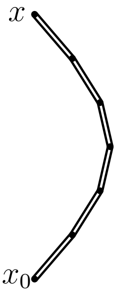
QCD cusp anomalous dimension

Andrey Grozin

Wilson lines

$$W = \left\langle P \exp\left[ig \int_C dx^\mu A_\mu(x) \right] \right\rangle \qquad A_\mu(x) = A^a_\mu(x) t^a_R$$

R — *some* representation of the color group



$$W(x, x_0) = W(x, x_{N-1}) \cdots W(x_2, x_1) W(x_1, x_0)$$

$$A_\mu(x) \to U(x) A_\mu(x) U^{-1}(x) + \frac{i}{g} (\partial_\mu U(x)) U^{-1}(x)$$

$$W(x_{i+1}, x_i) \to U(x_{i+1}) W(x_{i+1}, x_i) U^{-1}(x_i)$$

$$W(x, x_0) \to U(x) W(x, x_0) U^{-1}(x_0)$$

$$\varphi^+(x) W(x, x_0) \varphi'(x_0) = \text{inv}$$

$$W_0 = Z_W(\alpha_s(\mu), a(\mu))W(\mu)$$

IR divergences of a scattering amplitude can be found in the eikonal approximation. It turns the amplitude to a product of straight semi-infinite Wilson lines along its external momenta. However, it introduces UV divergences which were absent in the original amplitude. These UV divergences are equal to the IR divergences of the amplitude with the opposite sign.

HQET

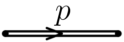
- QED: Bloch, Nordsieck (1937)
- Wilson lines: Gervais, Neveu (1980); Aref'eva (1980)
- HQET: (1990)

QCD with n_f flavors plus a single heavy colored particle (R)
$P = Mv + p$, $p \ll M$, $p_i \ll M$, $m_i \ll M$

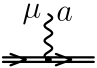
$$L = h^*_{v0} iD_0 \cdot v h_{v0} + L_{\text{QCD}} \qquad \left[\sum_i h^*_{v_i 0} iD_0 \cdot v_i h_{v_i 0} \right]$$

At the leading order in $1/M$ the heavy-particle spin does not interact with gluons and can be freely rotated (heavy quark symmetry). Moreover, it can be switched off (superflavor symmetry).

Renormalization



$$= iS_{h0}(p \cdot v) \qquad S_{h0}(\omega) = \frac{1}{\omega}$$



$$= ig_0 v^\mu t_R^a$$

$$h_{v0} = Z_h^{1/2}(\alpha_s(\mu), a(\mu)) h_v(\mu)$$

$$= iS_h(\omega) \qquad S_h(\omega) = \frac{1}{\omega - \Sigma_h(\omega)}$$

$$\log \frac{S_h(\omega)}{S_{h0}(\omega)} = \log Z_h + \mathcal{O}(\varepsilon^0)$$

$$\gamma_h(\alpha_s(\mu)) = \frac{d \log Z_h(\alpha_s(\mu), a(\mu))}{d \log \mu}$$

Z_h does not depend on ω

Coordinate space

$$= \theta(x \cdot v)\delta(x_\perp)$$

v rest frame $\delta(\vec{x}\,)S_{h0}(x^0)$, $iS_{h0}(t) = \theta(t)$

$$=$$

$$\times \left\langle P \exp\left[i g_0 \int_x^y dx_\mu A_0^{a\mu}(x) t_R^a \right] \right\rangle$$

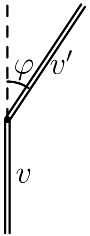
v rest frame $S_h(t) = S_{h0}(t)W(t)$

$$W(t) = \left\langle P \exp\left[i g_0 \int_0^t dt\, v_\mu A_0^{a\mu}(vt) t^a \right] \right\rangle$$

$$\log W(t) = \log Z_h + \mathcal{O}(\varepsilon^0)$$

Z_h does not depend on t

Cusp



$$J_0 = h^*_{v'0} h_{v0} = Z_J(\alpha_s(\mu), \varphi) J(\mu) \qquad \cosh\varphi = v \cdot v'$$

$$\Gamma(\alpha_s(\mu), \varphi) = \frac{d \log Z_J(\alpha_s(\mu), \varphi)}{d \log \mu}$$

Dependence of the Isgur–Wise function on μ

$1/\varepsilon$ IR divergence of massive QCD form factors

Renormalization

$$V(\omega,\omega',\varphi) = 1 + \Lambda(\omega,\omega',\varphi)$$

$$\log V(\omega,\omega',\varphi) = \log Z_J(\varphi) - \log Z_h + \mathcal{O}(\varepsilon^0)$$

$\varphi = 0$: Ward identity

$$\Lambda(\omega,\omega',0) = -\frac{\Sigma_h(\omega) - \Sigma_h(\omega')}{\omega - \omega'}$$

$$V(\omega,\omega',0) = \frac{S_h^{-1}(\omega) - S_h^{-1}(\omega')}{\omega - \omega'}$$

$$\log V(\omega,\omega',0) = -\log Z_h + \mathcal{O}(\varepsilon^0) \qquad Z_J(\alpha_s,0) = 1$$

$$\Gamma(\alpha_s,0) = 0$$

Coordinate space

$$\langle h_{v'0}(x') J_0(0) h^*_{v0}(x) \rangle = (\delta\text{-functions}) \times W(t, t', \varphi)$$
$$\log W(t, t', \varphi) = \log Z_J(\varphi) + \log Z_h + \mathcal{O}(\varepsilon^0)$$

$\varphi = 0$: $W(t, t', 0) = W(t + t')$, $Z_J(0) = 1$

$\varphi \to 0$

$$\Gamma(\alpha_s, \varphi) = \sum_{n=1}^{\infty} B_n(\alpha_s)\varphi^{2n}$$

$B_1(\alpha_s)$ — Bremsstrahlung function
A classical pointlike charge + free electromagnetic field

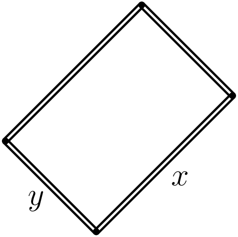
$$\Delta E = 2\pi B_1(\alpha) \int_{-\infty}^{+\infty} dt\,(-a^2(t))$$
$$B_1(\alpha) = \frac{\alpha}{3\pi}$$

Also in $\mathcal{N} = 4$ SYM Correa, Henn, Maldacena, Sever (2012)

$\varphi \to \infty$

$$\Gamma(\alpha_s, \varphi) = K(\alpha_s)\varphi + \mathcal{O}(\varphi^0)$$

K — light-like cusp anomalous dimension
Related to renormalization of Wilson lines with light-like segments: Korchemskaya, Korchemsky (1992)



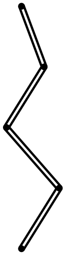
$$x^2 = y^2 = 0$$

$$\frac{d\log W}{d\log\mu} + 2K(\alpha_s)\left[\log(\mu^2(x\cdot y) + i0) + \log(-\mu^2(x\cdot y) + i0)\right] + \gamma(\alpha_s) = 0$$

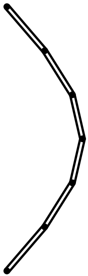
In SCET many anomalous dimensions have log parts with the coefficient $K(\alpha_s)$ plus non-log parts

The DGLAP kernels $P(x)$ have singularities at $x \to 1$:
$\delta(1-x)$, $1/(1-x)_+$ plus weaker ones like $\log^n(1-x)$
The coefficient of $1/(1-x)_+$ is $K(\alpha_s)$
Similarly, the coefficient of $1/(x-y)_+$ in the ERBL kernels $V(x,y)$ is $K(\alpha_s)$

IR $1/\varepsilon^2$ divergences of form factor of massless particles are also determined by $K(\alpha_s)$



$$Z = Z_h \prod Z_J(\varphi_i)$$



$\varphi_i \sim 1/N$

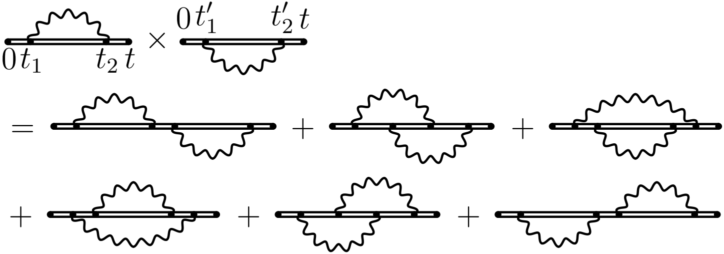
$\log Z$: N contributions $\sim 1/N^2$ each

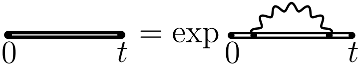
UV — small distances, any smooth line is straight
The only UV divergence — residual mass $\Sigma_h(0)$
Linear $\Rightarrow 0$

Exponentiation

QED $n_f = 0$

$0 < t_1 < t_2 < t$, $0 < t'_1 < t'_2 < t$





$W(t) = e^{w(t)}$

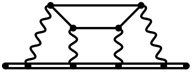
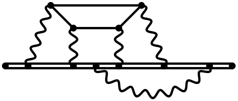
$w = \log Z_h + \mathcal{O}(\varepsilon^0)$

$$\gamma_h(\alpha) = 2(a - 3)\frac{\alpha}{4\pi}$$

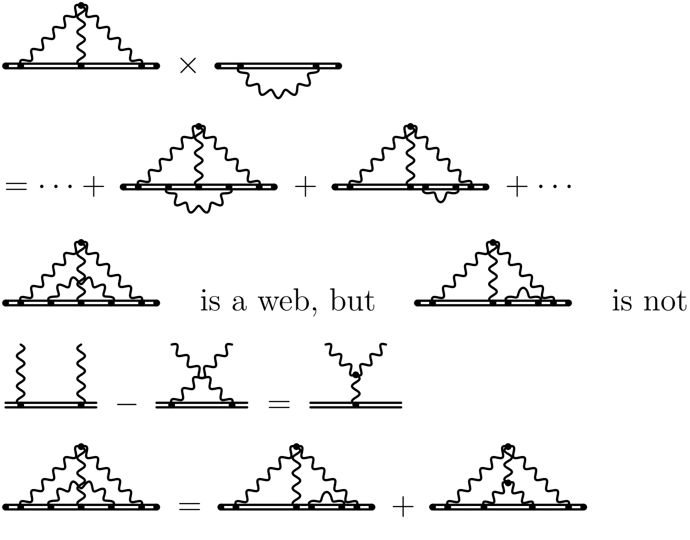
QED $n_f \neq 0$

 $= \exp\Big[$  $+$  $+ \cdots \Big]$

$$W = \exp\big[w_2 + w_4 + \cdots\big]$$

 $\times$  $= \cdots +$  $+ \cdots$

Nonabelian exponentiation



Color factors

$$\mathrm{Tr}\, t_R^a t_R^b = T_R \delta^{ab} \qquad t_R^a t_R^a = C_R \mathbf{1}_R \qquad N_R = \mathrm{Tr}\, \mathbf{1}_R$$

$$d_{RR'} = \frac{d_R^{abcd} d_{R'}^{abcd}}{N_R} \qquad d_R^{abcd} = \mathrm{Tr}\, t_R^{(a} t_R^b t_R^c t_R^{d)}$$

$$\mathrm{Tr}\, t_R^a t_R^a = C_R N_R = T_R N_A \Rightarrow C_R = \frac{T_R N_A}{N_R} \quad \text{e.g.} \quad C_A = T_A$$

γ_h, Γ without n_f:

1 loop C_R

2 loops $C_R C_A$

3 loops $C_R C_A^2$

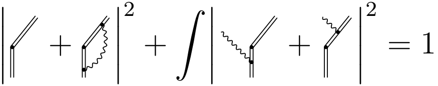
4 loops $C_R C_A^3$, d_{RA}

At least 1 quark loop (n_f) — anything

Up to 3 loops — Casimir scaling

4 loops — d_{RA}, $d_{RF} n_f$

1 loop



Classical electrodynamics (Landau, Lifshitz; Jackson)

$$dE = \frac{e^2}{2\pi^2}(\varphi \coth \varphi - 1)\, d\omega$$

In any frame!

$$\int \frac{d^4k\, \delta(k^2)\delta(k \cdot v - \omega)}{(k \cdot v_1)(k \cdot v_2)} \quad \text{does not depend on } v$$

In textbooks

$$\varphi \coth \varphi - 1 = \frac{1}{2u} \log \frac{1+u}{1-u} - 1 \qquad u = \tanh \varphi$$

Dim. reg.

$$dE = C(\varepsilon, \varphi) \frac{e^2 \omega^{-2\varepsilon}}{2\pi^2} (\varphi \coth \varphi - 1)\, d\omega \qquad C(0, \varphi) = 1$$

Photon emission probability dE/ω

$$F = 1 - \frac{C(\varepsilon)}{2} \int_\lambda^\infty \frac{e^2}{2\pi^2} (\varphi \coth \varphi - 1) \frac{d\omega}{\omega^{1+2\varepsilon}}$$
$$= 1 - 2\frac{\alpha}{4\pi\varepsilon} (\varphi \coth \varphi - 1 + \mathcal{O}(\varepsilon))$$

$$\Gamma = 4\frac{\alpha}{4\pi} (\varphi \coth \varphi - 1)$$

2 loops

γ_h

- Aoyama (1982) — wrong!
- Knauss, Scharhorst (1984) ($n_f = 0$)
- Broadhurst, Gray, Schilcher (1991) (Z_Q^{os})
- Ji, Musolf (1991); Broadhurst, Grozin (1991) (HQET)

2 loops

$\Gamma(\varphi)$

- Knauss, Scharhorst (1984) ($n_f = 0$)
  complicated double and triple integrals
- Korchemsky, Radyushkin
  Yad. Fiz. (1986), Phys. Lett. B (1986),
  Nucl. Phys. B (1987) Referee: nobody will ever need this; now 789 citations
  3 single integrals
- Kilian, Manakos, Mannel (1993)
- Grozin (2004) 2 integrals via $\mathrm{Li}_{2,3}$
- Kidonakis (2009) (only $\mathrm{Li}_{2,3}$)

3 loops

γ_h

- Melnikov, van Ritbergen (2000) (Z_Q^{os})
- Chetyrkin, Grozin (2003) (HQET)

K

- Moch, Vermaseren, Vogt (2004) (DGLAP)
- Moch, Vermaseren, Vogt (2005) (form factors)

$\Gamma(\varphi)$

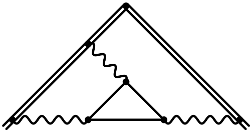
Grozin, Henn, Korchemsky, Marquard (2015–16)

$$K(\alpha_s) = 4 C_R A$$

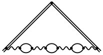
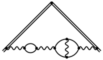
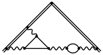
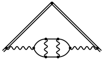
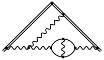
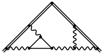
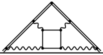
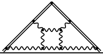
$$\Gamma(\alpha_s, \varphi) = \Omega(A, \varphi)$$

$$= C_R \left[\Omega_1(\varphi) A + C_A \Omega_A(\varphi) A^2 + C_A^2 \Omega_{AA}(\varphi) A^3 + \cdots \right]$$

$C_R C_A T_F n_f$



4 loops

color	example	γ_h	$\varphi \ll 1$	$\Gamma(\varphi)$	1L	$\varphi \gg 1$
$C_R(T_F n_f)^3$		[1]		[2,3]	✓	
$C_R C_F (T_F n_f)^2$		[4,5]		[4,5]	✓	[4,5,6]
$C_R C_A (T_F n_f)^2$		[7,8]		[8]	+	[6,9,10]
$C_R C_F^2 T_F n_f$		[11]		[11]	✓	
$C_R C_F C_A T_F n_f$		[8]		[8]	+	[12,8,13]
$C_R C_A^2 T_F n_f$		[8]	[8]		−	[12,13]
$d_{RF} n_f$		[14]	[14,8]	[15]	−	[16,17]
$C_R C_A^3$		[18]	[18]			[12,13]
d_{RA}		[18]	[18]			[12,13]

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Conjecture

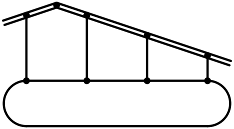
+ $C_R C_A (T_F n_f)^2$ 3 terms at $\varphi \to 0$ and $\varphi \to \infty$ known
+ $C_R C_F C_A T_F n_f$ 2 terms at $\varphi \to 0$ known; $\varphi \to \infty$ was known numerically. Analytical expression was predicted and later confirmed.
– $C_R C_A^2 T_F n_f$ 2 terms at $\varphi \to 0$
  Brüser, Grozin, Henn, Stahlhofen (2019)
– $d_{RF} n_f$

$$A = \cdots + \frac{d_{RF}}{C_R} n_f \left(\frac{\alpha_s}{4\pi}\right)^4 + \cdots$$

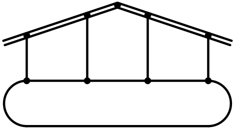
Grozin, Henn, Stahlhofen (2017)

A sad story

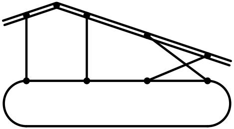
A. G. Grozin, R. N. Lee, A. V. Smirnov, V. A. Smirnov, M. Steinhauser (2019–20)



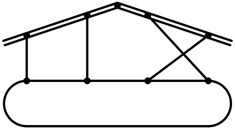
a1



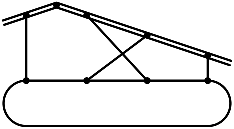
a2



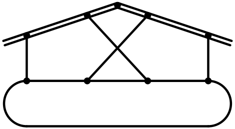
b1



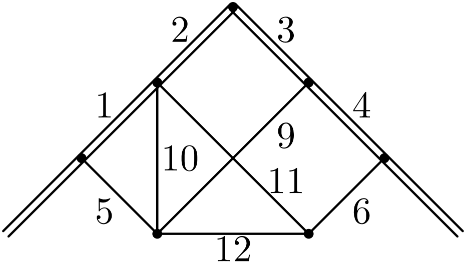
b2



c1



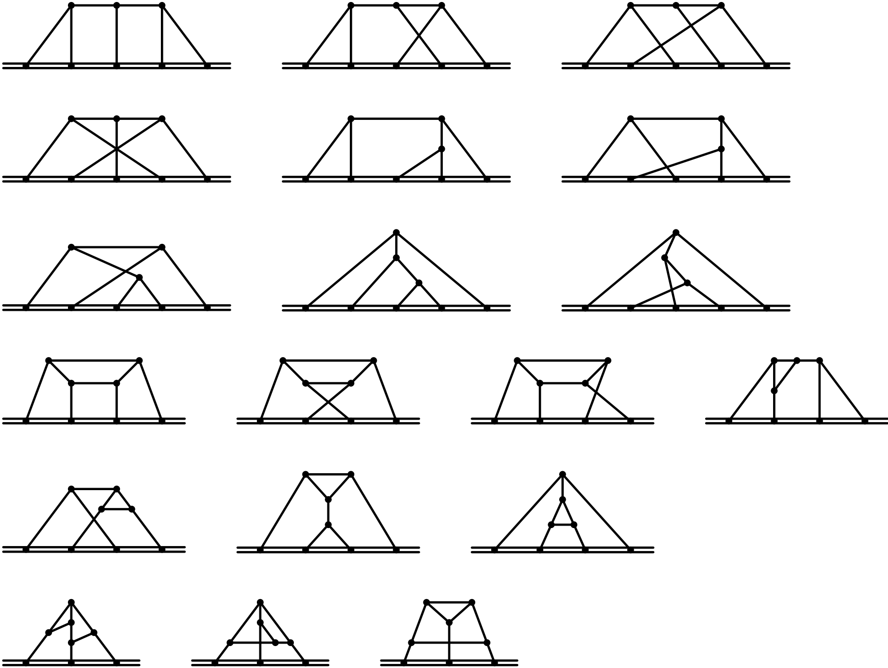
c2



Sector in b2: DEs cannot be reduced to ε form

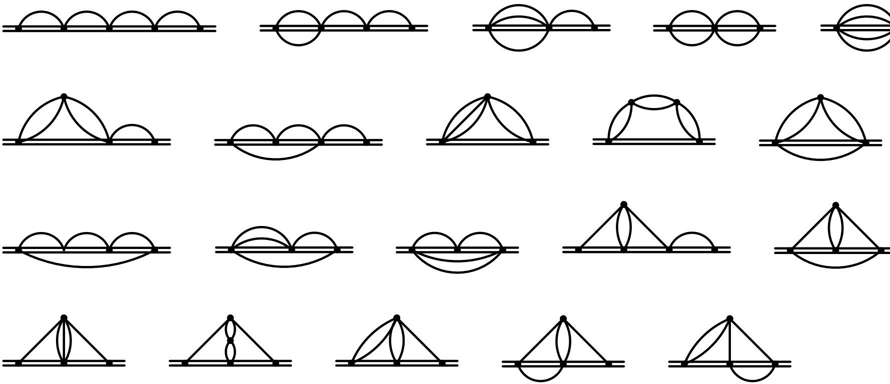
γ_h

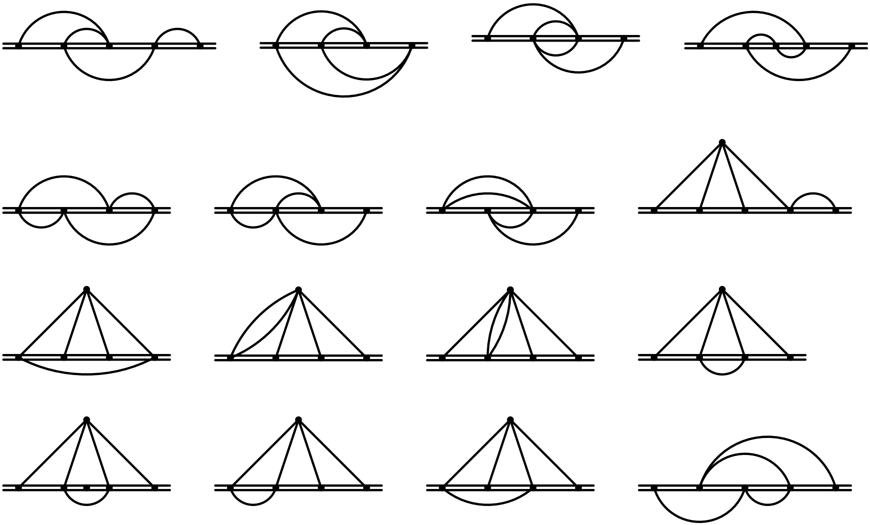
- Generate diagrams (qgraf)
- Partial fractioning $\Rightarrow$ 19 topologies

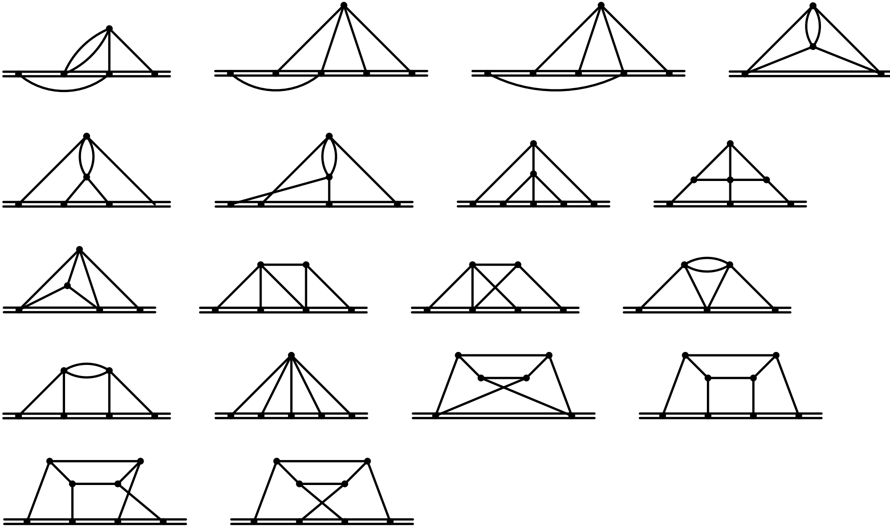


10–12	Grozin, Henn, Stahlhofen (2017)
1	Brüser, Grozin, Henn, Stahlhofen (2019)
all	Lee, Pikelner (2023)

54 master integrals







- 13 recursively 1-loop
- 10 ${}_3F_2(1)$ Beneke, Braun (1994); Grozin (2000, 2004)
  1 reduces to Γ functions
- all: Lee, Pikelner (2023) DRA, ε exlansions

$\gamma_h - \gamma_q$

γ_h, γ_q at L loops: up to a^L

$\gamma_h - \gamma_q$ $(R = F)$:

- up to 2 loops — gauge invariant
- 3 loops — up to a
- 4 loops — up to a^2 (γ_q — more color structures)

QED

- γ_h — only 1 loop gauge dependent (c-webs)
- γ_q — only 1 loop gauge dependent (LKF)
  Grozin (2010)

$\gamma_h - \gamma_q$ gauge invariant to all orders

The same for Z_Q^{os}

Heavy-quark field: QCD/HQET matching

$$Q(\mu) = z(\mu) h_v(\mu)$$

$$z(\mu) = \frac{Z_h(\alpha_s^{(n_l)}(\mu), a^{(n_l)}(\mu)) Z_Q^{\text{os}}(g_0^{(n_f)}, a_0^{(n_f)})}{Z_Q(\alpha_s^{(n_f)}(\mu), a^{(n_f)}(\mu)) Z_h^{\text{os}}(g_0^{(n_l)}, a_0^{(n_l)})} = \text{finite}$$

$$Z_h^{\text{os}}(g_0^{(n_l)}, a_0^{(n_l)}) = 1$$

$Z_Q^{\text{os}}(g_0^{(n_f)}, a_0^{(n_f)})$ at 4 loops: $1/\varepsilon^4 \ldots 1/\varepsilon$ all known analytically

$z(\mu)$ — the same pattern of a dependence

$\Gamma(\varphi)$ at $\varphi \ll 1$

$$v' = v + \delta v \qquad \delta v = v(\cosh\varphi - 1) + n\sinh\varphi$$
$$v \cdot n = 0 \qquad n^2 = -1$$

Expand in δv and average over n direction in the $(d-1)$-dimensional subspace orthogonal to v

$$
\begin{aligned}
\Gamma(\varphi) = 4\frac{\alpha_s}{4\pi}(\varphi\coth\varphi - 1)\Bigg\{ & C_R + \frac{\alpha_s}{4\pi}\left[C_A() + T_F n_f()\right] \\
& + C_R\left(\frac{\alpha_s}{4\pi}\right)^2\left[C_A^2() + C_A T_F n_f() + C_F T_F n_f() + (T_F n_f)^2()\right] \\
& + \left(\frac{\alpha_s}{4\pi}\right)^3\Big[C_R C_A^3() + d_{RA}() + C_R C_A^2 T_F n_f() \\
& \quad + C_R C_F C_A T_F n_f() + C_R C_F^2 T_F n_f() + d_{RF} n_f() \\
& \quad + C_R C_A (T_F n_f)^2() + C_R C_F (T_F n_f)^2() + C_R (T_F n_f)^3()\Big]\Bigg\} \\
+ \varphi^4\left(\frac{\alpha_s}{4\pi}\right)^2\Bigg\{ & C_R C_A() + C_R C_A\frac{\alpha_s}{4\pi}\left[C_A() + T_F n_f()\right] \\
& + \left(\frac{\alpha_s}{4\pi}\right)^2\Bigg[C_R C_A^3() + d_{RA}() + C_R C_A^2 T_F n_f() \\
& \quad + C_R C_F C_A T_F n_f() + d_{RF} n_f() + C_R C_A (T_F n_f)^2()\Bigg]\Bigg\} + \mathcal{O}(\varphi^6, \alpha
\end{aligned}
$$

$\mathcal{N} = 4$ SYM

Supersymmetric cusp $\vartheta = 0$

$SU(N_c)$, $N_c \to \infty$

- B_1 *exactly* in $N_c \alpha_s$
  Correa, Henn, Maldacena, Sever (2012)
-

$$\Gamma = \sum_{L=1}^{\infty} \Gamma_L \left(\frac{N_c \alpha_s}{2\pi} \right)^L \qquad \Gamma_L = \sum_{n=1}^{L} \Gamma_{Ln} \tanh^n \frac{\varphi}{2}$$

Γ_L known to $L = 4$
Henn, Huber (2013)

Arbitrary group B_1
Fiol, Martinez-Montoya, Fukelman (2019)

- Γ_{L1} $\mathcal{O}(\varphi)$: weight $2(L-1)$, homogeneous
- Γ_{L1} $\mathcal{O}(\varphi^3)$, Γ_{L2} $\mathcal{O}(\varphi^2)$: lower weights, non-homogeneous

$$\Gamma = \frac{\alpha_s}{2\pi}\varphi \tanh\frac{\varphi}{2}\left\{C_R - \frac{1}{6}C_R C_A \pi\alpha_s + \frac{1}{24}C_R C_A^2(\pi\alpha_s)^2 - \left(\frac{5}{24}C_R C_A^3 - \frac{d_{RA}}{5}\right)\frac{(\pi\alpha_s)^3}{18}\right\}$$

$$+ \frac{N_c\alpha_s\varphi^4}{2\pi}\left\{\frac{1}{6} - \frac{3}{2}\zeta_3\frac{N_c\alpha_s}{2\pi} + \left(\frac{45}{4}\zeta_5 + \frac{2}{3}\pi^2\zeta_3 - \frac{2}{45}\pi^4\right)\left(\frac{N_c\alpha_s}{2\pi}\right)^2\right\}$$

$$+ \mathcal{O}(\varphi^6, \alpha_s^5)$$

Maximal transcendentality: Kotikov, Lipatov (2003); Kotikov, Lipatov, Onishchenko, Velizhanin (2004)

Euclidean $\phi = \pi - \delta$

$$\Gamma(\pi - \delta) = \frac{r\,V(r)}{\delta} = \frac{\vec{q}^{\,2} V(\vec{q})}{4\pi\delta}$$

at 2 loops: Kilian, Mannel, Ohl (1993)

Conformal symmetry

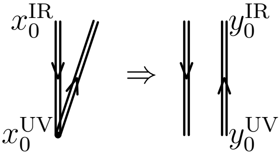
$ds^2 = dx_0^2 + d\vec{x}^{\,2}$

$x_0 = r\cos\delta \qquad \vec{x} = r\vec{n}\sin\delta \qquad ds^2 = dr^2 + r^2(d\delta^2 + \sin^2\delta\, d\vec{n}^{\,2})$

$\delta \ll 1 \qquad r = e^{y_0} \qquad \vec{y} = \delta\vec{n} \quad ds^2 = e^{2y_0}\left(dy_0^2 + d\vec{y}^{\,2}\right)$

conformally flat

Conformal symmetry



$$\log W = \Gamma \log \frac{x_0^{\mathrm{IR}}}{x_0^{\mathrm{UV}}} = V(\vec{y}) \left(y_0^{\mathrm{IR}} - y_0^{\mathrm{UV}} \right)$$

$$\Gamma(\pi - \delta) = \frac{y V(y)}{\delta} \qquad V(y) = \frac{\mathrm{const}}{y}$$

$$\Gamma(\pi - \delta) = \frac{\vec{q}^{\,2} V(\vec{q})}{4\pi\delta} \qquad V(\vec{q}) = \frac{\mathrm{const}}{\vec{q}^{\,2}}$$

$\mathcal{N} = 4$ SYM checked up to 3 loops

Conformal anomaly

$$4\pi\Delta(\alpha_s(|\vec{q}\,|)) = \left[\delta\,\Gamma(\alpha_s(|\vec{q}\,|), \pi - \delta)\right]_{\delta\to 0} - \frac{\vec{q}^{\,2} V(\alpha_s(|\vec{q}\,|), \vec{q}\,)}{4\pi}$$

$$\Delta(\alpha_s) = \frac{4}{27}\beta_0 C_R(47C_A - 28T_F n_f)\left(\frac{\alpha_s}{4\pi}\right)^3 + \mathcal{O}(\alpha_s^4)$$

vanishes when $\beta_0 = 0$.

Conjecture (like the Crewther–Broadhurst–Kataev relation)

$$\Delta(\alpha_s) = \beta(\alpha_s) C(\alpha_s)$$

$$C(\alpha_s) = \frac{4}{27} C_R (47 C_A - 28 T_F n_f) \left(\frac{\alpha_s}{4\pi}\right)^2$$

$$+ 4 C_R \Bigg[? C_A^2 - \left(5\zeta_3 + \frac{\pi^4}{6} - \frac{79}{648}\right) C_A T_F n_f$$

$$+ \frac{2}{3}\left(19\zeta_3 + \frac{\pi^4}{10} - \frac{1711}{48}\right) C_F T_F n_f$$

$$+ \frac{8}{9}\left(\zeta_3 + \frac{58}{27}\right)(T_F n_f)^2 \Bigg] \left(\frac{\alpha_s}{4\pi}\right)^3 + \mathcal{O}(\alpha_s^4)$$

Brüser, Grozin, Henn, Stahlhofen (2019)

- $C_F\alpha_s^2$, $C_F^2\alpha_s^3$ vanish
- Known $\Gamma_{fff} \Rightarrow (T_F n_f)^2\alpha_s^3$ in $C \Rightarrow \Gamma_{Aff}$ at $\delta \to 0$ — agrees with the *conjecture*
- Known $\Gamma_{Fff} \Rightarrow C_F T_F n_f \alpha_s^3$ in $C \Rightarrow \Gamma_{FAf}$ at $\delta \to 0$ — agrees with the *conjecture*
- *Conjectured* Γ_{Aff} at $\delta \to 0 \Rightarrow C_A T_F n_f \alpha_s^3$ in C
- No $d_{FF} n_f \alpha_s^4$ in Δ — not *explicitely* checked yet

Kataev, Molokoedov (2022) *conjectured*

$$C(\alpha_s) = \sum_{n=0}^{\infty} C_n(\alpha_s) \left[\beta(\alpha_s)\right]^n$$

$C_n(\alpha_s)$ don't contain $T_F n_f$

An arbitrary series

$$C(\alpha_s) = \sum_{n=0}^{\infty} P_n(T_F n_f) \left(\frac{\alpha_s}{4\pi}\right)^{n+1} \qquad (*)$$

can be written in this form

- In P_1 express $T_F n_f$ via

$$\beta(\alpha_s) - \sum_{n=1}^{\infty} \beta_n \left(\frac{\alpha_s}{4\pi}\right)^{n+1}$$

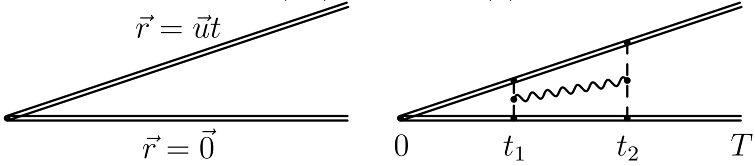
  ($\beta_{n\geq 1}$ is a polynomial in $T_F n_f$ of degree n)
  and update $P_{\geq 2}(T_F n_f)$ by incorporating this sum
- Repeat for P_2 and so on

At the N-th step, 3 kinds of terms appear:

- $\big[\beta(\alpha_s)\big]^m$ with coefficients not containing $\beta_{\geq 1}$ ($m \in [0, N]$); these terms become a part of the final result.
- $\big[\beta(\alpha_s)\big]^m$ times some series having the form (*) ($m \in [1, N-1]$); for these series, we call the algorithm recursively.
- terms without $\beta(\alpha_s)$ containing $\beta_{\geq 1}$; they are absorbed into $P_{\geq N+1}(T_F n_f)$.

Logarithmic term

$C_R C_A^3 \alpha_s^4 \log(\delta)/\delta$ in $\Gamma(\pi - \delta)$
similar to 3-loop $\log(\mu r)$ term in $V(r)$

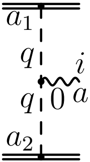


Coulomb gauge

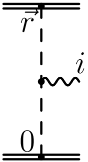
$$V(\vec{q}) = -C_F \frac{g_0^2}{\vec{q}^2} \qquad V(\vec{r}) = -C_F \kappa_0 \frac{g_0^2}{4\pi} \frac{1}{r^{1-2\varepsilon}}$$

- Ultrasoft $t_1 \sim t_2 \sim t_2 - t_1$: Coulomb $q \sim 1/(u t_{1,2})$, transverse $k \sim 1/t_{1,2} \ll q$
- Soft $t_2 - t_1 \sim u t_{1,2}$: $k \sim 1/(t_2 - t_1) \sim q$

Ultrasoft: neglect k



$$= f^{aa_1a_2} g_0^3 \frac{2q^i}{(\vec{q}^2)^2}$$



$$= i f^{aa_1a_2} \kappa_0 \frac{g_0^3}{4\pi} \frac{r^i}{r^{1-2\varepsilon}}$$

The ratio of W with/without transverse gluon

$1 + R_{\rm us} + R_{\rm soft}$

$$R_{\rm us} = \int_0^T dt_2 \int_0^{t_2} dt_1\, K(t_1,t_2)$$

$$K(t_1,t_2) = \frac{1}{4} C_F C_A^2 \kappa_0^2 \frac{g_0^6}{(4\pi)^2} \frac{r_1^i}{r_1^{1-2\varepsilon}} \frac{r_2^j}{r_2^{1-2\varepsilon}}$$

$$\times D^{ij}(v(t_2-t_1)) \exp\left[-i \int_{t_1}^{t_2} dt\, \Delta V(ut)\right]$$

$$\Delta V(r) = V_o(r) - V(r) \qquad V_o(r):\ V(r) \text{ with } C_F \to C_F - C_A/2$$

$$D^{ij}(vt) = 8(i/2)^{2\varepsilon} \frac{\Gamma(2-\varepsilon)}{3-2\varepsilon} \frac{t^{-2+2\varepsilon}}{(4\pi)^{2-\varepsilon}} \delta^{ij}$$

$$K(t_1, t_2) = \frac{2}{3} C_F C_A^2 \kappa_1 \frac{g_0^6}{(4\pi)^4} u^{4\varepsilon} t_1^{2\varepsilon} t_2^{2\varepsilon} (t_2 - t_1)^{-2+2\varepsilon}$$

$$\times \exp\left[-\frac{i}{4} C_A \kappa_0 \frac{g_0^2}{4\pi} \frac{t_2^{2\varepsilon} - t_1^{2\varepsilon}}{\varepsilon u^{1-2\varepsilon}} \right]$$

1 Coulomg gluon between t_1 and t_2

$$K^{(1)}(t_1, t_2) = -\frac{i}{6} C_F C_A^3 \kappa_2 \frac{g_0^8}{(4\pi)^5} \frac{t_1^{2\varepsilon} t_2^{2\varepsilon} (t_2^{2\varepsilon} - t_1^{2\varepsilon})(t_2 - t_1)^{-2+2\varepsilon}}{\varepsilon u^{1-6\varepsilon}}$$

$$R_{\text{us}}^{(1)} = -\frac{i}{48} C_F C_A^3 \kappa_3 \frac{g_0^8}{(4\pi)^5} \frac{T^{8\varepsilon}}{\varepsilon^2 u^{1-6\varepsilon}}$$

Soft $t_2 - t_1 \sim u t_{1,2} \ll t_{1,2}$

$$V^{(1)}_{\text{soft}}(r) = c C_F C_A^3 \frac{g_0^8}{r^{1-8\varepsilon}}$$
$$R^{(1)}_{\text{soft}} = -i \int_0^T dt\, V^{(1)}_{\text{soft}}(ut) = -i c C_F C_A^3 \frac{g_0^8 T^{8\varepsilon}}{8\varepsilon u^{1-8\varepsilon}}$$

$R^{(1)} = R^{(1)}_{\text{us}} + R^{(1)}_{\text{soft}}$: $1/\varepsilon^2$ cancels

$$R^{(1)} = -\frac{i}{48} C_F C_A^3 \frac{g_0^8 T^{8\varepsilon}}{(4\pi)^5} \frac{\kappa_3 u^{6\varepsilon} - \kappa_4 u^{8\varepsilon}}{\varepsilon^2 u}$$
$$= \frac{i}{24} C_F C_A^3 \frac{\alpha_s^4(\mu)(\mu T)^{8\varepsilon}}{4\pi} \frac{\log u + \text{const}}{\varepsilon u}$$
$$\Delta\Gamma = -\frac{i}{3} C_F C_A^3 \frac{\alpha_s^4}{4\pi} \frac{\log u + \text{const}}{u}$$

To Euclidean $\varphi_E = \pi + i\varphi_M$, $\varphi_M = u$

$$\Delta\Gamma(\pi - \delta) = -\frac{1}{3} C_F C_A^3 \frac{\alpha_s^4}{4\pi} \frac{\log\delta + \text{const}}{\delta}$$

Grozin, Stahlhofen (2018)

Abelian structures: 2-leg c-web

Set S of color structures

1. the contributions of the color structures $C_R \times S$ to bare $\log W$ are given by 2-leg c-webs only
2. no color factor $C \notin S$ being multiplied by a color factor in Z_α can produce a color factor $C' \in S$

$$D_S^{\mu\nu}(k) = \sum_{L=0}^{\infty} d_S^{(L)} D_L^{\mu\nu}(k) A_0^L \qquad A_0 = e^{-\gamma\varepsilon} \frac{e_0^2}{(4\pi)^{d/2}}$$

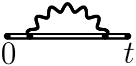
$$D_L^{\mu\nu}(k) = \frac{1}{(-k^2)^{1+L\varepsilon}} \left(g^{\mu\nu} + \frac{k^\mu k^\nu}{-k^2} \right)$$

$$D_{L-1}^{\mu\nu}(x) = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(1-u)}{\Gamma(2+u-\varepsilon)} \frac{2^{1-2u}}{(-x^2)^{1-u}}$$

$$\times \left[(1 + 2(u-\varepsilon)) g^{\mu\nu} - 2(1-u) \frac{x^\mu x^\nu}{-x^2} \right] \qquad u = L\varepsilon$$

HQET field

Coordinate space



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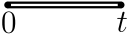


$\times\, w_S(t)$

$$w_S(\tau) = \sum_{L=1}^{\infty} d_S^{(L-1)} w_L \left[A_0 (\tau/2)^{2\varepsilon} e^{\gamma\varepsilon} \right]^L$$

Momentum space

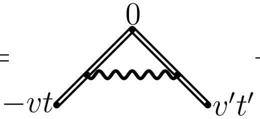
$=$

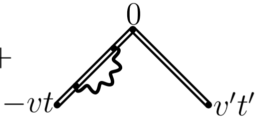

$\times\, \tilde{w}_S(\omega)$

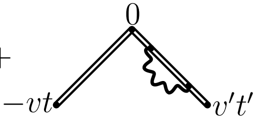
$$\tilde{w}_S(\omega) = \sum_{L=1}^{\infty} d_S^{(L-1)} \tilde{w}_L \left[A_0 (-2\omega)^{-2\varepsilon} \right]^L$$

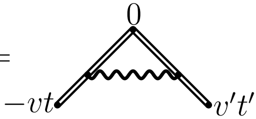
Cusp

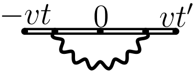
$$(\log W(t,t',\varphi))_S = w_S(t,t',\varphi)$$

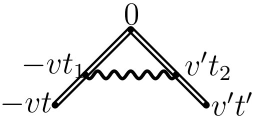
$=$ 

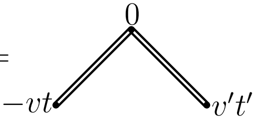
$+$ 

$+$ 

$w_S(t,t',\varphi) - w_S(t,t',0) =$ 

$-$ 



$=$ 

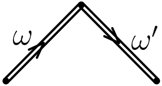
$\times V_S(t,t',\varphi)$

$$V_S(\tau,\tau',\varphi) = \sum_{L=1}^{\infty} d_S^{(L-1)} V_L(\tau'/\tau,\varphi) \left[A_0(\tau\tau'/4)^{\varepsilon} e^{\gamma\varepsilon}\right]^L$$

$V_L(1,\varphi)$ via ${}_2F_1$ Grozin (2018)

Momentum space

$=$



$\times \tilde{V}_S(\omega, \omega', \varphi)$

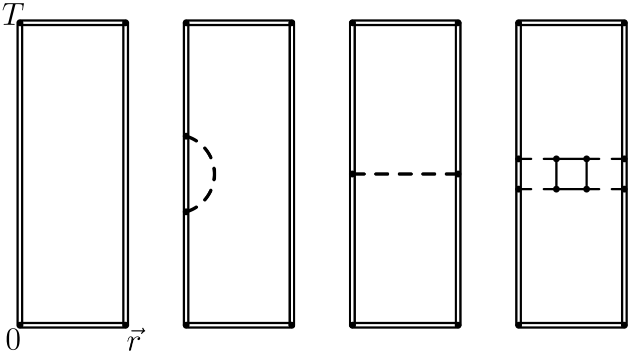
$\tilde{V}_S(\omega, \omega', \varphi) =$

$= \sum_{L=1}^{\infty} d_S^{(L-1)} \tilde{V}_L(\omega'/\omega, \varphi) \left[A_0 (4\omega\omega')^{-\varepsilon} \right]^L$

$\tilde{V}_L(1, \varphi)$ via ${}_2F_1$ Grozin, Kotikov (2011)

Potential

$$\log W = -iV(\vec{r})T$$

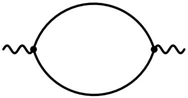


$$V_S(\vec{q}) = -(4\pi)^{d/2} e^{\gamma\varepsilon} \sum_{L=1}^{\infty} \frac{d_S^{(L-1)}}{(\vec{q}^2)^{1+(L-1)\varepsilon}} A_0^L$$

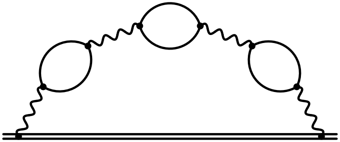
Leading large β_0

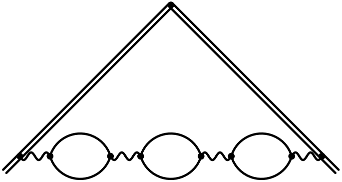
$S = \{(T_F n_f)^L, L \geq 0\}$

QED $\qquad b = \beta_0 \dfrac{\alpha}{4\pi} \sim 1 \qquad 1/\beta_0 \ll 1$

 $\Rightarrow \Pi_0(k^2) = \Pi_0 A_0 (-k^2)^{-\varepsilon} \sim 1 \qquad \Pi_0 = \beta_0 \dfrac{D(\varepsilon)}{\varepsilon}$

$$\frac{d \log Z_\alpha(b)}{d \log b} = -\frac{b}{\varepsilon + b} \qquad Z_\alpha(b) = \frac{1}{1 + b/\varepsilon}$$





HQET field

$$d_S^{(L)} = \Pi_0^L \qquad k^2 = (-2\omega)^2$$

$$\tilde{w}(\omega) = \frac{1}{\beta_0} \sum_{L=1}^{\infty} \frac{\tilde{f}(\varepsilon, L\varepsilon)}{L} \left[\Pi_0(k^2)\right]^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$$= \frac{1}{\beta_0} \sum_{L=1}^{\infty} \frac{\tilde{f}(\varepsilon, L\varepsilon)}{L} \left(\frac{b}{\varepsilon + b}\right)^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$$\mu = (-2\omega) D(\varepsilon)^{-1/(2\varepsilon)} \to (-2\omega) e^{-5/6}$$

$$\tilde{f}(\varepsilon, u) = \frac{u\tilde{w}_L}{D(\varepsilon)} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \tilde{f}_{nm} \varepsilon^n u^m$$

$$z_{h1}(b) = -\frac{1}{\beta_0} \sum_{n=0}^{\infty} \frac{\tilde{f}_{n0}}{n+1} (-b)^{n+1} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$$\gamma_h(b) = -2\frac{b}{\beta_0} \tilde{f}(-b, 0) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

Coordinate space

$$w(\tau) = \frac{1}{\beta_0} \sum_{L=1}^{\infty} \frac{f(\varepsilon, L\varepsilon)}{L} \left[\Pi_0(k^2) e^{\gamma\varepsilon}\right]^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right) \qquad (k^2 = (2/\tau)^2)$$

$$f(\varepsilon, u) = \frac{u w_L}{D(\varepsilon)} \qquad \mu = \frac{2}{\tau} e^{-\gamma} D(\varepsilon)^{1/(2\varepsilon)} \to \frac{2}{\tau} e^{-\gamma - 5/6}$$

$$w(\tau) = \frac{1}{\beta_0} \sum_{L=1}^{\infty} \frac{f(\varepsilon, L\varepsilon)}{L} \left(\frac{b}{\varepsilon + b}\right)^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$$f(-b, 0) = \tilde{f}(-b, 0)$$

$$\gamma_h(b) = -6 \frac{b}{\beta_0} \gamma_0(b) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$$\gamma_0(b) = \frac{\left(1 + \frac{2}{3} b\right)^2 \Gamma(2 + 2b)}{(1 + b)^2 \Gamma^3(1 + b) \Gamma(1 - b)}$$

Broadhurst, Grozin (1994)

Cusp

$$\tilde{\tilde{V}}(\omega,\omega,\varphi) = \tilde{V}(1,\varphi) - \tilde{V}(1,0)$$
$$= \frac{1}{\beta_0} \sum_{L=1}^{\infty} \frac{\tilde{f}(\varepsilon, L\varepsilon, \varphi)}{L} \left[\Pi_0(k^2)\right]^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$
$$= \frac{1}{\beta_0} \sum_{L=1}^{\infty} \frac{\tilde{f}(\varepsilon, L\varepsilon, \varphi)}{L} \left(\frac{b}{\varepsilon + b}\right)^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$
$$\Gamma(b,\varphi) = 4(\varphi \coth\varphi - 1)\frac{b}{\beta_0}\Gamma_0(b) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$
$$\Gamma_0(b) = \hat{f}(-b) = \frac{\left(1+\frac{2}{3}b\right)\Gamma(2+2b)}{(1+b)\Gamma^3(1+b)\Gamma(1-b)}$$

Gracey (1994); Beneke, Braun (1995)
coordinate space — similar

Potential

$$\mu = |\vec{q}| \qquad V(\vec{q}) = -\frac{(4\pi)^{d/2} e^{\gamma\varepsilon}}{\beta_0 D(\varepsilon) (\vec{q}^{\,2})^{1-\varepsilon}} \varepsilon S$$

$$S = \sum_{L=1}^{\infty} g(\varepsilon, L\varepsilon) \left(\frac{b}{\varepsilon + b} \right)^L = \frac{b}{\varepsilon} \sum_{n=0}^{\infty} n!\, g_{0n} b^n + \mathcal{O}(\varepsilon^0)$$

$$g(\varepsilon, u) = D(\varepsilon)^{u/\varepsilon} = \sum_{n,m=0}^{\infty} g_{nm} \varepsilon^n u^m$$

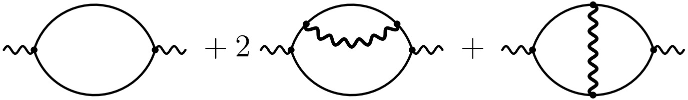
$$g(0, u) = e^{\frac{5}{3}u} \qquad g_{0n} = \frac{1}{n!} \left(\frac{5}{3} \right)^n$$

$$V(\vec{q}) = -\frac{(4\pi)^2}{\vec{q}^{\,2}} \frac{b}{\beta_0} V_0(b) + \mathcal{O}\left(\frac{1}{\beta_0^2} \right) \qquad V_0(b) = \frac{1}{1 - \frac{5}{3} b}$$

$$C = \frac{b^2}{\beta_0} C_0(b) + \mathcal{O}\left(\frac{1}{\beta_0^2} \right) \qquad C_0(b) = \frac{V_0(b) - \Gamma_0(b)}{b^2}$$

Next-to-leading large β_0

$$S = \{C_F (T_F n_f)^{L-1}, L \geq 2\}$$



$$\Rightarrow \Pi_0(k^2) + \frac{\Pi_1(k^2)}{\beta_0} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

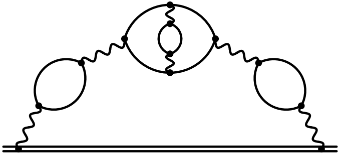
$$\Pi_1(k^2) = 3\varepsilon \sum_{L=2}^{\infty} \frac{F(\varepsilon, L\varepsilon)}{L} \Pi_0(k^2)^L$$

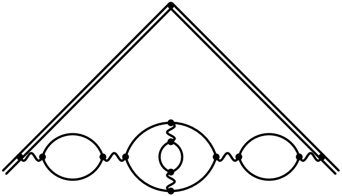
${}_3F_2(1)$ Kotikov (1995); Broadhurst, Gracey, Kreymer (1996)

$$\beta(b) = b + \frac{b^2}{\beta_0} B_1(b) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right) \qquad B_1(b) = 3 \sum_{n=0}^{\infty} \frac{F_{n0}(-b)^n}{n+1}$$

Palanques-Mestre, Pascual (1984)

HQET field





$$\tilde{w}(\omega) = \frac{1}{\beta_0} \sum_{L=1}^{\infty} \frac{\tilde{f}(\varepsilon, L\varepsilon)}{L} \left(\frac{b}{\varepsilon + b} \right)^L$$

$$\times \left[1 + L \frac{Z_{\alpha 1}}{\beta_0} + \frac{3\varepsilon}{\beta_0} \sum_{L'=2}^{L-1} \frac{L - L'}{L'} F(\varepsilon, L'\varepsilon) \right] + \mathcal{O}\left(\frac{1}{\beta_0^3} \right)$$

$$\gamma_h(b) = -6 \left[\frac{b}{\beta_0} \gamma_0(b) - \frac{b^3}{\beta_0^2} \gamma_1(b) \right] + \mathcal{O}\left(\frac{1}{\beta_0^3} \right)$$

$\gamma_1(b)$ up to b^5 (8 loops) — F_{nm} up to $n + m = 6$ ($\zeta_{5,3}$, but cancels in $F_{51} + F_{42} + F_{33} + F_{24} + F_{15}$). Grozin (2016)

Cusp

$$\Gamma(b,\varphi) = 4(\varphi\cot\varphi - 1)\left[\frac{b}{\beta_0}\Gamma_0(b) - \frac{b^3}{\beta_0^2}\Gamma_1(b)\right] + \mathcal{O}\left(\frac{1}{\beta_0^3}\right)$$

$\Gamma_1(b)$ up to b^5 (8 loops), the same combination of F_{nm} with $n+m=6$ — $\zeta_{5,3}$ cancels. Grozin (2016)

Potential

$$V(\vec{q}) = -\frac{(4\pi)^2}{\beta_0 \vec{q}^2}\varepsilon \sum_{L=1}^{\infty} g(\varepsilon, L\varepsilon)\left(\frac{b}{\varepsilon + b}\right)^L$$

$$\times \left[1 + L\frac{Z_{\alpha 1}}{\beta_0} + \frac{3\varepsilon}{\beta_0}\sum_{L'=2}^{L-1}\frac{L-L'}{L'}F(\varepsilon, L'\varepsilon)\right] + \mathcal{O}\left(\frac{1}{\beta_0^3}\right)$$

$$V(\vec{q}) = -\frac{(4\pi)^2}{\vec{q}^2}\left[\frac{b}{\beta_0}V_0(b) - \frac{b^3}{\beta_0^2}V_1(b)\right] + \mathcal{O}\left(\frac{1}{\beta_0^3}\right)$$

$V_1(b)$ contains only g_{0n}, F_{n0}, F_{0m} — $\Gamma \Rightarrow \zeta_n$
Highest weights: 3,3,5,5,7,7

$$C = \frac{b^2}{\beta_0}C_0(b) - \frac{b^3}{\beta_0^2}C_1(b) + \mathcal{O}\left(\frac{1}{\beta_0^3}\right)$$

Abelian $(T_F n_f)^1$

$$S = \{1\} \cup \{C_L^{L-1} T_F n_f, L \geq 1\}$$

$$\Pi_{L-1} = \tilde{\Pi}_{L-1} n_f + (n_f^{>1} \text{ terms})$$

$$D^{\mu\nu}(k) = D_0^{\mu\nu}(k) + n_f \sum_{L=1}^{\infty} \tilde{\Pi}_{L-1} D_L^{\mu\nu}(k) A_0^L + (n_f^{>1} \text{ terms})$$

$$\tilde{\Pi}_{L-1} = \frac{\bar{\beta}_{L-1}}{L\varepsilon} + \bar{\Pi}_{L-1} + \mathcal{O}(\varepsilon)$$

$$\bar{\beta}_0 = -\frac{4}{3} \qquad \bar{\beta}_1 = -4 \qquad \bar{\beta}_2 = 2 \qquad \bar{\beta}_3 = 46$$

Gorishnii, Kataev, Larin, Surguladze (1991)

$$\bar{\Pi}_0 = -\frac{20}{9} \qquad \bar{\Pi}_1 = 16\zeta_3 - \frac{55}{3}$$

$$\bar{\Pi}_2 = -2\left(80\zeta_5 - \frac{148}{3}\zeta_3 - \frac{143}{9}\right)$$

$$\bar{\Pi}_3 = 2240\zeta_7 - 1960\zeta_5 - 104\zeta_3 + \frac{31}{3}$$

Ruijl, Ueda, Vermaseren, Vogt (2017)

HQET field

$$\tilde{w}(\omega) = \tilde{w}_1 A_0(-2\omega)^{-2\varepsilon} + n_f \sum_{L=2}^{\infty} \tilde{\Pi}_{L-2} \tilde{w}_L \left[A_0(-2\omega)^{-2\varepsilon} \right]^L$$

$$+ (n_f^{>1} \text{ terms}) + (w_{>2 \text{ legs}} \text{ terms})$$

$$\tilde{w}_L = \frac{3}{L\varepsilon} + \frac{1}{L} + 3 + \mathcal{O}(\varepsilon)$$

$$Z_\alpha = 1 - \frac{n_f}{\varepsilon} \sum_{L=1}^{\infty} \frac{\bar{\beta}_{L-1}}{L} \left(\frac{\alpha}{4\pi} \right)^L + (n_f^{>1} \text{ terms})$$

$$\gamma_h = -2C_R \frac{\alpha_s}{4\pi} \left[3 + T_F n_f \frac{\alpha_s}{4\pi} \sum_{L=0}^{\infty} (3\bar{\Pi}_L - \bar{\beta}_L) \left(C_F \frac{\alpha_s}{4\pi} \right)^L \right] + \cdots$$

coordinate space similar. Grozin (2018)

Cusp

$$\tilde{\tilde{V}}(\varphi) = -2\frac{\varphi\coth\varphi - 1}{L\varepsilon} + V(\varphi) + \mathcal{O}(\varepsilon)$$

$V(\varphi) = V(-\varphi)$ does not depend on L $\Rightarrow$ $\bar{\beta}_{L-2}$ cancels

$$\Gamma(\varphi) = 4C_R(\varphi\coth\varphi - 1)\frac{\alpha_s}{4\pi}\left[1 + T_F n_f \frac{\alpha_s}{4\pi}\sum_{L=0}^{\infty}\bar{\Pi}_L\left(C_F\frac{\alpha_s}{4\pi}\right)^L\right] + \cdots$$

coordinate space similar. Grozin (2018)

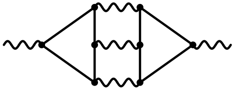
Potential

$$V(\vec{q}) = -C_R\frac{4\pi\alpha_s}{\vec{q}^{\,2}}\left[1 + T_F n_f \frac{\alpha_s}{4\pi}\sum_{L=0}^{\infty}\bar{\Pi}_L\left(C_F\frac{\alpha_s}{4\pi}\right)^L\right] + \cdots$$

Δ: $C_F^{L-1}T_F n_f$ absent to all orders $\Rightarrow$
$C(\alpha_s)$: $C_R C_F^{L-1}\alpha_s^L$ absent to all orders

2 more 5-loop terms

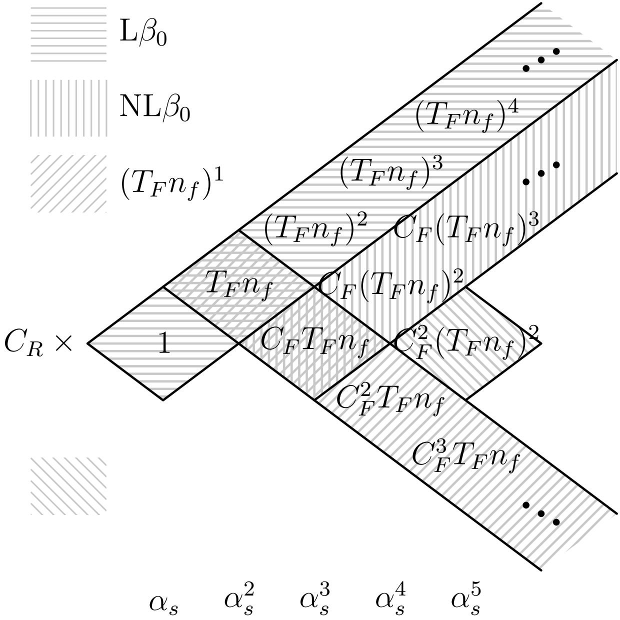
We considered $C_R C_F^{L-n-1}(T_F n_f)^n \alpha_s^L$
The only missing 5-loop term $C_R C_F^2 (T_F n_f)^2 \alpha_s^5$
$S = \{1, T_F n_f, C_F T_F n_f, C_F^2 T_F n_f, C_F^2 (T_F n_f)^2\}$



$S = \{\bar{d}_{FF} n_f^2\}$

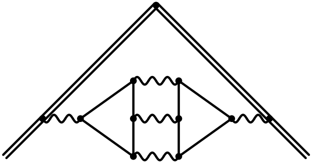
$\bar{d}_{FF} n_f^2$

$$\bar{d}_{FF} = \frac{d_F^{abcd} d_F^{abcd}}{N_A}$$



$\mathrm{L}\beta_0$
$\mathrm{NL}\beta_0$
$(T_F n_f)^1$
$C_R \times$
1
$T_F n_f$
$(T_F n_f)^2$
$(T_F n_f)^3$
$(T_F n_f)^4$
$C_F T_F n_f$
$C_F (T_F n_f)^2$
$C_F (T_F n_f)^3$
$C_F^2 (T_F n_f)^2$
$C_F^2 T_F n_f$
$C_F^3 T_F n_f$
α_s
α_s^2
α_s^3
α_s^4
α_s^5

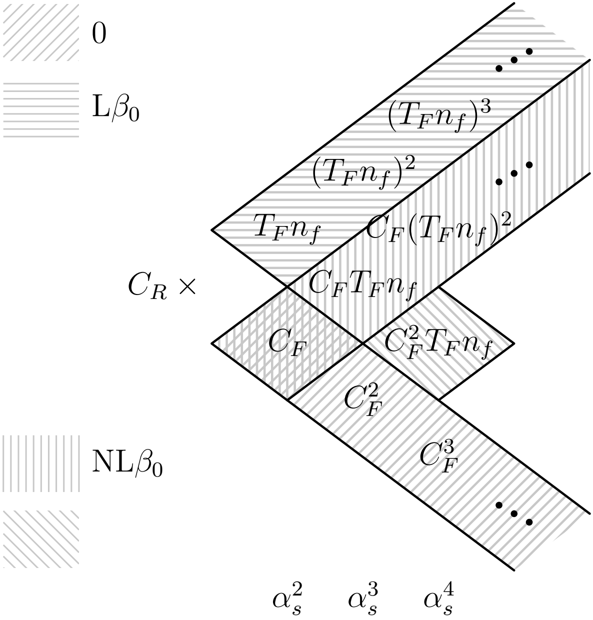
γ_h, $\Gamma(\varphi)$, V

- Lβ_0 from time immemorable
- NLβ_0 Grozin (2016)
- $C_R C_F^{L-2} T_F n_f \alpha_s^L$ Grozin (2018–19)
- $C_R C_F^2 (T_F n_f)^2 \alpha_s^5$
- 

$$C_R \frac{d_F^{abcd} d_F^{abcd}}{N_A} n_f^2 \alpha_s^5$$

$C(\alpha_s)$

- $C_R(T_F n_f)^3\alpha_s^4$ in $\Gamma(\delta\to 0) \Rightarrow C_R(T_F n_f)^2\alpha_s^3$ in $C \Rightarrow$ $C_R C_A (T_F n_f)^2\alpha_s^4$ in $\Gamma(\delta\to 0)$ agrees with the conjecture
- $C_R C_F (T_F n_f)^2\alpha_s^4$ in $\Gamma(\delta\to 0) \Rightarrow C_R C_F T_F n_f\alpha_s^3$ in $C \Rightarrow$ $C_R C_F C_A T_F n_f\alpha_s^4$ in $\Gamma(\delta\to 0)$ agrees with the conjecture
- $C_R C_A T_F n_f\alpha_s^3$ in C follows from conjectured $C_R C_A (T_F n_f)^2\alpha_s^4$ in $\Gamma(\delta\to 0)$
- $d_{RF} n_f\alpha_s^4$ cancels in Δ
- $C_R C_F^{L-2} T_F n_f\alpha_s^L$ cancel in Δ $\forall L$
- $C_R d_F^{abcd} d_F^{abcd}/N_A n_f^2\alpha_s^5$ cancel in Δ
- $C_R C_F^2 (T_F n_f)^2\alpha_s^5$ in $\Gamma(\delta\to 0) \Rightarrow C_R C_F^2 T_F n_f\alpha_s^4$ in C


0
$\mathrm{L}\beta_0$
$\mathrm{NL}\beta_0$
$C_R \times$
$(T_F n_f)^3$
$(T_F n_f)^2$
$T_F n_f$
$C_F (T_F n_f)^2$
$C_F T_F n_f$
C_F
$C_F^2 T_F n_f$
C_F^2
C_F^3
α_s^2
α_s^3
α_s^4

Conclusion

- γ_h is known to 4 loops
- $\Gamma(\varphi)$ is known to 4 loops up to φ^4 (for some color structures, up to φ^6)
- K is known up to 4 loops
- $d_{RF} n_f \alpha_s^4$ in $\Gamma(\varphi)$ is known
- $C_R C_A (T_F n_f)^2 \alpha_s^4$, $C_R C_F C_A T_F n_f \alpha_s^4$ are known from the conjecture
- $C_R C_A^2 T_F n_f \alpha_s^4$, $C_R C_A^3 \alpha_s^4$, $d_{RA} \alpha_s^4$ are not known
- $C_R C_F^2 (T_F n_f)^2 \alpha_s^5$, $C_R d_F^{abcd} d_F^{abcd} / N_A n_f^2 \alpha_s^5$ are known in γ_h, Γ at 5 loops and in V at 4 loops
- $C_R (T_F n_f)^{L-1} \alpha_s^L$ in γ_h, Γ, V are known $\forall L$
- $C_R C_F (T_F n_f)^{L-2} \alpha_s^L$ in γ_h, Γ, V are known, in principle, $\forall L$
- $C_R C_F^{L-2} T_F n_f \alpha_s^L$ in γ_h, Γ, V are known up to $L = 5$

- How to formulate the conjecture consistently beyond Casimir scaling? Why does it work for $C_R C_A T_F n_f \alpha_s^3$, $C_R C_A (T_F n_f)^2 \alpha_s^4$, $C_R C_F C_A T_F n_f \alpha_s^4$? Why does it fail for $C_R C_A^2 T_F n_f \alpha_s^4$, $d_{RF} n_f \alpha_s^4$?
- Why are Z_Q^{os}, $\gamma_h - \gamma_q$, z gauge-nivariant up to 2 loops, linear in a at 3 loops, and quadratic in a at 4 loops? Does it continue at higher loops?
- In Z_Q^{os} at 4 loops all coefficients of ε^{-n} are known analytically (some ε^0 coefficients are still only known numerically)
- Principle of maximal transcendentality works for the Bremsstrahlung function and light-like K up to 4 loops
- Why does it work? Is the maximum-weight part of γ_h related to renormalization of an end of a Wilson line in $\mathcal{N} = 4$ SYM?

- How to define Δ consistently, in spite of $\log\delta$ terms? How to sum highest powers of $\log\delta$ in $\delta\,\Gamma(\pi-\delta)$? What if we sum over the number of Coulomb exchanges? Is $\Delta(\alpha_s) = \beta(\alpha_s)C(\alpha_s)$? Is it possibly to calculate $C(\alpha_s)$ (or $\Delta(\alpha_s)$) more directly?