dz}\Sigma_4(s,t,z) = 2s \int_0^1 dx \int_0^x dy \; (\Sigma_3(s,t',z) + \Sigma_3(t',s,z))|_{t'=xt+yu}$$

$$\Sigma_4(s,t,z) = \Sigma_3(s,t,z) + S_3(s,t)z^3 \qquad \Sigma(s,t,z) = z^{-2}\Sigma_3(s,t,z)$$

All loop Exact Recurrence Relation

D=6 N=2

s-channel term $S_n(s,t)$ **t-channel term** $T_n(s,t)$ $T_n(s,t) = S_n(t,s)$

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$n \geq 4$

$t' = t(x-y) - sy$

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Summation

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Diff eqn

$$\frac{d}{dz}\Sigma_4(s,t,z) = 2s\int_0^1 dx \int_0^x dy \, \left(\Sigma_3(s,t',z) + \Sigma_3(t',s,z)\right)|_{t'=xt+yu}$$

$$\Sigma_4(s,t,z) = \Sigma_3(s,t,z) + S_3(s,t)z^3 \qquad \Sigma(s,t,z) = z^{-2}\Sigma_3(s,t,z)$$

$$\frac{d}{dz}\Sigma(s,t,z) = s - \frac{2}{z}\Sigma(s,t,z) + 2s\int_0^1 dx \int_0^x dy \, \left(\Sigma(s,t',z) + \Sigma(t',s,z)\right)|_{t'=xt+yu}$$

All loop Exact Recurrence Relation

D=8 N=1

s-channel term $S_n(s,t)$ **t-channel term** $T_n(s,t)$ $T_n(s,t) = S_n(t,s)$

Exact relation for ALL diagrams

$$nS_n(s,t) = -2s^2 \int_0^1 dx \int_0^x dy\ y(1-x)\ (S_{n-1}(s,t') + T_{n-1}(s,t'))|_{t'=tx+yu}$$

$$+ \quad s^4 \int_0^1 dx\ x^2(1-x)^2 \sum_{k=1}^{n-2} \sum_{p=0}^{2k-2} \frac{1}{p!(p+2)!}\ \frac{d^p}{dt'^p}(S_k(s,t') + T_k(s,t')) \times$$

$$S_1 = \frac{1}{12},\ T_1 = \frac{1}{12} \qquad \times \frac{d^p}{dt'^p}(S_{n-1-k}(s,t') + T_{n-1-k}(s,t'))|_{t'=-sx}\ (tsx(1-x))^p$$

summation $\Sigma_3(s,t,z) = \Sigma_1(s,t,z) - S_2(s,t)z^2 + S_1(s,t)z,\ \Sigma_2(s,t,z) = \Sigma_1(s,t,z) + S_1(s,t)z$

Diff eqn

$$\frac{d}{dz}\Sigma(s,t,z) = -\frac{1}{12} + 2s^2 \int_0^1 dx \int_0^x dy\ y(1-x)\ (\Sigma(s,t',z) + \Sigma(t',s,z))|_{t'=tx+yu}$$

$$-s^4 \int_0^1 dx\ x^2(1-x)^2 \sum_{p=0}^{\infty} \frac{1}{p!(p+2)!}(\frac{d^p}{dt'^p}(\Sigma(s,t',z) + \Sigma(t',s,z))|_{t'=-sx})^2\ (tsx(1-x))^p.$$

All loop Exact Recurrence Relation

D=10 N=1

s-channel term $S_n(s,t)$ **t-channel term** $T_n(s,t)$ $T_n(s,t) = S_n(t,s)$

Exact relation for ALL diagrams

$$nS_n(s,t) = -s^3 \int_0^1 dx \int_0^x dy\; y^2(1-x)^2\; (S_{n-1}(s,t') + T_{n-1}(s,t'))|_{t'=tx+yu}$$

$$+ \quad s^5 \int_0^1 dx\; x^3(1-x)^3 \sum_{k=1}^{n-2} \sum_{p=0}^{3k-2} \frac{1}{p!(p+3)!} \frac{d^p}{dt'^p}(S_k(s,t') + T_k(s,t')) \times$$

$$\times \frac{d^p}{dt'^p}(S_{n-1-k}(s,t') + T_{n-1-k}(s,t'))|_{t'=-sx}\; (tsx(1-x))^p$$

$S_1 = \frac{s}{5!}, \; T_1 = \frac{t}{5!}$

summation

$$\Sigma_3(s,t,z) = \Sigma_1(s,t,z) - S_2(s,t)z^2 + S_1(s,t)z, \;\; \Sigma_2(s,t,z) = \Sigma_1(s,t,z) + S_1(s,t)z$$

Diff eqn

$$\frac{d}{dz}\Sigma(s,t,z) = -\frac{s}{5!} + s^3 \int_0^1 dx \int_0^x dy\; y^2(1-x)^2\; (\Sigma(s,t',z) + \Sigma(t',s,z))|_{t'=tx+yu}$$

$$-s^5 \int_0^1 dx\; x^3(1-x)^3 \sum_{p=0}^{\infty} \frac{1}{p!(p+3)!} (\frac{d^p}{dt'^p}(\Sigma(s,t',z) + \Sigma(t',s,z))|_{t'=-sx})^2\; (tsx(1-x))^p$$

The Fixed Point and Finiteness

D=6 N=2

Diff eqn for the sum of two channels

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The fixed point

$\epsilon \to 0$

$$\Sigma(s,t,\infty)+\Sigma(t,s,\infty) = -1$$

Finite value

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Finiteness:

s-t channel s+t<0

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t-u channel t+u<0

The Fixed Point and Finiteness

D=6 N=2

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Finiteness:

- **s-t channel s+t<0**
- **s-u channel s+u<0**
- **t-u channel t+u<0**

} **incompatible since s+t+u=0**

Summary

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D=6 N=2

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D=6 N=2 D=8 N=1 D=10 N=1

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- **This equation possesses the fixed point. The STABLE fixed point would imply the FINITENESS of the theory when** $\epsilon \to +0$

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D=6 N=2

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Summary

D=6 N=2

- **Equation for the total sum has a fixed point** $\Sigma(s,t,\infty) + \Sigma(t,s,\infty) = -1$
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- Existence of a fixed point does not contradict the equation
- Example of the horizontal boxes demonstrates that the limit $\epsilon\to+0$ might be similar to a gauge theory in D=4