# Majorana neutrinos and other Majorana particles: Theory and experiment 

Evgeny Kh. Akhmedov* ${ }^{* \dagger}$<br>Max-Planck-Institut für Kernphysik, Saupfercheckweg 1<br>D-69117 Heidelberg, Germany


#### Abstract

This is a somewhat modified version of Chapter 15 of the book "The Physics of Ettore Majorana", by Salvatore Esposito with contributions by Evgeny Akhmedov (Ch. 15) and Frank Wilczek (Ch. 14), Cambridge University Press, 2014.


[^0]What are Majorana particles? These are massive fermions that are their own antiparticles. In this chapter we will concentrate on spin- $1 / 2$ Majorana particles, though fermions of higher spin can also be of Majorana nature. Obviously, Majorana particles must be genuinely neutral, i.e. they cannot possess any conserved charge-like quantum number that would allow one to discriminate between the particle and its antiparticle. In particular, they must be electrically neutral. Among the known spin-1/2 particles, only neutrinos can be of Majorana nature. Another known quasi-stable neutral fermion, the neutron, has non-zero magnetic moment which disqualifies it for being a Majorana particle: the antineutron exists, and its magnetic moment is negative of that of the neutron ${ }^{1}$.

Neutrinos are exactly massless in the original version of the standard model of electroweak interaction, and are massive Majorana particles in most its extensions. Although massive Dirac neutrinos is also a possibility, most economical and natural models of neutrino mass lead to Majorana neutrinos. Since only massive neutrinos can oscillate, the interest to the possibility of neutrinos being Majorana particles rose significantly after the first hints of neutrino oscillations obtained in the solar and atmospheric neutrino experiments. It has greatly increased after the oscillations were firmly established in the experiments with solar, atmospheric, accelerator and reactor neutrinos [1-3]. In addition to being the simplest and most economical possibility, Majorana neutrinos bring in two important added bonuses: they can explain the smallness of the neutrino mass in a very natural way through the so-called seesaw mechanism, and can account for the observed baryon asymmetry of the Universe through 'baryogenesis via leptogenesis'. We shall discuss both in this chapter.

In the limit of vanishingly small mass the difference between Dirac and Majorana fermions disappears. Therefore the observed smallness of the neutrino mass makes it very difficult to discriminate between different types of massive neutrinos, and it is not currently known if neutrinos are Majorana or Dirac particles. The most promising means of finding this out is through the experiments on neutrinoless double beta decay. Such experiments are currently being conducted in a number of laboratories.

In this chapter we review the properties of Majorana neutrinos and other Majorana particles. We start with discussing Weyl, Dirac and Majorana fermions and comparing the Dirac and Majorana mass terms. We then proceed to discuss C, P, CP and CPT properties of Majorana particles in sec. 15.2. This is followed by a discussion of mixing and oscillations of neutrinos in the Majorana and general Dirac + Majorana cases in sec. 15.3. In sec. 15.4 we discuss the seesaw mechanism of the neutrino mass generation, which is the leading candidate for the explanation of the smallness of the neutrino mass. Next, we consider electromagnetic properties of Majorana neutrinos in sec. 15.5. Section 15.6 contains a brief

[^1]discussion of Majorana particles predicted by supersymmetric theories. In sec. 15.7 we review theoretical foundations and the experimental status of the neutrinoless $2 \beta$-decay as well as of other processes that could distinguish between Majorana and Dirac neutrinos. Our next topic is baryogenesis via leptogenesis due to lepton number violating processes caused by Majorana neutrinos (sec. 15.8). Finally, in sec. 15.9 we collect a few assorted remarks on Majorana particles and in sec. 15.10 summarize the main points of our discussion.

### 15.1 Weyl, Dirac and Majorana fermions

Being dissatisfied with the interpretation of antifermions as holes in the Dirac sea, in his famous paper [4] Majorana sought to cast the Dirac equation in a form that would be completely symmetric with respect to particles and antiparticles. He succeeded to do that by finding a new form of the Dirac equation, in which all coefficients were real. While it led to only formal improvement for charged fermions, the Majorana form of the Dirac equation opened up a very important new possibility for neutral ones - they can be their own antiparticles. The Majorana particles are thus fermionic analogues of genuinely neutral bosons, such as the $\pi^{0}$-meson or the photon.

Recall that a free spin- $1 / 2$ fermion field in general satisfies the Dirac equation ${ }^{2}$

$$
\begin{equation*}
\left(i \gamma^{\mu} \partial_{\mu}-m\right) \psi(x)=0, \tag{1}
\end{equation*}
$$

where $\partial_{\mu} \equiv \partial / \partial x^{\mu}, \psi(x)$ is a 4-component spinor field, $m$ is the mass of the fermion, and $\gamma^{\mu}(\mu=0,1,2,3)$ are $4 \times 4$ matrices satisfying

$$
\begin{equation*}
\left\{\gamma^{\mu}, \gamma^{\nu}\right\}=2 g^{\mu \nu} \cdot \mathbb{1}, \quad \gamma^{0} \gamma^{\mu \dagger} \gamma^{0}=\gamma^{\mu} \tag{2}
\end{equation*}
$$

with $g^{\mu \nu}=\operatorname{diag}(1,-1,-1,-1)$ and $\mathbb{1}$ being the flat space-time metric tensor and the $4 \times 4$ unit matrix, respectively. Note that the Dirac equation (1) can be cast in the Schrödinger form $i(\partial / \partial t) \psi(x)=H_{D} \psi(x)$, where $H_{D}=-i \gamma^{0} \boldsymbol{\gamma} \cdot \nabla+\gamma^{0} m$. The first equality in eq. (2) follows from the requirement that the solutions of the Dirac equation obey the usual dispersion law of free relativistic particles $E^{2}=\mathbf{p}^{2}+m^{2}$, while the second equality follows from hermiticity of the Dirac Hamiltonian $H_{D}$. In addition to the matrices $\gamma^{\mu}$, a very important role is played by the matrix $\gamma_{5} \equiv i \gamma^{0} \gamma^{1} \gamma^{2} \gamma^{3}$ which satisfies

$$
\begin{equation*}
\left\{\gamma_{5}, \gamma^{\mu}\right\}=0, \quad \gamma_{5}^{\dagger}=\gamma_{5}, \quad \gamma_{5}^{2}=\mathbb{1} \tag{3}
\end{equation*}
$$

There are infinitely many unitarily equivalent representations of the Dirac matrices. In this chapter, unless otherwise specified, we will use the so-called chiral (or Weyl) representation

$$
\gamma^{0}=\left(\begin{array}{ll}
0 & \mathbb{1}  \tag{4}\\
\mathbb{1} & 0
\end{array}\right), \quad \gamma^{i}=\left(\begin{array}{cc}
0 & \sigma^{i} \\
-\sigma^{i} & 0
\end{array}\right), \quad \gamma_{5}=\left(\begin{array}{cc}
-\mathbb{1} & 0 \\
0 & \mathbb{1}
\end{array}\right)
$$

[^2]where $\mathbb{1}$ and 0 are the unit and zero $2 \times 2$ matrices, and $\sigma^{i}(i=1,2,3)$ are the standard Pauli matrices.

The left-handed and right-handed chirality projector operators $P_{L, R}$ are defined as

$$
\begin{equation*}
P_{L}=\frac{\mathbb{1}-\gamma_{5}}{2}, \quad P_{R}=\frac{\mathbb{1}+\gamma_{5}}{2} \tag{5}
\end{equation*}
$$

They have the following properties:

$$
\begin{equation*}
P_{L}^{2}=P_{L}, \quad P_{R}^{2}=P_{R}, \quad P_{L} P_{R}=P_{R} P_{L}=0, \quad P_{L}+P_{R}=\mathbb{1} \tag{6}
\end{equation*}
$$

Any spin- $1 / 2$ fermion field $\psi$ can be decomposed into the sum of its left-handed and righthanded components according to

$$
\begin{equation*}
\psi=\psi_{L}+\psi_{R}, \quad \text { where } \quad \psi_{L, R}=P_{L, R} \psi=\frac{\mathbb{1} \mp \gamma_{5}}{2} \psi \tag{7}
\end{equation*}
$$

Note that the chiral fields $\psi_{L, R}$ are eigenstates of $\gamma_{5}: \gamma_{5} \psi_{L, R}=\mp \psi_{L, R}$. The terms 'lefthanded' and 'right-handed' originate from the fact that for relativistic particles chirality almost coincides with helicity defined as the projection of the spin of the particle on its momentum. More precisely, in the relativistic limit, for positive-energy solutions of the Dirac equation the left- and right-handed chirality fields approximately coincide with those of negative and positive helicity, respectively. The helicity projection operators are

$$
\begin{equation*}
P_{ \pm}=\frac{1}{2}\left(1 \pm \frac{\boldsymbol{\sigma} \mathbf{p}}{|\mathbf{p}|}\right) \tag{8}
\end{equation*}
$$

They satisfy relations similar to (6). For a free fermion, helicity is conserved but chirality in general is not; it is only conserved in the limit $m=0$, when it coincides with helicity. However, for relativistic particles chirality is nearly conserved, and the description in terms of chiral states is useful.

For our discussion we will need the particle - antiparticle conjugation operator $\hat{C}$. Its action on a fermion field $\psi$ is defined as

$$
\begin{equation*}
\hat{C}: \psi \rightarrow \psi^{c}=\mathcal{C} \bar{\psi}^{T} \tag{9}
\end{equation*}
$$

where $\bar{\psi} \equiv \psi^{\dagger} \gamma^{0}$ is the adjoint field and the matrix $\mathcal{C}$ satisfies

$$
\begin{equation*}
\mathcal{C}^{-1} \gamma^{\mu} \mathcal{C}=-\gamma^{\mu T}, \quad \mathcal{C}^{-1} \gamma_{5} \mathcal{C}=\gamma_{5}^{T}, \quad \mathcal{C}^{\dagger}=\mathcal{C}^{-1}=-\mathcal{C}^{*} \tag{10}
\end{equation*}
$$

Note that the second equality here follows from the first one and the definition of $\gamma_{5}$. For free particles, the $\hat{C}$-conjugate field $\psi^{c}(x)$ satisfies the same Dirac equation as $\psi(x)$. Some useful relations that follow from (9) and (10) are

$$
\begin{equation*}
\left(\psi^{c}\right)^{c}=\psi, \quad \overline{\psi^{c}}=-\psi^{T} \mathcal{C}^{-1}, \quad \overline{\psi_{k}} \psi_{i}^{c}=\overline{\psi_{i}} \psi_{k}^{c}, \quad \overline{\psi_{k}} A \psi_{i}=\overline{\psi_{i}^{c}}\left(\mathcal{C} A^{T} \mathcal{C}^{-1}\right) \psi_{k}^{c} \tag{11}
\end{equation*}
$$

where $\psi, \psi_{i}, \psi_{k}$ are anticommuting 4-component fermion fields and $A$ is an arbitrary $4 \times 4$ matrix. Note that the third equality in (10) means that the matrix $\mathcal{C}$ is antisymmetric. In the representation (4) as well as in a number of other representations of the Dirac matrices one can choose e.g. $\mathcal{C}=i \gamma^{2} \gamma^{0}$. In this case $\mathcal{C}$ is real, $\mathcal{C}^{-1}=-\mathcal{C}$, and $\overline{\psi^{c}}=\psi^{T} \mathcal{C}$. For future use, we give here the expressions $\mathcal{C} A^{T} \mathcal{C}^{-1}$ for several matrices $A$ :

$$
\begin{array}{ll}
\mathcal{C} \gamma^{\mu T} \mathcal{C}^{-1}=-\gamma^{\mu}, & \mathcal{C}\left(\gamma^{\mu} \gamma_{5}\right)^{T} \mathcal{C}^{-1}=\gamma^{\mu} \gamma_{5} \\
\mathcal{C}\left(\sigma^{\mu \nu}\right)^{T} \mathcal{C}^{-1}=-\sigma^{\mu \nu}, & \mathcal{C}\left(\sigma^{\mu \nu} \gamma_{5}\right)^{T} \mathcal{C}^{-1}=-\sigma^{\mu \nu} \gamma_{5} \tag{12}
\end{array}
$$

where $\sigma^{\mu \nu} \equiv \frac{i}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right]$.
Using the anticommutation properties of the Dirac $\gamma$-matrices it is easy to see that, acting on a chiral field, $\hat{C}$ flips its chirality:

$$
\begin{equation*}
\hat{C}: \psi_{L} \rightarrow\left(\psi_{L}\right)^{c}=\left(\psi^{c}\right)_{R}, \quad \psi_{R} \rightarrow\left(\psi_{R}\right)^{c}=\left(\psi^{c}\right)_{L}, \tag{13}
\end{equation*}
$$

i.e. the antiparticle of a left-handed fermion is right-handed. This fact plays a very important role in the theory of Majorana particles.

The particle - antiparticle conjugation operation $\hat{C}$ must not be confused with the charge conjugation operation C which, by definition, flips all the charge-like quantum numbers of a field (electric charge, baryon number $B$, lepton number $L$, etc.) but leaves all the other quantum numbers (including chirality) intact. In particular, charge conjugation would take a left-handed neutrino into a left-handed antineutrino that does not exist, which is a consequence of maximal C-violation in weak interactions. At the same time, $\hat{C}$-conjugation converts a left-handed neutrino into a right-handed antineutrino which does exist and is the antiparticle of the left-handed neutrino.

> A little caveat should be added to the above. Strictly speaking, a particle and its antiparticle are related by the CPT transformation, as only this combination of the charge conjugation C, space parity P and time reversal T is exactly conserved in any 'normal' theory (i.e. local Poincaré invariant Lagrangian quantum field theory with the usual relation between spin and statistics). However, the CP conjugation does essentially the same job as far as (typically very small) effects of CP-violation can be neglected. The $\hat{C}$ conjugation introduced in eq. (9) acts very similarly to the CP conjugation as it flips all the non-zero charges of the fermion as well as its chirality, which is odd under P transformation. We discuss these points in more detail in sec. 15.2 . It should be added that when we say that the charge conjugation C flips the baryon and lepton numbers of the particles we assume that these numbers are well defined, i.e. that small effects of $B$ and $L$ violation can be ignored.

Let us now return to the discussion of the Dirac equation. Adopting the Weyl representation of the Dirac $\gamma$-matrices (4) and writing the 4 -component spinor field $\psi(x)$ in terms
of the 2-component spinors $\phi(x)$ and $\xi(x)$ as

$$
\begin{equation*}
\psi=\binom{\phi}{\xi} \tag{14}
\end{equation*}
$$

one can rewrite the Dirac equation (1) as a set of two coupled equations for $\phi$ and $\xi$ :

$$
\begin{array}{r}
\left(i \partial_{0}-i \boldsymbol{\sigma} \cdot \boldsymbol{\nabla}\right) \phi-m \xi=0 \\
\left(i \partial_{0}+i \boldsymbol{\sigma} \cdot \boldsymbol{\nabla}\right) \xi-m \phi=0 \tag{15}
\end{array}
$$

From the expression for $\gamma_{5}$ in eq. (4) and eq. (14) one obtains

$$
\begin{equation*}
\psi_{L}=\binom{\phi}{0}, \quad \psi_{R}=\binom{0}{\xi} \tag{16}
\end{equation*}
$$

i.e. the 2 -component spinor fields $\phi$ and $\xi$ determine, respectively, the left- and right-handed components of the 4 -component field $\psi$. Thus, the chiral fields are actually 2 -component rather than 4-component objects.

From eq. (15) it follows that in the limit $m=0$ the equations for $\phi$ and $\xi$ decouple, i.e. the left-handed and right-handed components of $\psi$ evolve independently. The resulting equations are called the Weyl equations, and the corresponding chiral solutions describe massless spin- $1 / 2$ particles called Weyl fermions. At the same time, as follows from (15), to describe a massive fermion one needs both left-handed and right-handed chiral fields.

The latter statement can also be demonstrated as follows. The Dirac equation for a free spin- $1 / 2$ particle can be obtained as the Euler-Lagrange equation applied to the Dirac Lagrangian

$$
\begin{equation*}
\mathcal{L}=\bar{\psi}\left(i \gamma^{\mu} \partial_{\mu}-m\right) \psi \tag{17}
\end{equation*}
$$

The mass term of this Lagrangian can be written as

$$
\begin{equation*}
-\mathcal{L}_{m}=m \bar{\psi} \psi=m \overline{\left(\psi_{L}+\psi_{R}\right)}\left(\psi_{L}+\psi_{R}\right)=m\left(\overline{\psi_{L}} \psi_{R}+\overline{\psi_{R}} \psi_{L}\right) \tag{18}
\end{equation*}
$$

i.e. only the cross terms survive while the $\overline{\psi_{L}} \psi_{L}$ and $\overline{\psi_{R}} \psi_{R}$ terms vanish identically. Thus, one needs both left-handed and right-handed chiral fields to construct the mass term of the Lagrangian, and a massive fermion field must be a sum of them: $\psi=\psi_{L}+\psi_{R}{ }^{3}$

Now, there are essentially two possibilities. First, the right-handed component of a massive field can be completely independent of the left-handed one; in this case we have a

[^3]Dirac field. The second, and the most important for us possibility, is based on the discussed above fact that the particle-antiparticle conjugate of a left-handed field is right-handed. Therefore the right-handed component of a massive spin- $1 / 2$ field can be just the $\hat{C}$ conjugate of its left-handed component: $\psi_{R}=\left(\psi_{L}\right)^{c}=\left(\psi^{c}\right)_{R}$, or

$$
\begin{equation*}
\psi=\psi_{L}+\left(\psi_{L}\right)^{c}=\psi_{L}+\left(\psi^{c}\right)_{R} \tag{19}
\end{equation*}
$$

In this case we have a Majorana field; one can construct it with just one chiral field. From (19) it immediately follows that the $\hat{C}$ - conjugate field coincides with the original one:

$$
\begin{equation*}
\psi^{c}=\psi \tag{20}
\end{equation*}
$$

This means that particles associated with Majorana fields are genuinely neutral, i.e. they are their own antiparticles. The condition in eq. (20) is called the Majorana condition.

In his paper [4] Majorana found a representation of the $\gamma$-matrices in which they were all pure imaginary, so that the Dirac equation (1) did not contain any complex coefficients. As a result, the equation admitted real solutions

$$
\begin{equation*}
\psi^{*}=\psi \tag{21}
\end{equation*}
$$

which describe genuinely neutral particles. Eq. (20) generalizes the Majorana condition (21) to the case of an arbitrary representation of the $\gamma$-matrices (see e.g. [5] for a formal proof).
It is easy to see that the general self-conjugacy condition (20) indeed reduces to (21) in the Majorana basis. In the Majorana representation the $\gamma$-matrices satisfying eq. (2) can be chosen as

$$
\gamma_{\mathrm{M}}^{0}=\left(\begin{array}{cc}
0 & \sigma^{2} \\
\sigma^{2} & 0
\end{array}\right), \quad \gamma_{\mathrm{M}}^{1}=i\left(\begin{array}{cc}
\sigma^{3} & 0 \\
0 & \sigma^{3}
\end{array}\right), \quad \gamma_{\mathrm{M}}^{2}=\left(\begin{array}{cc}
0 & -\sigma^{2} \\
\sigma^{2} & 0
\end{array}\right), \quad \gamma_{\mathrm{M}}^{3}=-i\left(\begin{array}{cc}
\sigma^{1} & 0 \\
0 & \sigma^{1}
\end{array}\right)
$$

where the subscript $M$ stands for the Majorana basis. All the $\gamma$-matrices are pure imaginary, as required (note that this representation is not unique). The matrix $\gamma_{5 \mathrm{M}}$ is then

$$
\gamma_{5 \mathrm{M}}^{0}=\left(\begin{array}{cc}
\sigma^{2} & 0 \\
0 & -\sigma^{2}
\end{array}\right)
$$

Notice that $\gamma_{\mathrm{M}}^{0}$ is antisymmetric, whereas $\gamma_{\mathrm{M}}^{i}(i=1,2,3)$ are symmetric; the particle-antiparticle conjugation matrix $\mathcal{C}$ satisfying eq. (10) can therefore be chosen as

$$
\mathcal{C}_{\mathrm{M}}=-\gamma_{\mathrm{M}}^{0}=-\left(\begin{array}{cc}
0 & \sigma^{2}  \tag{22}\\
\sigma^{2} & 0
\end{array}\right)
$$

From eq. (9) we then find

$$
\begin{equation*}
\psi_{\mathrm{M}}^{c}=\mathcal{C}_{\mathrm{M}} \bar{\psi}_{\mathrm{M}}^{T}=\mathcal{C}_{\mathrm{M}} \gamma_{\mathrm{M}}^{0 T} \psi_{\mathrm{M}}^{*}=-\gamma_{\mathrm{M}}^{0} \gamma_{\mathrm{M}}^{0 T} \psi_{\mathrm{M}}^{*}=\psi_{\mathrm{M}}^{*} \tag{23}
\end{equation*}
$$

i.e. the condition that the particle is its own antiparticle $\psi^{c}=\psi$ reduces in the Majorana basis to the requirement that the field $\psi_{\mathrm{M}}$ be real.

As was discussed above, to construct a massive Dirac field one needs two independent 2component chiral fields, $\psi_{L}$ and $\psi_{R}$; this gives four degrees of freedom. In contrast with this,
a Majorana fermion has only two degrees of freedom, because its right-handed component is constructed from the left-handed one. Thus, Majorana fermions are actually simpler and more economical constructions than the Dirac ones.

While Majorana fields are essentially 2-component objects, it is often useful to write them in the 4-component notation, especially when considering processes in which Majorana particles participate along with Dirac ones. It is easy to see that in the chiral representation of the Dirac matrices the Majorana field can be written in the 4-component form as

$$
\begin{equation*}
\psi=\binom{\phi}{-i \sigma^{2} \phi^{*}} \tag{24}
\end{equation*}
$$

Indeed, from $\mathcal{C}=i \gamma^{2} \gamma^{0}$ and eq. (9) we have

$$
\psi^{c}=i \gamma^{2} \psi^{*}=\left(\begin{array}{cc}
0 & i \sigma^{2}  \tag{25}\\
-i \sigma^{2} & 0
\end{array}\right)\binom{\phi^{*}}{-i \sigma^{2} \phi}=\binom{\phi}{-i \sigma^{2} \phi^{*}}=\psi .
$$

To understand better the difference between the Dirac and Majorana particles it is instructive to look at the expansions of their quantum fields in terms of the plane-wave modes. Recall that for a Dirac field the expansion has the form

$$
\begin{equation*}
\psi(x)=\int \frac{d^{3} p}{(2 \pi)^{3} \sqrt{2 E_{\mathbf{p}}}} \sum_{s}\left[b_{s}(\mathbf{p}) u_{s}(\mathbf{p}) e^{-i p x}+d_{s}^{\dagger}(\mathbf{p}) v_{s}(\mathbf{p}) e^{i p x}\right] \tag{26}
\end{equation*}
$$

where $s= \pm 1 / 2$ is the projection of the particle's spin on a fixed spatial direction, $E_{\mathbf{p}}=$ $p^{0}=+\sqrt{\mathbf{p}^{2}+m^{2}}, u_{s}(\mathbf{p})$ and $v_{s}(\mathbf{p})$ are the positive- and negative-energy solutions of the Dirac equation in the momentum space, and $b_{s}(\mathbf{p})$ and $d_{s}^{\dagger}(\mathbf{p})$ are the annihilation operator for the particle and the creation operator for the antiparticle, respectively. The field $\psi$ thus annihilates the particle and creates its antiparticle, whereas the hermitian conjugate field annihilates the antiparticle and creates the particle. Because for Majorana fermions particle and antiparticle coincide, for them one has to identify $b_{s}(\mathbf{p})$ and $d_{s}(\mathbf{p})$, i.e. the Fourier expansion of Majorana fields takes the form ${ }^{4}$

$$
\begin{equation*}
\psi(x)=\int \frac{d^{3} p}{(2 \pi)^{3} \sqrt{2 E_{\mathbf{p}}}} \sum_{s}\left[b_{s}(\mathbf{p}) u_{s}(\mathbf{p}) e^{-i p x}+b_{s}^{\dagger}(\mathbf{p}) v_{s}(\mathbf{p}) e^{i p x}\right] . \tag{27}
\end{equation*}
$$

It is possible (and convenient) to choose the phases of the spinors $u_{s}(\mathbf{p})$ and $v_{s}(\mathbf{p})$ in such a way that

$$
\begin{equation*}
v_{s}(\mathbf{p})=\mathcal{C} \bar{u}_{s}^{T}(\mathbf{p}), \quad u_{s}(\mathbf{p})=\mathcal{C} \bar{v}_{s}^{T}(\mathbf{p}) \tag{28}
\end{equation*}
$$

[^4]From these relations it immediately follows that the field (27) satisfies the Majorana selfconjugacy condition (20). The plane-wave decomposition of the Majorana fields (27) is reminiscent of the familiar Fourier expansion of the photon field $A_{\mu}(x)$, which also contains the creation and annihilation operators of only one kind, $a_{\lambda}(\mathbf{p})$ and $a_{\lambda}^{\dagger}(\mathbf{p})$, because the photon is its own antiparticle.

The action of the charge conjugation operation C amounts to interchanging the particle with its antiparticle without changing its momentum or spin polarization state. For a Dirac fermion field (26) it can therefore be represented as

$$
\begin{equation*}
\mathrm{C} b_{s}(\mathbf{p}) \mathrm{C}^{-1}=d_{s}(\mathbf{p}), \quad \mathrm{C} d_{s}^{\dagger}(\mathbf{p}) \mathrm{C}^{-1}=b_{s}^{\dagger}(\mathbf{p}) \tag{29}
\end{equation*}
$$

With the help of eq. (28) one can readily make sure that applying to (26) the particleantiparticle conjugation defined in eq. (9) yields exactly the same result as the C conjugation (29). How about the Majorana fields? For them $d_{s}(\mathbf{p})=b_{s}(\mathbf{p})$, so that the operation in eq. (29) is just the trivial identity transformation which has no effect on the fields. The $\hat{C}$ operation also leaves the Majorana fields unchanged - we have actually defined them through this condition, eq. (20). Thus, we conclude that for free massive fermion fields, both of Dirac and Majorana nature, the C and $\hat{C}$ conjugations are equivalent. As we already pointed out, the two operations are not equivalent when acting on chiral fields.

Consider now the equations of motion for the left-handed and right-handed components of a Majorana field. Eq. (24) tells us that in the 4-component notation (14) the lower 2 -spinor is given by $\xi=-i \sigma^{2} \phi^{*}$. Substituting this into (15) we find [6]

$$
\begin{align*}
\left(\partial_{0}-\boldsymbol{\sigma} \cdot \boldsymbol{\nabla}\right) \phi+m \sigma^{2} \phi^{*} & =0  \tag{30}\\
\left(\partial_{0}+\boldsymbol{\sigma} \cdot \boldsymbol{\nabla}\right) \sigma^{2} \phi^{*}-m \phi & =0 \tag{31}
\end{align*}
$$

It is easy to see that the second of these equations is equivalent to the first one. Indeed, taking the complex conjugate of (31), multiplying on the left by $\sigma^{2}$ and using the relation $\sigma^{2} \boldsymbol{\sigma}^{*} \sigma^{2}=-\boldsymbol{\sigma}$ we obtain eq. (30). Next, let us exclude $\phi^{*}$ from eqs. (30) and (31). By acting on (30) with $\left(\partial_{0}+\boldsymbol{\sigma} \cdot \boldsymbol{\nabla}\right)$ and making use of (31) we find that $\phi$ satisfies the Klein-Gordon equation

$$
\begin{equation*}
\left(\partial^{2}+m^{2}\right) \phi=0 . \tag{32}
\end{equation*}
$$

This means that free Majorana particles obey the standard dispersion relation $E^{2}=\mathbf{p}^{2}+m^{2}$. Thus, kinematically Dirac and Majorana fermions are indistinguishable. They can, however, in principle be told apart through their interactions, as we discuss below.

Let us now turn to the Lagrangian of a free Majorana field. From eqs. (18) and (19) we find that the mass term in the Lagrangian is

$$
\begin{equation*}
\mathcal{L}_{m}=-\frac{m}{2}\left[\overline{\left(\psi_{L}\right)^{c}} \psi_{L}+\overline{\psi_{L}}\left(\psi_{L}\right)^{c}\right]=\frac{m}{2}\left[\psi_{L}^{T} \mathcal{C}^{-1} \psi_{L}+\overline{\psi_{L}} \mathcal{C}^{-1}{\overline{\psi_{L}}}^{T}\right]=\frac{m}{2}\left[\psi_{L}^{T} \mathcal{C}^{-1} \psi_{L}+\text { h.c. }\right] \tag{33}
\end{equation*}
$$

where we have used the second equality in eq. (11), and the factor $1 / 2$ was introduced because $\mathcal{L}_{m}$ is quadratic in $\psi_{L}$. Thus, the Majorana Lagrangian can be written as

$$
\begin{equation*}
\mathcal{L}=\overline{\psi_{L}} i \gamma^{\mu} \partial_{\mu} \psi_{L}+\frac{m}{2}\left[\psi_{L}^{T} \mathcal{C}^{-1} \psi_{L}+\text { h.c. }\right] . \tag{34}
\end{equation*}
$$

Note that it is expressed solely in terms of $\psi_{L}$. In particular, there is no kinetic term for the field $\psi_{R}$ because the left-handed and right-handed components of the Majorana field are not independent. The Lagrangian in (34) can be cast in a more familiar form if we use the notation $\psi=\psi_{L}+\left(\psi_{L}\right)^{c}$. Then, up to a total derivative term that does not contribute to the action, the Lagrangian (34) can be rewritten as ${ }^{5}$

$$
\begin{equation*}
\mathcal{L}=\frac{1}{2} \bar{\psi} i \gamma^{\mu} \partial_{\mu} \psi-\frac{m}{2} \bar{\psi} \psi . \tag{35}
\end{equation*}
$$

It is not difficult to write down the Majorana Lagrangian in the 2 -component notation. From $\psi_{L}=(\phi, 0)^{T}$ and (34) we have

$$
\begin{equation*}
\mathcal{L}=\phi^{\dagger} i\left(\partial_{0}-\boldsymbol{\sigma} \cdot \boldsymbol{\nabla}\right) \phi-\frac{1}{2}\left(\phi^{T} i \sigma^{2} \phi+\text { h.c. }\right) . \tag{36}
\end{equation*}
$$

By comparing the mass term in this expression with (33) one can see that in the 2 -component formalism the role of the particle-antiparticle conjugation matrix $\mathcal{C}$ is played by $i \sigma^{2}$.

From eq. (33) a very important difference between the Dirac and Majorana mass terms follows. The Dirac mass terms $\bar{\psi} \psi$ are invariant with respect to the $U(1)$ transformations

$$
\begin{equation*}
\psi \rightarrow e^{i \alpha} \psi, \quad \bar{\psi} \rightarrow \bar{\psi} e^{-i \alpha} \tag{37}
\end{equation*}
$$

i.e. they conserve the charges associated with the corresponding transformations (electric charge, lepton or baryon number, etc.). At the same time, the Majorana mass terms have the structure $\psi_{L} \psi_{L}+$ h.c. and therefore they break all $U(1)$-charges by two units. Since the electric charge is exactly conserved, this in particular means that no charged particle can have Majorana mass.

Another important point is that the Majorana mass term in eqs. (33) and (34) do not vanish even though the matrix $\mathcal{C}^{-1}$ is antisymmetric (because so is $\mathcal{C}$ ). This follows from the fact that the fermionic quantum fields anticommute, and so the interchange of the two $\psi_{L}$ in $\psi_{L}^{T} \mathcal{C}^{-1} \psi_{L}$ yields an extra minus sign. Similar argument applies to the Majorana mass term in the 2 -component formalism in eq. (36) (note that the matrix $\sigma^{2}$ is antisymmetric). Thus, the Majorana mass is of essentially quantum nature. ${ }^{6}$

[^5]In the massless limit the difference between Dirac and Majorana particles disappears as both actually become Weyl particles. In particular, vanishing Majorana mass means that the free Lagrangian now conserves a $U(1)$ charge corresponding to the transformations (37).

Let us now briefly review the Feynman rules for Majorana particles [7-12]. Unlike for a Dirac fermion, whose quantum field $\psi$ annihilates the particle and creates its antiparticle while $\psi^{\dagger}$ annihilates the antiparticle and creates the particle, in the Majorana case the same field $\chi$ creates and annihilates the corresponding Majorana fermion. This leads to the existence of Wick contractions that are different from the standard ones. As a result, in addition to the usual Feynman propagator

$$
\begin{equation*}
S_{F}\left(x-x^{\prime}\right) \equiv\langle 0| T \chi(x) \bar{\chi}\left(x^{\prime}\right)|0\rangle=\int \frac{d^{4} p}{(2 \pi)^{4}} \frac{i(\not p+m)}{p^{2}-m^{2}+i \varepsilon} e^{-i p\left(x-x^{\prime}\right)} \tag{38}
\end{equation*}
$$

which coincides with the propagator of the Dirac fermion, there exist new types of propagators [7, 8],

$$
\begin{equation*}
\langle 0| T \chi(x) \chi^{T}\left(x^{\prime}\right)|0\rangle=-S_{F}\left(x-x^{\prime}\right) \mathcal{C} \quad \text { and } \quad\langle 0| T \bar{\chi}^{T}(x) \bar{\chi}\left(x^{\prime}\right)|0\rangle=\mathcal{C}^{-1} S_{F}\left(x-x^{\prime}\right), \tag{39}
\end{equation*}
$$

where $\not p \equiv \gamma^{\mu} p_{\mu}$ and we have used the second equality in (11) and the Majorana condition $\chi^{c}=\chi$. Recall that Dirac fermions carry a conserved additive charge which is generically called the fermion number. The flow of this number is usually indicated on Feynman diagrams by arrows on the fermion lines which correspond to the standard propagator (38). If a diagram contains a chain of fermion lines, the fermion number flow is continuous through this chain. As Majorana particles do not carry any conserved additive quantum number, there is no continuous flow of fermion number through Feynman diagrams in the Majorana case. This is reflected in the existence of the fermion number violating propagators (39), which can be graphically represented as lines with two arrows pointing in opposite directions (outwards for the first propagator in (39) and inwards for the second one). In addition, each term of the interaction Lagrangian that contains Majorana fields gives rise to several vertices, depending on the direction of the arrows on the incoming and outgoing lines of Majorana particles. Some of these vertices also contain the particle-antiparticle conjugation matrix $\mathcal{C}[7,8]$. Special care should be taken to get the correct relative signs between different diagrams contributing coherently to the same amplitude. The rules are completed by requiring that diagrams with Majorana fermion loops have an extra factor $1 / 2$ due to the permutation symmetry of the Majorana particles.

The resulting Majorana Feynman rules are rather complicated. They can, however, be simplified by noting that the matrices $\mathcal{C}\left(\right.$ or $\left.\mathcal{C}^{-1}\right)$ that are present in some vertices are always either canceled by the corresponding matrices in the propagators in eq. (39) or eliminated through the proper attribution of the spinors to the external fermionic legs of
the diagram with the help of eq. (28). This leads to much simpler Majorana Feynman rules, with propagators and vertices not containing explicitly the matrix $\mathcal{C}$ [9-11]. In this case the Feynman rules include just the usual propagator (38) for Majorana fermions, and the number of vertices corresponding to each term of the interaction Lagrangian is at most two. The Majorana fermion propagators are depicted by lines with no arrows. Instead of the fermion number flow (which is not conserved) the notion of a fermion flow is introduced. To each diagram a certain (but arbitrary) direction of the fermion flow is attributed, which is used simply as a bookkeeping device; the analytic expressions for the amplitudes are independent of the chosen direction of this flow.

Finally, in appendix B of ref. [12] a very simple set of Majorana Feynman rules is suggested, based on the elimination of the adjoint Majorana fields $\bar{\chi}$ from the kinetic as well as the interaction terms of the Lagrangian through the relation $\bar{\chi}=\overline{\chi^{c}}=-\chi^{T} \mathcal{C}^{-1}$.

For more detailed discussions of Majorana Feynman rules we refer the reader to refs. [712].

## 15.2 $\mathrm{C}, \mathrm{P}, \mathrm{CP}$ and CPT properties of Majorana fermions

Since Majorana fermions are their own antiparticles, they are expected to have special properties with respect to C, CP and CPT transformations.

Consider first the charge conjugation C. ${ }^{7}$ As was discussed above, for massive spinor fields this operation coincides with the particle-antiparticle conjugation $\hat{C}$ defined in eq. (9). The latter, however, without loss of generality can be modified by introducing an arbitrary phase factor $\eta_{\mathrm{C}}^{*}$ on the right-hand side. That is, instead of eq. (9) we can define

$$
\begin{equation*}
\psi^{c} \equiv \eta_{\mathrm{C}}^{*} \mathcal{C} \bar{\psi}^{T}=\eta_{\mathrm{C}}^{*} i \gamma^{2} \psi^{*} \tag{40}
\end{equation*}
$$

Indeed, this will not affect the evolution equation satisfied by $\psi^{c}(x)$ as well as the relation $\left(\psi^{c}\right)^{c}=\psi$. The charge conjugation transformation C (29) can be modified accordingly, so that the equivalence between the C and $\hat{C}$ conjugations is maintained:

$$
\begin{equation*}
\mathrm{C} b_{s}(\mathbf{p}) \mathrm{C}^{-1}=\eta_{\mathrm{C}}^{*} d_{s}(\mathbf{p}), \quad \mathrm{C} d_{s}^{\dagger}(\mathbf{p}) \mathrm{C}^{-1}=\eta_{\mathrm{C}}^{*} b_{s}^{\dagger}(\mathbf{p}) \tag{41}
\end{equation*}
$$

Eqs. (40) and (41) apply to arbitrary spin-1/2 fermions; let us now discuss Majorana fields. Because for them $\mathcal{C} \bar{\psi}^{T}=\psi$, the new definition of $\hat{C}$-conjugation (40) implies that the Majorana condition (20) now takes the form

$$
\begin{equation*}
\psi^{c}(x)=\eta_{\mathrm{C}}^{*} \psi(x) \tag{42}
\end{equation*}
$$

[^6]Since for Majorana fields one has to identify $d_{s}(\mathbf{p})=b_{s}(\mathbf{p})$, eq. (41) becomes

$$
\begin{equation*}
\mathrm{C} b_{s}(\mathbf{p}) \mathrm{C}^{-1}=\eta_{\mathrm{C}}^{*} b_{s}(\mathbf{p}), \quad \mathrm{C} b_{s}^{\dagger}(\mathbf{p}) \mathrm{C}^{-1}=\eta_{\mathrm{C}}^{*} b_{s}^{\dagger}(\mathbf{p}) \tag{43}
\end{equation*}
$$

Hermitian conjugation of the first of these two equalities yields $\mathrm{Cb}_{s}^{\dagger}(\mathbf{p}) \mathrm{C}^{-1}=\eta_{\mathrm{C}} b_{s}^{\dagger}(\mathbf{p})$. The consistency of this relation and the second one in eq. (43) requires that $\eta_{C}$ be real, i.e. $\eta_{C}= \pm 1$. Next, let us apply the second relation in (43) to the vacuum state. Assuming the vacuum to be even under the charge conjugation, we find

$$
\begin{equation*}
\mathrm{C}|\mathbf{p}, s\rangle=\eta_{\mathrm{C}}|\mathbf{p}, s\rangle \tag{44}
\end{equation*}
$$

where $|\mathbf{p}, s\rangle$ is the 1-particle Majorana state with momentum $\mathbf{p}$ and spin projection $s$.
The Majorana condition (42) and eq. (44) imply that the Majorana state is an eigenstate of charge conjugation $C$, and $\eta_{C}$ is its charge parity. It should be stressed, however, that this is, strictly speaking, only valid when C is exactly conserved. The above description certainly applies to free Majorana fermions, since the corresponding action is charge conjugation invariant. ${ }^{8}$ However, the charge parity $\eta_{\mathrm{C}}$, apart from being real, is completely arbitrary and therefore unphysical in this case. A physical (i.e. interacting) Majorana particle is an eigenstate of C only when all its interactions are C-invariant. ${ }^{9}$ The C-parity of a Majorana particle is then constrained by the C-transformation properties of the other fields that enter its interaction Lagrangian. If C is only an approximate symmetry of the theory, the Majorana particle will be an approximate eigenstate of C , to the same extent to which charge conjugation invariance is satisfied.

The situation is completely different for neutrinos, which are the prime candidates for being Majorana particles. The point is that their charged-current weak interactions are maximally C-violating. Indeed, these interactions are left-handed (i.e. of the $V-A$ form), whereas charge conjugation would transform them into the right-handed $(V+A)$ interactions which do not exist in the standard model based on the gauge group $S U(2)_{L} \times U(1)$. Thus, for Majorana neutrinos C-parity does not bear any physical sense.

However, CP is a good approximate symmetry of the leptonic sector of the standard model. Indeed, it is is an exact symmetry of the gauge interactions, and in the minimally extended (to include non-zero neutrino mass) standard model it can only be violated by the neutrino mass generating sector. The corresponding CP-violation effects are very difficult to observe - in particular, they have not been unambiguously observed by the time of

[^7]publication of this book. Therefore, in many situations CP violation in the leptonic sector can be ignored. In other words, in some regards Majorana neutrinos can be considered as CP-eigenstates with certain CP-parities. This, however, is not in general true when possible CP-violating effects play a major role. We discuss these effects in secs. 15.3, 15.7 and 15.8.

The properties of Majorana particles with respect to CP (assuming that it is a good symmetry) and CPT can be studied similarly to their properties under C transformation [13]. The results are summarized in Table 1.

In deriving the properties of Majorana neutrinos under the discrete symmetries one can make use of the following properties of the spinors $u_{s}(\mathbf{p})$ and $v_{s}(\mathbf{p})$ :

$$
\begin{array}{ll}
\gamma^{0} u_{s}(\mathbf{p})=u_{s}(-\mathbf{p}), & \gamma^{0} v_{s}(\mathbf{p})=-v_{s}(-\mathbf{p}) \\
u_{s}^{*}(\mathbf{p})=(-1)^{s+1 / 2} \gamma^{1} \gamma^{3} u_{-s}(-\mathbf{p}), & v_{s}^{*}(\mathbf{p})=(-1)^{s+1 / 2} \gamma^{1} \gamma^{3} v_{-s}(-\mathbf{p}) \tag{45}
\end{array}
$$

where the sign factors in the second line correspond to the phase convention $\gamma_{5} u_{s}(\mathbf{p})=$ $(-1)^{s-1 / 2} v_{-s}(\mathbf{p})$. This choice of the phases is consistent with that in eq. (28). It should also be kept in mind that, while C and P are unitary operators, T is antiunitary, and so is CPT.

| Symmetry <br> operation | Effect on $\psi(t, \mathbf{x})$ | Effect on $\|\mathbf{p}, s\rangle$ | Restriction |
| :--- | :--- | :--- | :--- |
| C | $\eta_{\mathrm{C}}^{*} i \gamma^{2} \psi^{*}(t, \mathbf{x})$ | $\eta_{\mathrm{C}}\|\mathbf{p}, s\rangle$ | $\eta_{\mathrm{C}}= \pm 1$ |
| CP | $\eta_{\mathrm{CP}}^{*} i \gamma^{0} \gamma^{2} \psi^{*}(t,-\mathbf{x})$ | $\eta_{\mathrm{CP}}\|-\mathbf{p}, s\rangle$ | $\eta_{\mathrm{CP}}= \pm i$ |
| CPT | $-\eta_{\mathrm{CPT}}^{*} \gamma_{5} \psi^{*}(-t,-\mathbf{x})$ | $\eta_{\mathrm{CPT}}^{s}\|\mathbf{p},-s\rangle$ | $\eta_{\mathrm{CPT}}= \pm i$ |

Table 1: Effects of C, CP and CPT operations on a Majorana field $\psi(t, \mathbf{x})$ and on the corresponding one-particle Majorana state $|\mathbf{p}, s\rangle$. Here $\eta_{\mathrm{CPT}}^{s} \equiv(-1)^{s-1 / 2} \eta_{\mathrm{CPT}}$.

### 15.3 Mixing and oscillations of Majorana neutrinos

In sec. 15.1 we considered the mass term of a lone Majorana particle, eq. (33). This expression is readily generalized to the case when there are $n$ Majorana fermions which in general can mix with each other:

$$
\begin{equation*}
\mathcal{L}_{m}=-\frac{1}{2}\left[\overline{\left(\psi_{L}\right)^{c}} m \psi_{L}+\overline{\psi_{L}} m^{\dagger}\left(\psi_{L}\right)^{c}\right]=\frac{1}{2}\left[\psi_{L}^{T} \mathcal{C}^{-1} m \psi_{L}+\text { h.c. }\right] . \tag{46}
\end{equation*}
$$

Here $\psi=\left(\psi_{1}, \ldots, \psi_{n}\right)^{T}$ and $m$ is an $n \times n$ matrix. Using the anticommutation property of the fermion fields and $\mathcal{C}^{T}=-\mathcal{C}$, it is easy to show that the matrix $m$ must be symmetric: $m^{T}=m$. Eq. (46) applies to any set of Majorana particles; in the rest of this section we shall specifically consider Majorana neutrinos and their oscillations.

### 15.3.1 Neutrinos with a Majorana mass term

Consider first the case of $n$ standard lepton generations, consisting each of an $S U(2)_{L^{-}}$ doublet of left-handed neutrino and charged lepton fields $l_{\alpha}=\left(\nu_{\alpha L}, e_{\alpha L}\right)^{T}$ and an $S U(2)_{L^{-}}$ singlet right-handed charged lepton field $e_{\alpha R}(\alpha=e, \mu, \tau, \ldots)[16,17]$. In the standard model extended to include the mass generation mechanism for Majorana neutrinos, the terms of the Lagrangian that are relevant to neutrino oscillations include the charged-current (CC) weak interaction term and the mass terms of the charged leptons and neutrinos:

$$
\begin{equation*}
\mathcal{L}_{w+m}=-\frac{g}{\sqrt{2}}\left(\bar{e}_{L}^{\prime} \gamma^{\mu} \nu_{L}^{\prime}\right) W_{\mu}-\bar{e}_{L}^{\prime} m_{l}^{\prime} e_{R}^{\prime}+\frac{1}{2} \nu_{L}^{\prime T} \mathcal{C}^{-1} m^{\prime} \nu_{L}^{\prime}+\text { h.c. } \tag{47}
\end{equation*}
$$

Here $g$ is the CC gauge coupling constant, $W_{\mu}$ is the $W^{-}$-boson field, and all leptons are assembled in vectors in the generation space: $\nu_{L}^{\prime}=\left(\nu_{e L}, \nu_{\mu L}, \nu_{\tau L}, \ldots\right)^{T}$ and similarly for $e_{L}^{\prime}$ and $e_{R}^{\prime}$. Since the CC weak interaction Lagrangian in eq. (47) is diagonal in the chosen basis, the latter is called the weak-eigenstate basis. The matrices $m_{l}^{\prime}$ and $m^{\prime}$ are, however, in general not diagonal in this basis. For $n$ leptonic generations, the mass matrix of the charged leptons $m_{l}^{\prime}$ is a general complex $n \times n$ matrix, whereas the Majorana mass matrix of neutrinos $m^{\prime}$ is a complex symmetric $n \times n$ matrix. Recall now that an arbitrary square matrix $A$ can be diagonalized by a bi-unitary transformation according to $A_{\text {diag }}=V_{1}^{\dagger} A V_{2}$, where $A_{\text {diag }}$ is a diagonal matrix with non-negative diagonal elements. Similarly, a symmetric square matrix $B$ is diagonalized by a transformation with a single unitary matrix: $B_{d i a g}=U^{T} B U$, where all the diagonal elements of $B_{\text {diag }}$ are non-negative. We therefore perform the basis transformations of the lepton fields according to

$$
\begin{equation*}
e_{L}^{\prime}=V_{L} e_{L}, \quad e_{R}^{\prime}=V_{R} e_{R}, \quad \nu_{L}^{\prime}=U_{L} \nu_{L} \tag{48}
\end{equation*}
$$

with $V_{L}, V_{R}$ and $U_{L}$ being unitary matrices chosen such that they diagonalize the chargedlepton and neutrino mass matrices:

$$
\begin{equation*}
V_{L}^{\dagger} m_{l}^{\prime} V_{R}=m_{l}, \quad U_{L}^{T} m^{\prime} U_{L}=m \quad\left(m_{l}, m-\text { diagonal mass matrices }\right) \tag{49}
\end{equation*}
$$

Note that the kinetic Lagrangians of neutrinos and of the left- and right-handed charged leptons are invariant under these transformations. In the new (unprimed) basis eq. (47) takes the form

$$
\begin{equation*}
\mathcal{L}_{w+m}=-\frac{g}{\sqrt{2}} \sum_{\alpha, i} \bar{e}_{\alpha L} \gamma^{\mu} U_{\alpha i} \nu_{i L} W_{\mu}-\sum_{\alpha} m_{l \alpha} \bar{e}_{\alpha L} e_{\alpha R}+\frac{1}{2} \sum_{i} m_{i} \nu_{i L}^{T} \mathcal{C}^{-1} \nu_{i L}+h . c . \tag{50}
\end{equation*}
$$

Here $m_{l \alpha}(a=e, \mu, \tau, \ldots)$ and $m_{i}(i=1,2,3, .$.$) are the diagonal elements of the mass$ matrices $m_{l}$ and $m$ respectively, i.e. they are the masses of the charged leptons and of the mass-eigenstate neutrinos. The matrix

$$
\begin{equation*}
U \equiv V_{L}^{\dagger} U_{L} \tag{51}
\end{equation*}
$$

is the leptonic mixing matrix, also called the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix $[18,19]$. The flavour-eigenstate neutrino fields are defined as

$$
\begin{equation*}
\nu_{\alpha L}=\sum_{i=1}^{n} U_{\alpha i} \nu_{i L} \tag{52}
\end{equation*}
$$

In terms of these fields the CC-interaction part of the Lagrangian (50) takes the form $\mathcal{L}_{w}=-\frac{g}{\sqrt{2}} \sum_{\alpha} \bar{e}_{\alpha L} \gamma^{\mu} \nu_{\alpha L} W_{\mu}+h . c$. , i.e. the flavour eigenstates $\nu_{e}, \nu_{\mu}, \nu_{\tau}, \ldots$ are the neutrinos emitted or absorbed together with the charged leptons $e, \mu, \tau, \ldots$ respectively.

Let us introduce the 4 -component neutrino fields

$$
\begin{equation*}
\chi_{i}=\nu_{i L}+\left(\nu_{i L}\right)^{c}, \quad i=1, \ldots, n \tag{53}
\end{equation*}
$$

Using the relations $\nu_{i L}^{T} \mathcal{C}^{-1} \nu_{i L}=-\overline{\left(\nu_{i L}\right)^{c}} \nu_{i L},\left(\nu_{i L}^{T} \mathcal{C}^{-1} \nu_{i L}\right)^{\dagger}=-\overline{\nu_{i L}}\left(\nu_{i L}\right)^{c}$ which follow from eq. (11), one can rewrite the neutrino mass term in eq. (50) as

$$
\begin{equation*}
\mathcal{L}_{\nu m}=-\frac{1}{2} \sum_{i=1}^{n} m_{i} \bar{\chi}_{i} \chi_{i} \tag{54}
\end{equation*}
$$

Eqs. (52) and (53) mean that the neutrino flavour eigenstates $\nu_{\alpha L}$ are linear superpositions of the left-handed components of the $n$ mass eigenstates $\chi_{i}$. From eq. (53) it follows that $\chi_{i}^{c}=\chi_{i}$, i.e. the massive neutrino fields are Majorana fields in the case we consider.

The Lagrangian (47) (or equivalently (50)) implies that massive neutrinos are in general mixed and leads to the phenomenon of neutrino flavour oscillations [18, 19]. The oscillation probability, i.e. the probability that a relativistic neutrino produced as a flavour eigenstate $\nu_{\alpha}$ will be in a flavour eigenstate $\nu_{\beta}$ after having propagated a distance $L$ in vacuum, is $[20,21]$

$$
\begin{equation*}
P\left(\nu_{\alpha} \rightarrow \nu_{\beta} ; L\right)=\left|\sum_{i} U_{\beta i} e^{-i \frac{\Delta m_{i j}^{2}}{2 p} L} U_{\alpha i}^{*}\right|^{2} \tag{55}
\end{equation*}
$$

where $p$ is the modulus of the mean momentum of the neutrino state, $\Delta m_{i j}^{2}=m_{i}^{2}-m_{j}^{2}$ and for the index $j$ one can take any fixed value between 1 and $n$. Eq. (55) can be equivalently written as
$P\left(\nu_{\alpha} \rightarrow \nu_{\beta} ; L\right)=\sum_{i, j} \operatorname{Re}\left(U_{\alpha j} U_{\beta j}^{*} U_{\alpha i}^{*} U_{\beta i}\right) \cos \left(\frac{\Delta m_{i j}^{2}}{2 p} L\right)+\sum_{i, j} \operatorname{Im}\left(U_{\alpha j} U_{\beta j}^{*} U_{\alpha i}^{*} U_{\beta i}\right) \sin \left(\frac{\Delta m_{i j}^{2}}{2 p} L\right)$.

Let us now discuss the general properties of the leptonic mixing matrix $U$. Being an $n \times n$ unitary matrix, it depends on $n^{2}$ independent parameters, of which $n(n-1) / 2$ are mixing angles and $n(n+1) / 2$ are complex phases. Not all of these phases are physical,
though. As follows from eq. (50), one can always remove $n$ phases from $U$ by rephasing the left-handed fields of charged leptons according to $e_{\alpha L} \rightarrow e^{i \varphi_{\alpha}} e_{\alpha L}$, which allows one to fix the phases of one column of the matrix $U$. If one also rephases the right-handed charged lepton fields in the same way, i.e. $e_{\alpha R} \rightarrow e^{i \varphi} e_{\alpha R}$, the phase change of the fields $e_{\alpha L}$ in the mass term of the charged leptons in (50) will be compensated, and therefore eq. (50) will remain unchanged. It is easy to see that the kinetic terms of the Lagrangian of $e_{L}$ and $e_{R}$ are also invariant under the field rephasing. Thus, the leptonic Lagrangian is invariant with respect to the above rephasing of the charged lepton fields, which means that $n$ out of the $n(n+1) / 2$ phases in $U$ are unphysical.

How about rephasing the neutrino fields? Consider first the case when neutrinos are Dirac particles (which means that we should add $n$ right-handed neutrino fields to our model). Then their mass term is similar to that of the charged leptons. In that case it is possible to similarly rephase the left-handed and right-handed neutrino fields without modifying their mass term, which could be used to fix the phases of the elements of one line of the matrix $U$. However, the number of the phases that can be removed from $U$ in this way is $n-1$ rather than $n$, because the phase of one element which is at the intersection of the selected line of $U$ and the column whose phases have already been fixed by the rephasing of the charged lepton fields can no longer be modified. Thus, the total number of physical phases characterizing the leptonic matrix $U$ in the case of Dirac neutrinos is

$$
\begin{equation*}
N_{p h}^{\mathrm{D}}=\frac{n(n+1)}{2}-n-(n-1)=\frac{(n-1)(n-2)}{2} . \tag{57}
\end{equation*}
$$

Note that the quantities $U_{\alpha j} U_{\beta j}^{*} U_{\alpha i}^{*} U_{\beta i}$ that enter the expression (56) for the oscillation probabilities are invariant with respect to the rephasing of the charged-lepton and neutrino fields.

Let us now return to Majorana neutrinos. In this case the neutrino fields cannot be rephased since the Majorana mass term is of the type $\nu_{L} \nu_{L}+h . c$. rather than $\bar{\nu}_{L} \nu_{R}+h . c .$, and therefore it is not rephasing-invariant. As a result, the total number of physical phases characterizing the leptonic matrix $U$ in the case of the Majorana neutrinos is

$$
\begin{equation*}
N_{p h}^{\mathrm{M}}=\frac{n(n+1)}{2}-n=\frac{n(n-1)}{2} . \tag{58}
\end{equation*}
$$

The extra $n-1$ physical phases that are present in the Majorana neutrino case can be collected in a diagonal matrix of phases which is factored out of $U$ as a right-hand side multiplier. Indeed, from eq. (50) it is seen that such a factorization isolates in a diagonal factor the phases that could have been absorbed into the rephasing of the $\nu_{L}$ fields if the neutrinos were Dirac particles. Thus, one can write

$$
\begin{equation*}
U=\tilde{U} \cdot \operatorname{diag}\left(1, e^{i \varphi_{1}}, e^{i \varphi_{2}}, \ldots, e^{i \varphi_{n-1}}\right) \equiv \tilde{U} K \tag{59}
\end{equation*}
$$

Here the matrix $\tilde{U}$ contains only $(n-1)(n-2) / 2$ phases that are relevant also in the Dirac neutrino case, whereas the factor $K$ contains the extra 'Majorana-type' phases. The position of the unit element in $K$ is irrelevant, as the overall phase of the matrix $U$ is unobservable. From eqs. (57) and (58) it follows that in the Dirac neutrino case the mixing matrix $U$ contains physical complex phases only for $n \geq 3$ generations, whereas in the Majorana case the physical phases are in general there for $n \geq 2$.

The reason why we paid so much attention to the phases of the leptonic mixing matrix $U$ is that they lead to CP-violating effects in the leptonic sector. CP-conjugation transforms left-handed neutrinos into their antiparticles - right-handed antineutrinos. Since CP-conjugation of a fermionic field includes complex conjugation (see Table 1), the oscillation probability $P\left(\bar{\nu}_{a} \rightarrow \bar{\nu}_{b} ; L\right)$ is described by the right-hand side of eq. (55) with the matrix $U$ substituted by $U^{*}$. Complex conjugation means that the signs of all the phases characterizing $U$ must be flipped; for this reason these phases are called CP-violating phases. In particular, if $U$ bears non-removable complex phases beyond those contained in the matrix $K$, neutrino oscillations violate CP-invariance, i.e.

$$
\begin{equation*}
\Delta P_{\alpha \beta}^{\mathrm{CP}}(L) \equiv P\left(\nu_{\alpha} \rightarrow \nu_{\beta} ; L\right)-P\left(\bar{\nu}_{\alpha} \rightarrow \bar{\nu}_{\beta} ; L\right) \neq 0 \tag{60}
\end{equation*}
$$

The quantity $\Delta P_{\alpha \beta}^{\mathrm{CP}}(L)$ coincides (up to the factor of two) with the CP-odd part of the $\nu_{\alpha} \rightarrow \nu_{\beta}$ oscillation probability, which is given by the second term on the right-hand side of eq. (56), whereas the first term in (56) corresponds to the CP-even part of the probability.

Can one find out whether neutrinos are Dirac or Majorana particles by studying neutrino oscillations? Unfortunately, this is not possible. It turns out that for Dirac neutrinos the oscillation probabilities are given by exactly the same formulas, eqs. (55) or (56), as those for Majorana neutrinos. Moreover, the extra Majorana-type phases that enter the leptonic mixing matrix $U$ in the Majorana neutrino case are not observable in neutrino oscillations. Indeed, the matrix $K$ in eq. (59) and the matrix $\exp \left[-i\left(\Delta m_{i j}^{2} / 2 p\right) L\right]$ for fixed $j$ that enters eq. (55) are both diagonal and therefore commute, so that $K$ drops out of the expression for the oscillation probability (55). A similar argument applies to neutrino oscillations in matter. The Majorana-type phases, however, enter some other physical observables and so are in general observable quantities. We discuss these observables in secs. 15.7 and 15.8.

For future reference, we give here the leptonic mixing matrix $U$ for the case of three leptonic generations with Majorana neutrinos in the so-called standard parameterization:

$$
U=\left(\begin{array}{ccc}
c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i \delta_{\mathrm{CP}}}  \tag{61}\\
-s_{12} c_{23}-c_{12} s_{13} s_{23} e^{i \delta_{\mathrm{CP}}} & c_{12} c_{23}-s_{12} s_{13} s_{23} e^{i \delta_{\mathrm{CP}}} & c_{13} s_{23} \\
s_{12} s_{23}-c_{12} s_{13} c_{23} e^{i \delta_{\mathrm{CP}}} & -c_{12} s_{23}-s_{12} s_{13} c_{23} e^{i \delta_{\mathrm{CP}}} & c_{13} c_{23}
\end{array}\right)\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & e^{i \varphi_{1}} & 0 \\
0 & 0 & e^{i \varphi_{2}}
\end{array}\right) .
$$

Here $c_{i j} \equiv \cos \theta_{i j}, s_{i j} \equiv \sin \theta_{i j}$, where $\theta_{i j}$ are the mixing angles, $\delta_{\mathrm{CP}}$ is the Dirac-type CP-violating phase and $\varphi_{1,2}$ are the Majorana-type CP-violating phases.

### 15.3.2 General case of Dirac + Majorana mass term

Consider now the case in which, in addition to $n$ standard leptonic generations with lefthanded neutrino fields $\nu_{\alpha L}$ being parts of $S U(2)_{L}$ leptonic doublets, there exist $k$ righthanded neutrino fields $\nu_{\sigma R}$ that are electroweak singlets, i.e. singlets with respect to both weak isospin group $S U(2)_{L}$ and hypercharge $U(1)$ [22-27]. Such neutrinos don't have electroweak gauge interactions (though may have, e.g., Yukawa interactions with leptonic and Higgs doublets) and therefore are often called 'sterile neutrinos'. In contrast to this, the usual $S U(2)_{L}$-doublet neutrinos $\nu_{L}$ are called 'active neutrinos'.

> Since $\nu_{R}$ are electroweak singlets, they do not contribute to the so-called chiral gauge anomalies and so their number is not fixed by the requirement of the anomaly cancellation. In particular, their number need not coincide with the number of the leptonic generations $n$, i.e. in general $k \neq n$. Let us stress that these extra neutrinos are sterile not because they are right-handed their $\hat{C}$-conjugates are left-handed and yet also sterile - but because they are electroweak singlets.

In the considered case the most general neutrino mass term contains the Majorana masses $m_{L}$ and $m_{R}$ for the left-handed and right-handed neutrino fields respectively, as well as the Dirac mass $m_{D}$ that couples the $\nu_{L}$ 's with the $\nu_{R}$ 's:

$$
\begin{equation*}
\mathcal{L}_{m}=\frac{1}{2} \nu_{L}^{\prime T} \mathcal{C}^{-1} m_{L} \nu_{L}^{\prime}-\overline{\nu_{R}^{\prime}} m_{D} \nu_{L}^{\prime}+\frac{1}{2} \nu_{R}^{\prime T} \mathcal{C}^{-1} m_{R}^{*} \nu_{R}^{\prime}+\text { h.c. } \tag{62}
\end{equation*}
$$

Here we have assembled the $n$ left-handed and $k$ right-handed neutrino fields into the vectors $\nu_{L}^{\prime}$ and $\nu_{R}^{\prime}$. The quantities $m_{L}$ and $m_{R}$ are complex symmetric $n \times n$ and $k \times k$ matrices respectively, $m_{D}$ is a complex $k \times n$ matrix, and we have introduced the right-handed Majorana mass matrix through its complex conjugate to simplify the further notation. Introducing the vector of $n+k$ left-handed fields

$$
\begin{equation*}
n_{L}=\binom{\nu_{L}^{\prime}}{\left(\nu_{R}^{\prime}\right)^{c}}=\binom{\nu_{L}^{\prime}}{\nu_{L}^{\prime c}} \tag{63}
\end{equation*}
$$

we can rewrite eq. (62) as

$$
\begin{equation*}
\mathcal{L}_{m}=\frac{1}{2} n_{L}^{T} \mathcal{C}^{-1} \mathcal{M} n_{L}+\text { h.c. } \tag{64}
\end{equation*}
$$

where the matrix $\mathcal{M}$ has the form

$$
\mathcal{M}=\left(\begin{array}{ll}
m_{L} & m_{D}^{T}  \tag{65}\\
m_{D} & m_{R}
\end{array}\right)
$$

In deriving eq. (64) we have used the relations

$$
\begin{equation*}
\left(\psi_{R}^{T} \mathcal{C}^{-1} m^{*} \psi_{R}\right)^{\dagger}=\left(\psi_{L}^{c}\right)^{T} \mathcal{C}^{-1} m \psi_{L}^{c}, \quad \overline{\psi_{R}} m \psi_{L}=-\left(\psi_{L}^{c}\right)^{T} \mathcal{C}^{-1} m \psi_{L}=-\psi_{L}^{T} \mathcal{C}^{-1} m^{T} \psi_{L}^{c} \tag{66}
\end{equation*}
$$

which follow from eqs. (28) and (10). The matrix $\mathcal{M}$ is complex symmetric, so it can be diagonalized with a single unitary matrix. We therefore write

$$
\begin{equation*}
n_{a L}=\sum_{i=1}^{n+k} \mathcal{U}_{a i} \chi_{i L}, \quad \mathcal{U}^{T} \mathcal{M} \mathcal{U}=\mathcal{M}_{d} \tag{67}
\end{equation*}
$$

where $\mathcal{M}_{d}$ is a diagonal $(m+n) \times(m+n)$ matrix with non-negative diagonal elements $\mathcal{M}_{d i}$. In terms of the fields $\chi_{i L}$ the neutrino mass term (64) reads

$$
\begin{equation*}
\mathcal{L}_{m}=\frac{1}{2} \sum_{i=1}^{n+k} \mathcal{M}_{d i} \chi_{i L}^{T} \mathcal{C}^{-1} \chi_{i L}+\text { h.c. } \tag{68}
\end{equation*}
$$

Introducing the 4-component massive neutrino fields $\chi_{i}$ as

$$
\begin{equation*}
\chi_{i}=\chi_{i L}+\left(\chi_{i L}\right)^{c}, \quad i=1, \ldots, n+k \tag{69}
\end{equation*}
$$

we can rewrite the neutrino mass term in the mass eigenstate basis (68) in the standard form:

$$
\begin{equation*}
\mathcal{L}_{m}=-\frac{1}{2} \sum_{i=1}^{n+k} m_{i} \bar{\chi}_{i} \chi_{i} \tag{70}
\end{equation*}
$$

In eq. (67) the index $a$ can take $n+k$ values; we will denote collectively the first $n$ of them with $\alpha$ or $\beta$ and the last $k$ with $\sigma$ or $\rho$. Eq. (67) yields

$$
\begin{equation*}
\nu_{\alpha L}=\sum_{i=1}^{n+k} \mathcal{U}_{\alpha i} \chi_{i L}, \quad\left(\nu_{\sigma R}\right)^{c}=\sum_{i=1}^{n+k} \mathcal{U}_{\sigma i} \chi_{i L} \tag{71}
\end{equation*}
$$

This means that left-handed active neutrinos and left-handed sterile antineutrinos are linear combinations of the left-handed components of all $n+k$ mass-eigenstate fields $\chi_{i}$.

From eq. (69) it follows that $\chi_{i}^{c}=\chi_{i}$, i.e. the neutrino mass eigenstates are Majorana fermions in the case we currently consider, just as in the pure Majorana mass term case discussed in the previous subsection. This is a general feature of fermion mass models: if the fermions possess Majorana mass terms, then, independently of whether or not the Dirac
mass terms are also present, the mass eigenstates are always Majorana particles.
This is actually easy to understand by counting the number of the field degrees of freedom. In the Majorana mass case studied in sec. 15.3 .1 one has $n$ two-component neutrino fields, and the neutrino mass matrix has in general $n$ distinct eigenvalues. Each massive neutrino field then has two degrees of freedom, i.e. it should be a Majorana field. In the pure Dirac case there are $2 n$ two-component fields ( $n$ left-handed and $n$ right-handed), and the mass matrix has $n$ eigenvalues. This means that each mass eigenstate has four degrees of freedom, i.e. is a Dirac field. In the Dirac + Majorana mass case there are $n+k 2$-component fields, $n$ left-handed and $k$ right-handed. The matrix $\mathcal{M}$ has $n+k$ in general distinct eigenvalues, which means that each neutrino mass eigenstate is characterized by two degrees of freedom, i.e. is a Majorana field.

If some of the mass eigenvalues coincide, the corresponding 2 -component Majorana fields can merge into 4 -component Dirac ones. We will consider this phenomenon in the next subsection.

Let us now discuss neutrino oscillations in the Dirac + Majorana ( $\mathrm{D}+\mathrm{M}$ ) mass scheme that we are now considering. Unlike in the pure Dirac or pure Majorana mass cases, in the $\mathrm{D}+\mathrm{M}$ scheme two new types of neutrino oscillations become possible: active - sterile and sterile - sterile oscillations. The oscillations between the active neutrino species are described by the same expression as in eq. (55) but with the matrix $U$ replaced by $\mathcal{U}$ and the summation over $i$ extending from 1 to $n+k$. The probability of oscillations between the active and sterile neutrinos is

$$
\begin{equation*}
P\left(\nu_{\alpha L} \rightarrow \nu_{\sigma L}^{c} ; L\right)=\left|\sum_{i=1}^{n+k} \mathcal{U}_{\sigma i} e^{-i \frac{\Delta m_{i j}^{2}}{2 p} L} \mathcal{U}_{\alpha i}^{*}\right|^{2} \tag{72}
\end{equation*}
$$

If a mechanism by which sterile neutrinos can be produced and detected exists, ${ }^{10}$ one can in principle observe sterile - sterile neutrino oscillations, whose probability is

$$
\begin{equation*}
P\left(\nu_{\sigma L}^{c} \rightarrow \nu_{\rho L}^{c} ; L\right)=\left|\sum_{i=1}^{n+k} \mathcal{U}_{\rho i} e^{-i \frac{\Delta m_{i j}^{2}}{2 p} L} \mathcal{U}_{\sigma i}^{*}\right|^{2} \tag{73}
\end{equation*}
$$

Eq. (73) describes the oscillations between the left-handed sterile neutrino states $\nu_{L \sigma}^{c}=$ $\left(\nu_{R \sigma}\right)^{c}$ and $\nu_{L \rho}^{c}=\left(\nu_{R \rho}\right)^{c}$; the oscillations between the corresponding right-handed states $\nu_{R \sigma}$ and $\nu_{R \rho}$ can be obtained from eq. (73) by replacing $\mathcal{U} \leftrightarrow \mathcal{U}^{*}$.

If the sterile neutrinos are completely undetectable, one can only observe active - active and active - sterile oscillations, the latter manifesting themselves through disappearance of the active neutrinos.

[^8]
### 15.3.3 Dirac and pseudo-Dirac neutrino limits in the $\mathrm{D}+\mathrm{M}$ case

As we have pointed out above, if Majorana mass terms are present in a fermion mass model, the mass-eigenstate fermions are always Majorana particles, even when the Dirac mass terms are present as well. One can expect, however, that in the limit when the Majorana mass terms are much smaller than the Dirac ones, the properties of the mass eigenstates would become close to those of Dirac fermions. To see how this happens, it is instructive to consider the simple one-generation neutrino case with Majorana and Dirac mass terms. The quantities $m_{L}, m_{R}$ and $m_{D}$ are then just numbers, and $\mathcal{M}$ is a $2 \times 2$ matrix. For simplicity we shall assume all the mass parameters to be real. This, in particular, means that CP is conserved in the neutrino mass sector, i.e. the free mass eigenstates are also eigenstates of CP . The matrix $\mathcal{M}$ can be diagonalized by the transformation $O^{T} \mathcal{M} O=\mathcal{M}_{d}$ where $O$ is a real orthogonal $2 \times 2$ matrix and $\mathcal{M}_{d}=\operatorname{diag}\left(m_{1}, m_{2}\right)$. We introduce the fields $\chi_{L}$ through $n_{L}=O \chi_{L}$, or

$$
n_{L}=\binom{\nu_{L}}{\nu_{L}^{c}}=\left(\begin{array}{cc}
\cos \theta & \sin \theta  \tag{74}\\
-\sin \theta & \cos \theta
\end{array}\right)\binom{\chi_{1 L}}{\chi_{2 L}}
$$

Here $\chi_{1 L}$ and $\chi_{2 L}$ are the left-handed components of the neutrino mass eigenstates. The mixing angle $\theta$ is given by

$$
\begin{equation*}
\tan 2 \theta=\frac{2 m_{D}}{m_{R}-m_{L}} \tag{75}
\end{equation*}
$$

and the neutrino mass eigenvalues are

$$
\begin{equation*}
m_{1,2}=\frac{m_{R}+m_{L}}{2} \mp \sqrt{\left(\frac{m_{R}-m_{L}}{2}\right)^{2}+m_{D}^{2}} \tag{76}
\end{equation*}
$$

They are real but can be of either sign. The neutrino mass term can now be rewritten as

$$
\begin{align*}
\mathcal{L}_{m} & =\frac{1}{2} n_{L}^{T} \mathcal{C}^{-1} \mathcal{M} n_{L}+\text { h.c. }=\frac{1}{2} \chi_{L}^{T} \mathcal{C}^{-1} \mathcal{M}_{d} \chi_{L}+\text { h.c. } \\
& =\frac{1}{2}\left(m_{1} \chi_{1 L}^{T} \mathcal{C}^{-1} \chi_{1 L}+m_{2} \chi_{2 L}^{T} \mathcal{C}^{-1} \chi_{2 L}\right)+\text { h.c. }=\frac{1}{2}\left(\left|m_{1}\right| \bar{\chi}_{1} \chi_{1}+\left|m_{2}\right| \bar{\chi}_{2} \chi_{2}\right) . \tag{77}
\end{align*}
$$

Here we have defined

$$
\begin{equation*}
\chi_{1}=\chi_{1 L}+\eta_{1}\left(\chi_{1 L}\right)^{c}, \quad \chi_{2}=\chi_{2 L}+\eta_{2}\left(\chi_{2 L}\right)^{c} \tag{78}
\end{equation*}
$$

with $\eta_{i}=1$ or -1 for $m_{i}>0$ or $<0$ respectively. It follows immediately from eq. (78) that the mass-eigenstates fields $\chi_{1}$ and $\chi_{2}$ describe Majorana neutrinos. The relative signs of the
mass eigenvalues ( $\eta_{1}$ and $\eta_{2}$ ) determine the relative CP parities of $\chi_{1}$ and $\chi_{2}$; the physical masses $\left|m_{1}\right|$ and $\left|m_{2}\right|$ are positive, as they should be.

Instead of using a real orthogonal matrix $O$ to diagonalize $\mathcal{M}$ one could employ a unitary matrix $U=O K$ with $K$ being a diagonal matrix of phases, as in eq. (59). Choosing for the positive mass eigenvalues of $\mathcal{M}$ the diagonal elements of $K$ to be 1 and for the negative ones $\pm i$, one can write $U^{T} \mathcal{M} U=\mathcal{M}_{d}$, where $\mathcal{M}_{d}$ now does not have any negative diagonal elements. Correspondingly, eq. (74) should be replaced with $n_{L}=U \chi_{L}$. In this way it is no longer necessary to introduce the factors $\eta_{1,2}$, i.e. instead of eq. (78) we have

$$
\begin{equation*}
\chi_{1}=\chi_{1 L}+\left(\chi_{1 L}\right)^{c}, \quad \chi_{2}=\chi_{2 L}+\left(\chi_{2 L}\right)^{c} \tag{79}
\end{equation*}
$$

That the neutrino mass eigenstates corresponding to opposite signs of the mass parameters $m_{1}$ and $m_{2}$ defined in eq. (76) have opposite CP-parities is now a consequence of the fact that the matrix $U$ has one complex column. Indeed, let $m_{1}$ defined in (76) be negative and $m_{2}$ positive. Then from $n_{L}=U \chi_{L}=O K \chi_{L}$ and eq. (79) we have

$$
\begin{equation*}
\chi_{1}=\mp i\left\{c\left[\nu_{L}-\left(\nu_{L}\right)^{c}\right]+s\left[\nu_{R}-\left(\nu_{R}\right)^{c}\right]\right\}, \quad \chi_{2}=s\left[\nu_{L}+\left(\nu_{L}\right)^{c}\right]+c\left[\nu_{R}+\left(\nu_{R}\right)^{c}\right] . \tag{80}
\end{equation*}
$$

Making use of the definition of the CP conjugation given in sec. 15.2 and taking into account that it is described by a linear operator, one can readily make sure that the CP-parities of $\chi_{1}$ and $\chi_{2}$ are opposite.

It is instructive to consider some limiting cases. In the limit of no Majorana masses ( $m_{L}=m_{R}=0$ ), pure Dirac case has to be recovered. Let us see how this limit can be obtained from the general $\mathrm{D}+\mathrm{M}$ formalism. For $m_{L}=m_{R}=0$ the mass matrix (65) takes the form

$$
\mathcal{M}=\left(\begin{array}{cc}
0 & m  \tag{81}\\
m & 0
\end{array}\right)
$$

This matrix corresponds to a conserved lepton number $L_{\nu_{L}}-L_{\nu_{L}^{c}}=L_{\nu_{L}}+L_{\nu_{R}}$ which can be identified with the total lepton number $L$. Thus, the lepton number is conserved in this limiting case, as expected. Let us now check that the usual Dirac mass term is recovered.

The matrix $\mathcal{M}$ in (81) is diagonalized by the rotation (74) with $\theta=45^{\circ}$, and its eigenvalues are $-m$ and $m$. This means that we have two Majorana neutrinos that have the same mass, opposite CP parities and are maximally mixed. Let us demonstrate that this is equivalent to having one Dirac neutrino of mass $m$. We have $\eta_{2}=-\eta_{1}=1$; from eqs. (74) and (78) it then follows $\chi_{1}+\chi_{2}=\sqrt{2}\left(\nu_{L}+\nu_{R}\right), \chi_{1}-\chi_{2}=-\sqrt{2}\left(\nu_{L}^{c}+\nu_{R}^{c}\right)=-\left(\chi_{1}+\chi_{2}\right)^{c}$. This gives

$$
\begin{equation*}
\frac{1}{2} m\left(\bar{\chi}_{1} \chi_{1}+\bar{\chi}_{2} \chi_{2}\right)=\frac{1}{4} m\left[\overline{\left(\chi_{1}+\chi_{2}\right)}\left(\chi_{1}+\chi_{2}\right)+\left[\overline{\left(\chi_{1}-\chi_{2}\right)}\left(\chi_{1}-\chi_{2}\right)\right]=m \bar{\nu}_{D} \nu_{D}\right. \tag{82}
\end{equation*}
$$

where

$$
\begin{equation*}
\nu_{D} \equiv \nu_{L}+\nu_{R} \tag{83}
\end{equation*}
$$

The counting of the degrees of freedom also shows that we must have a Dirac neutrino in this case - there are four degrees of freedom and just one physical mass. Thus, two maximally mixed degenerate Majorana neutrinos of opposite CP parities merge to form a Dirac neutrino. In sec. 15.7 we shall discuss neutrinoless double beta ( $0 \nu \beta \beta$ ) decay and show that this process can only take place if neutrinos are Majorana particles. We shall demonstrate there that in the limit when two degenerate in mass Majorana neutrinos merge into a Dirac neutrino, their contributions to the amplitude of the $0 \nu \beta \beta$ decay exactly cancel, as they should.

If the Majorana mass parameters $m_{L}$ and $m_{R}$ do not vanish but are small compared to $m_{D}$, the resulting pair of Majorana neutrinos will be quasi-degenerate with almost maximal mixing and opposite CP parities. The physical neutrino masses in this case are $\left|m_{1,2}\right| \simeq$ $m_{D} \mp\left(m_{L}+m_{R}\right) / 2 \simeq m_{D}$. Such a pair in many respects behaves as a Dirac neutrino and therefore sometimes is called a 'pseudo-Dirac' (or a 'quasi-Dirac') neutrino. In particular, its contribution to the $0 \nu \beta \beta$ decay amplitude is proportional to the mass difference $\left|m_{2}\right|-\left|m_{1}\right| \simeq$ $\left(m_{L}+m_{R}\right)$ which is much smaller than the mass of each component of the pair.

### 15.4 The seesaw mechanism of the neutrino mass generation

The seesaw mechanism [28] provides a very simple and attractive explanation of the smallness of neutrino mass by relating it with the existence of a very large mass scale. In the simplest and most popular version of this mechanism, the requisite large mass scale is given by the masses of heavy electroweak-singlet Majorana neutrinos. Although the seesaw mechanism is most natural in the framework of the grand unified theories (such as $S O(10)$ ) or left-right symmetric models [29], it also operates in the standard model extended to include the right-handed sterile neutrinos $\nu_{R}$. Indeed, as soon as the $\nu_{R}$ 's are introduced, one can add to the Lagrangian of the model the Majorana mass term $(1 / 2) \nu_{R}^{T} \mathcal{C}^{-1} m_{R} \nu_{R}+h . c$., which is allowed because $\nu_{R}$ are electroweak singlets. The Yukawa couplings of the right-handed neutrinos with the lepton doublets and the Higgs boson are also allowed, and after the spontaneous breaking of the electroweak symmetry they give rise to the Dirac mass term connecting the active and sterile neutrinos, similar to those that are present for the quarks and charged leptons. The resulting neutrino mass scheme is just the $\mathrm{D}+\mathrm{M}$ one discussed in secs. 15.3.2 and 15.3.3, see eqs. (62) - (65). Notice that in the standard model there is no Majorana mass term for left-handed neutrinos, i.e. $m_{L}=0$; however, $m_{L}$ is different from zero in some extensions of the standard model, so we shall keep it for generality.

Because the right-handed neutrinos $\nu_{R}$ are electroweak singlets, the scale of their Majorana mass term need not be related to the electroweak scale. In particular, $m_{R}$ can be very large, possibly even at the Planck scale $M_{\mathrm{Pl}} \simeq 1.2 \times 10^{19} \mathrm{GeV}$, grand unification scale
or at some intermediate scale $M_{I} \sim 10^{10}-10^{12} \mathrm{GeV}$ which may be relevant for the physics of parity breaking. The seesaw suppression of the masses of active neutrinos is realized just in this case of a very large $m_{R}$. We therefore consider the limit in which the characteristic scales of the Dirac and Majorana neutrino masses satisfy

$$
\begin{equation*}
m_{L}, m_{D} \ll m_{R} \tag{84}
\end{equation*}
$$

Let us first discuss the one-generation case studied in sec. 15.3.3. The diagonalization of the mass matrix is then performed by the simple rotation (74). A straightforward calculation gives for the rotation angle $\theta$ and eigenvalues of the mass matrix $\mathcal{M}$

$$
\begin{equation*}
\theta \simeq \frac{m_{D}}{m_{R}} \ll 1, \quad m_{1} \simeq m_{L}-\frac{m_{D}^{2}}{m_{R}}, \quad m_{2} \simeq m_{R} \tag{85}
\end{equation*}
$$

while the mass eigenstates are given by

$$
\begin{equation*}
\chi_{1} \simeq \nu_{L}+\eta_{1}\left(\nu_{L}\right)^{c}, \quad \chi_{2} \simeq\left(\nu_{R}\right)^{c}+\eta_{2} \nu_{R} \tag{86}
\end{equation*}
$$

Thus, we have a very light Majorana mass eigenstate $\chi_{1}$ predominantly composed of the active neutrino $\nu_{L}$ and its $\hat{C}$-conjugate $\left(\nu_{L}\right)^{c}$, and a heavy eigenstate $\chi_{2}$, mainly composed of the sterile $\nu_{R}$ and $\left(\nu_{R}\right)^{c}$. The admixture of the sterile neutrino state $\nu_{R}$ in $\chi_{1}$ and that of the usual active neutrinos $\nu_{L}$ in $\chi_{2}$ are of the order of $m_{D} / m_{R} \ll 1$. As follows from eq. (85), for $m_{L} \lesssim m_{D}^{2} / m_{R}$ it is the sterile neutrino being heavy that makes the usual active one light (which explains the name 'the seesaw mechanism').

Consider now the case of $n$ standard generations of left-handed leptons and $k$ sterile neutrinos $\nu_{R}$. This is actually the case discussed in sec. 15.3.2, but now we want to specifically consider the limit of very high $\nu_{R}$ mass scale. Let us first decouple the light and heavy neutrino degrees of freedom. To this end, we block-diagonalize the matrix $\mathcal{M}$ in eqs. (64), (65) according to

$$
n_{L}=V \chi_{L}^{\prime}, \quad V^{T} \mathcal{M} V=V^{T}\left(\begin{array}{cc}
m_{L} & m_{D}^{T}  \tag{87}\\
m_{D} & M_{R}
\end{array}\right) V=\left(\begin{array}{cc}
\tilde{m}_{L} & 0 \\
0 & \tilde{M}_{R}
\end{array}\right)
$$

where $V$ is a unitary $(n+k) \times(n+k)$ matrix, $\tilde{m}_{L}$ and $\tilde{M}_{R}$ are symmetric $n \times n$ and $k \times k$ matrices, respectively, and we have changed the notation $m_{R} \rightarrow M_{R}$. Note that the fields $\chi^{\prime}$ that block-diagonalize $\mathcal{M}$ are not the fields of mass-eigenstate neutrinos, since the matrices $\tilde{m}_{L}$ and $\tilde{M}_{R}$ are not in general diagonal. They can be diagonalized by further unitary transformations. Correspondingly, $V$ is not the leptonic mixing matrix.

We shall be looking for the matrix $V$ in the form [30]

$$
V=\left(\begin{array}{cc}
\sqrt{1-\rho \rho^{\dagger}} & \rho  \tag{88}\\
-\rho^{\dagger} & \sqrt{1-\rho^{\dagger} \rho}
\end{array}\right)
$$

where $\rho$ is an $n \times k$ matrix. Note that $V$ is unitary by construction. Treating $\rho$ as perturbation and performing block-diagonalization of $\mathcal{M}$ approximately, we find

$$
\begin{align*}
& \rho^{*} \simeq m_{D}^{T} M_{R}^{-1}, \quad \tilde{M}_{R} \simeq M_{R}  \tag{89}\\
& \tilde{m}_{L} \simeq m_{L}-m_{D}^{T} M_{R}^{-1} m_{D} \tag{90}
\end{align*}
$$

These relations generalize those of eq. (85) to the case of $n$ active and $k$ sterile neutrinos. The diagonalization of the effective mass matrix $\tilde{m}_{L}$ then yields $n$ light Majorana neutrino fields which are predominantly composed of the fields of the usual (active) neutrinos $\nu_{L}$ and their $\hat{C}$-conjugates $\left(\nu_{L}\right)^{c}$, with very small $\left(\sim m_{D} / M_{R}\right)$ admixture of sterile neutrinos $\nu_{R}$; the diagonalization of $\tilde{M}_{R}$ produces $k$ heavy Majorana neutrino fields which are mainly composed of $\nu_{R}$ and $\left(\nu_{R}\right)^{c}$. This, in particular, means that the oscillations between the active and sterile neutrinos are suppressed in this case.

It is important that the active neutrinos get Majorana masses $\tilde{m}_{L}$ even if they have no 'direct' masses, i.e. $m_{L}=0$, as it is the case in the standard model. The masses of the active neutrinos are then of the order of $m_{D}^{2} / M_{R}$. Generation of the effective Majorana mass of light neutrinos is diagrammatically illustrated in fig. 1.


Figure 1: Seesaw mechanism of $\tilde{m}_{L}$ generation. The vacuum expectation values of the Higgs field are denoted by $\langle H\rangle$.

What happens if $M_{R}$ has one or more zero eigenvalues? Obviously, in this case $M_{R}^{-1}$ does not exist, and the usual seesaw approximation fails. However, it can be readily modified to produce meaningful results. One can just go to the $\nu_{R}$ basis where $M_{R}$ is diagonal and include the lines and columns of $M_{R}$ that contain zero eigenvalues into a redefined matrix $m_{L}$. This situation is called the 'singular seesaw', and it generally leads to the existence of pseudo-Dirac light neutrinos.

What can one say about the expected mass scale of the right-handed neutrinos $M_{R}$ ? Let us take the mass of the heaviest among the light neutrinos to be $m_{\nu} \sim 5 \times 10^{-2} \mathrm{eV}$, as required by the data of the atmospheric and accelerator neutrino oscillation experiments under the assumption of the hierarchical neutrino masses. Then, assuming that the largest eigenvalue of the Dirac neutrino mass matrix is of the order of the electroweak scale, $m_{D} \sim 200 \mathrm{GeV}$, and that $m_{L} \lesssim m_{D}^{2} / M_{R}$, from eq. (90) we find $M_{R} \sim 10^{15} \mathrm{GeV}$. Interestingly, this is very close to the expected grand unification scale $m_{\mathrm{GUT}} \sim 10^{16} \mathrm{GeV}$. Thus, neutrino oscillations
together with the seesaw mechanism of neutrino mass generation may be giving us an indication in favour of grand unification of weak, electromagnetic and strong interactions.

The version of the seesaw mechanism discussed here, with heavy sterile neutrinos responsible for the small light neutrino masses, is sometimes called type I seesaw. There also exist other versions - those with heavy $S U(2)_{L}$-triplet Higgs scalars (type II seesaw), heavy triplet fermions (type III seesaw), as well as other realizations of the seesaw mechanism, see ref. [31] for a review. In all these cases neutrinos of definite mass are generically Majorana particles.

### 15.5 Electromagnetic properties of Majorana neutrinos

As neutrinos are electrically neutral, they have no direct coupling to the photon and their electromagnetic interactions arise entirely through loop effects (see ref. [32] for a review). It is interesting to compare the electromagnetic properties of Majorana neutrinos with those of Dirac neutrinos, which we discuss first. The matrix elements of the electromagnetic current $j^{\mu}(x)$ between the 1-particle on-shell states of a Dirac neutrino (or any other Dirac fermion) can be written as

$$
\begin{equation*}
\left\langle\mathbf{p}^{\prime}, s^{\prime}\right| j^{\mu}(x)|\mathbf{p}, s\rangle=e^{i q x}\left\langle\mathbf{p}^{\prime}, s^{\prime}\right| j^{\mu}(0)|\mathbf{p}, s\rangle=e^{i q x} \bar{u}_{s^{\prime}}\left(\mathbf{p}^{\prime}\right) \Lambda^{\mu}(q) u_{s}(\mathbf{p}) \tag{91}
\end{equation*}
$$

where $q \equiv p^{\prime}-p, u_{s}(\mathbf{p})$ and $u_{s^{\prime}}\left(\mathbf{p}^{\prime}\right)$ are the free-particle plane wave spinors, and

$$
\begin{equation*}
\Lambda^{\mu}(q)=F_{Q}\left(q^{2}\right) \gamma^{\mu}+F_{M}\left(q^{2}\right) i \sigma^{\mu \nu} q_{\nu}+F_{E}\left(q^{2}\right) \sigma^{\mu \nu} \gamma_{5} q_{\nu}+F_{A}\left(q^{2}\right)\left(q^{2} g^{\mu \nu}-q^{\mu} q^{\nu}\right) \gamma_{\nu} \gamma_{5} \tag{92}
\end{equation*}
$$

Here $F_{Q}\left(q^{2}\right)$ and $F_{M}\left(q^{2}\right)$ are the electric charge and magnetic dipole form factors, while $F_{E}\left(q^{2}\right)$ and $F_{A}\left(q^{2}\right)$ are the electric dipole form factor and anapole form factor, respectively.

Unlike the magnetic and electric dipole moments, the anapole moment, first proposed in [33], has no simple classical multipolar analogue. It can be modeled by a torus-shaped solenoid and therefore is sometimes called the toroidal moment. For a discussion of the properties of the anapole moment and its experimental manifestations see $[34,35]$ and references therein.

The form of $\Lambda^{\mu}(q)$ in (92) follows from the requirements of Lorentz covariance and electromagnetic current conservation. ${ }^{11}$ For interactions with real photons the anapole form factor does not contribute. The hermiticity of the Hamiltonian of the electromagnetic interaction $\mathcal{H}_{\text {int }}=j^{\mu}(x) A_{\mu}(x)$ implies that all the four form factors in eq. (92) are real. The vectorcurrent form factors $F_{Q}\left(q^{2}\right)$ and $F_{M}\left(q^{2}\right)$ are parity conserving, while the axial-vector ones $F_{E}\left(q^{2}\right)$ and $F_{A}\left(q^{2}\right)$ violate parity. In addition, $F_{E}\left(q^{2}\right)$ violates CP invariance. As electroweak

[^9]interactions that induce the effective neutrino electromagnetic current do violate parity (and possibly also CP), in general the form factors $F_{A}\left(q^{2}\right)$ and $F_{E}\left(q^{2}\right)$ need not vanish.

The charge form factor taken at zero squared 4 -momentum transfer yields the electric charge of the Dirac particle, $F_{Q}(0)=Q$, whereas $F_{M}(0), F_{E}(0)$ and $F_{A}(0)$ give, respectively, its anomalous magnetic moment $g-2$, electric dipole moment and anapole moment. The term $F_{Q}\left(q^{2}\right) \gamma^{\mu}$ in eq. (92) describes in the static limit $\left(q^{2} \rightarrow 0\right)$ not only the electric charge interaction of the Dirac fermion, but also its normal magnetic moment. This can be seen from the Gordon identity

$$
\begin{equation*}
\bar{u}_{s^{\prime}}\left(\mathbf{p}^{\prime}\right) \gamma^{\mu} u_{s}(\mathbf{p})=\bar{u}_{s^{\prime}}\left(\mathbf{p}^{\prime}\right)\left[\frac{p^{\mu}+p^{\prime \mu}}{2 m}+\frac{i \sigma^{\mu \nu} q_{\nu}}{2 m}\right] u_{s}(\mathbf{p}) . \tag{93}
\end{equation*}
$$

The first term in the square brackets here corresponds to the convective part of the current, while the second one describes its spin part, i.e. the normal magnetic moment. The total magnetic moment of the particle is the sum of the normal and the anomalous ones.

Because neutrinos have no electric charge (i.e. for them $F_{Q}(0)=0$ ), they do not have normal magnetic moments either. Note that electric neutrality does not mean that the entire charge form factor $F_{Q}\left(q^{2}\right)$ vanishes. At small $q^{2}$ one can write

$$
\begin{equation*}
F_{Q}\left(q^{2}\right)=F_{Q}(0)+F_{Q}^{\prime}(0) q^{2}+\cdots \equiv F_{Q}(0)+\frac{1}{6}\langle r\rangle^{2} q^{2}+\ldots \tag{94}
\end{equation*}
$$

The quantity $\left\langle r^{2}\right\rangle$ characterizes the charge distribution within the particle and is called its charge radius. It is in general different from zero even for neutral particles. Non-triviality of their charge distributions is related to the fact that interactions that 'dress' the particles produce clouds of virtual particles of opposite charges and in general different configurations.

Let us now discuss the electromagnetic properties of Majorana neutrinos [13, 36-41]. For a Majorana particle all the electromagnetic form factors but one vanish identically, and the matrix element of the electromagnetic current takes the form

$$
\begin{equation*}
\left\langle\mathbf{p}^{\prime}\right| j^{\mu}(0)|\mathbf{p}\rangle=\bar{u}\left(\mathbf{p}^{\prime}\right)\left[F_{A}\left(q^{2}\right)\left(q^{2} g^{\mu \nu}-q^{\mu} q^{\nu}\right) \gamma_{\nu} \gamma_{5}\right] u(\mathbf{p}) \tag{95}
\end{equation*}
$$

That is, while the electromagnetic properties of a Dirac fermion are in general described by four form factors, for a Majorana particle only the anapole form factor survives. The simplest way to see this is to note that each term in eq. (92) can be viewed as emerging from the matrix element of the corresponding effective operator $\bar{\psi} \Gamma^{\mu} \psi$, where $\psi$ is the free field operator and $\Gamma^{\mu}=\left(\gamma^{\mu}, \sigma^{\mu \nu} q_{\nu}, \sigma^{\mu \nu} \gamma_{5} q_{\nu}, \gamma^{\mu} \gamma_{5}\right)$, between the free one-particle states. For Majorana neutrinos from eqs. (11) and (12) we find

$$
\begin{equation*}
\bar{\psi}_{k} \gamma^{\mu} \psi_{i}=-\bar{\psi}_{i} \gamma^{\mu} \psi_{k}, \quad \bar{\psi}_{k} \gamma^{\mu} \gamma_{5} \psi_{i}=\bar{\psi}_{i} \gamma^{\mu} \gamma_{5} \psi_{k} \tag{96}
\end{equation*}
$$

$$
\begin{equation*}
\bar{\psi}_{k} \sigma^{\mu \nu} \psi_{i}=-\bar{\psi}_{i} \sigma^{\mu \nu} \psi_{k}, \quad \bar{\psi}_{k} \sigma^{\mu \nu} \gamma_{5} \psi_{i}=-\bar{\psi}_{i} \sigma^{\mu \nu} \gamma_{5} \psi_{k} \tag{97}
\end{equation*}
$$

where we have used the Majorana condition (20). In the case when the fields $\psi_{i}$ and $\psi_{k}$ correspond to the same particle, i.e. $k=i$, from eqs. (96) and (97) it follows that the only non-zero operator is the axial-vector one, $\bar{\psi}_{i} \gamma^{\mu} \gamma_{5} \psi_{i}$, whereas all the other operators vanish identically. Electromagnetic current conservation then implies that the axial-vector operator can enter the matrix element (91) only through the anapole interaction. This result has a simple interpretation. The charge radius and magnetic and electric dipole moments have opposite signs for neutrinos and antineutrinos and thus could be used to distinguish between them. Therefore, they must vanish if neutrinos are Majorana fermions. At the same time, the anapole moment does not change its sign under the particle-antiparticle conjugation [33, 34], and so is allowed. Note that the vanishing of the charge, magnetic dipole and electric dipole form factors of Majorana neutrinos has a deep reason - it is related to CPT invariance.

Expressions (91) and (92) describe the matrix elements of the electromagnetic current $j^{\mu}(x)$ between the 1-particle states of an individual neutrino. They are actually a special case of more general matrix elements of $j^{\mu}(x)$ between the states of neutrinos of different mass. These matrix elements have a form similar to eq. (91), except that the quantity $\Lambda^{\mu}(q)$ and the form factors are now matrices in the space of neutrino mass eigenstates. The requirements of Lorentz invariance and electromagnetic current conservation yield

$$
\begin{equation*}
\Lambda^{\mu}(q)_{k i}=\left[F_{1}\left(q^{2}\right)_{k i}-\gamma_{5} F_{A}\left(q^{2}\right)_{k i}\right]\left(q^{2} g^{\mu \nu}-q^{\mu} q^{\nu}\right) \gamma_{\nu}+\left[F_{M}\left(q^{2}\right)_{k i}-i \gamma_{5} F_{E}\left(q^{2}\right)_{k i}\right] i \sigma^{\mu \nu} q_{\nu} \tag{98}
\end{equation*}
$$

Eq. (92) corresponds to the diagonal elements of (98), with $F_{Q}\left(q^{2}\right)=F_{Q}\left(q^{2}\right)_{i} \equiv F_{1}\left(q^{2}\right)_{i i} q^{2}$ (note that for $i=k$ the expression $\bar{u}_{k s^{\prime}}\left(\mathbf{p}^{\prime}\right) \gamma^{\mu} q_{\mu} u_{i s}(\mathbf{p})$ vanishes identically since the spinors satisfy the Dirac equation). The off-diagonal matrix elements of the form factors in (98) are called the transition form factors. They describe transitions $\nu_{i} \rightarrow \nu_{k}$ caused by interaction of neutrinos with real or virtual photons or external electromagnetic fields. We will briefly discuss some of these processes towards the end of this section. Let us also note that, unlike for the diagonal elements of the form-factors, hermiticity of the electromagnetic interaction Hamiltonian $\mathcal{H}_{\text {int }}$ does not by itself mean that the transition form factors are real. However, hermiticity of $\mathcal{H}_{\text {int }}$ combined with the assumption of CP invariance would require the form factors to be relatively real, i.e. for given $i, k$ all $F_{a}\left(q^{2}\right)_{k i}(a=1, A, M, E)$ would be allowed to differ from their respective complex conjugates only by the same phase factor.

The expression for $\Lambda^{\mu}(q)_{k i}$ has the same form (98) for Dirac and Majorana neutrinos, though in the Majorana case the form factors must satisfy some additional constraints. We have found that for Majorana neutrinos the diagonal matrix elements of the electromagnetic current contain only one non-zero form factor, $F_{A}\left(q^{2}\right)_{i i}$. This is an immediate consequence of eqs. (96) and (97). However, for transition form factors the constraints are less severe.

Eqs. (96) and (97) then simply imply that the form factors $F_{1}\left(q^{2}\right)_{k i}, F_{M}\left(q^{2}\right)_{k i}$ and $F_{E}\left(q^{2}\right)_{k i}$ are antisymmetric with respect to the interchange of the indices $i$ and $k$, whereas $F_{A}\left(q^{2}\right)_{k i}$ is symmetric. Actually, this is a consequence of CPT invariance; one can readily check this by making use of the transformation properties of Majorana states under CPT given in Table I and taking into account that the CPT transformation is anti-unitary and that the electromagnetic current $j^{\mu}(0)$ is CPT-odd. ${ }^{12}$

Another important point is that non-vanishing transition electric dipole form factor $F_{E}\left(q^{2}\right)_{k i}$ would not necessarily signify leptonic CP violation. It is only in the case when both the transition magnetic dipole and electric dipole form factors $F_{M}\left(q^{2}\right)_{k i}$ and $F_{E}\left(q^{2}\right)_{k i}$ are simultaneously different from zero that one would have to conclude that CP is violated. In fact, if CP is conserved in the leptonic sector, the form factors $F_{1}\left(q^{2}\right)_{k i}$ and $F_{M}\left(q^{2}\right)_{k i}$ vanish when $\nu_{i}$ and $\nu_{k}$ have the same CP-parity (either $i$ or $-i$ ), whereas $F_{A}\left(q^{2}\right)_{k i}$ and $F_{E}\left(q^{2}\right)_{k i}$ vanish when $\nu_{i}$ and $\nu_{k}$ have opposite CP-parities [41]. This can be readily shown using the transformation properties of Majorana states under CP given in Table I. If CP is violated, neutrinos do not possess definite CP-parities, and the simultaneous existence of non-zero $F_{M}\left(q^{2}\right)_{k i}$ and $F_{E}\left(q^{2}\right)_{k i}$ (or $F_{1}\left(q^{2}\right)_{k i}$ and $\left.F_{A}\left(q^{2}\right)_{k i}\right)$ with $k \neq i$ is allowed.

We have mentioned in sec. 15.1 that in the limit of vanishing neutrino mass Dirac and Majorana neutrinos become indistinguishable (as both actually become Weyl neutrinos). It is therefore interesting to see how the electromagnetic properties of Dirac and Majorana neutrinos converge in the massless limit. Let us first note that the vector and axial-vector operators $\bar{\psi}_{i} \gamma^{\mu} \psi_{k}$ and $\bar{\psi}_{i} \gamma^{\mu} \gamma_{5} \psi_{k}$ are chirality-preserving, while $\bar{\psi}_{i} \sigma^{\mu \nu} \psi_{k}$ and $\bar{\psi}_{i} \sigma^{\mu \nu} \gamma_{5} \psi_{k}$ are chirality-flipping:

$$
\begin{equation*}
\bar{\psi}_{i} \sigma^{\mu \nu} \psi_{k}=\bar{\psi}_{i L} \sigma^{\mu \nu} \psi_{k R}+\bar{\psi}_{i R} \sigma^{\mu \nu} \psi_{k L} \tag{99}
\end{equation*}
$$

and similarly for $\bar{\psi}_{i} \sigma^{\mu \nu} \gamma_{5} \psi_{k}$. No left-right transitions can be induced by loop effects in the massless neutrino limit, and so the dipole form factors $F_{M}\left(q^{2}\right)$ and $F_{E}\left(q^{2}\right)$ must vanish in this limit identically.

> This result is quite general and is easy to understand. If there is a loop diagram giving a contribution to the chirality-flipping neutrino magnetic or electric dipole form factors, then the same diagram with the external photon line removed will give a contribution to the neutrino mass term, which is also chirality-flipping. Thus, for massless neutrinos their magnetic and electric dipole form factors must vanish identically. (It is possible to devise symmetries that allow neutrino magnetic moments but forbid neutrino masses [42], but such symmetries must be broken in the real world).

Thus, we are left with only vector and axial-vector operators. Next, we notice that for mass-

[^10]less neutrinos $\bar{u}\left(\mathbf{p}^{\prime}\right) \gamma^{\mu} q_{\mu} u(\mathbf{p})=\bar{u}\left(\mathbf{p}^{\prime}\right) \gamma^{\mu} \gamma_{5} q_{\mu} u(\mathbf{p})=0$. The quantity $\Lambda^{\mu}(q)_{k i}$ can therefore be written as
\[

$$
\begin{equation*}
\Lambda^{\mu}(q)_{k i}=F_{1}\left(q^{2}\right)_{k i} q^{2} \gamma^{\mu}+F_{A}\left(q^{2}\right)_{k i} q^{2} \gamma^{\mu} \gamma_{5} \tag{100}
\end{equation*}
$$

\]

The two terms here contain the neutrino charge form factor and the anapole form factor. For Majorana neutrinos, the former is antisymmetric and the latter is symmetric with respect to the indices $i$ and $k$, while no such constraints exist in the Dirac case. As massless neutrinos are chiral and $\gamma_{5} u_{L, R}=\mp u_{L, R}$, the two terms in (100) are actually indistinguishable and merge into one, which is neither symmetric nor antisymmetric. Thus, the restrictions on the neutrino electromagnetic interactions that are specific to the Majorana case disappear in the limit $m_{\nu} \rightarrow 0$.

The same conclusion can also be achieved in a different way [39]. As follows from the Majorana condition (20) (or equivalently from eq. (27)), for each loop diagram contributing to the electromagnetic vertex of a Dirac neutrino, in the Majorana case there is an additional diagram with all particles replaced by their $\hat{C}$-conjugates. ${ }^{13}$ If the original diagram is caused by left-handed currents, then the $\hat{C}$-conjugate one is due to the right-handed interactions of the antiparticles (see eq. (13)). The additional diagrams contribute to the electromagnetic vertices of massive Majorana neutrinos because Majorana fields contain both the left-handed and right-handed parts, the latter being $\hat{C}$-conjugates of the former. Decoupling of the lefthanded and right-handed neutrino states in the massless limit means that the contribution of these additional diagrams to the amplitude of a given electromagnetic transition becomes negligible as $m_{\nu} \rightarrow 0$. Vanishing contributions of the diagrams that are specific to the Majorana case means that the electromagnetic properties of Majorana neutrinos converge to those of Dirac neutrinos when $m_{\nu} \rightarrow 0$.

In sec. 15.3 we pointed out that a pair of mass-degenerate Majorana neutrinos with maximal mixing and opposite CP-parities merges into a Dirac neutrino. The electromagnetic properties of such a pair should then be the same as those of a Dirac neutrino. This is indeed the case; in particular, the transition magnetic moment of such a Majorana pair becomes the usual magnetic moment of the Dirac neutrino. We refer the reader to ref. [38] for details.

From the above discussion it follows that the electromagnetic properties of massive Dirac and Majorana neutrinos are very different. Can this be used to find out whether neutrinos are Dirac or Majorana particles? To answer this question, we should first examine how the neutrino electromagnetic properties can manifest themselves. First, through the the photon exchange diagrams, they can contribute to the cross sections of $\nu f$ scattering, where $f$ is a charged lepton or a quark. In principle, such contributions can probe the

[^11]neutrino magnetic and electric dipole moments, as well as the charge radius and anapole moment. Unfortunately, up to now experiment and observations (most notably, experiments on $\bar{\nu}_{e} e$ scattering with reactor antineutrinos as well as astrophysical data) have failed to discover neutrino electromagnetic properties and only produced upper limit on them [32]. This is actually not surprising, as in the standard model and its simplest extensions the neutrino electromagnetic interactions are expected to be extremely weak. As an example, the standard-model prediction for the neutrino charge radius is $\left\langle r^{2}\right\rangle \sim 10^{-33} \mathrm{~cm}^{2}$, whereas adding right-handed neutrinos to the model, for the diagonal magnetic moments of Dirac neutrinos one finds [43]
\[

$$
\begin{equation*}
\mu_{i} \approx \frac{3 e G_{F}}{8 \sqrt{2} \pi^{2}} m_{i} \approx 3.2 \times 10^{-19}\left(\frac{m_{i}}{\mathrm{eV}}\right) \mu_{B} \tag{101}
\end{equation*}
$$

\]

where $e$ is the absolute value of the electron charge, $G_{F}$ is the Fermi constant, $m_{i}$ is the mass of the $i$ th neutrino mass eigenstate and $\mu_{B}=e / 2 m_{e}$ is the electron Bohr magneton. Similar expressions can be obtained for transition magnetic moments [37,40]. In addition, as we discussed above, the smallness of neutrino mass makes it very difficult to tell Majorana neutrinos from Dirac ones through their electromagnetic properties. In particular, it is difficult to distinguish experimentally the neutrino charge radius (which is non-zero only for Dirac neutrinos) from the anapole moment, which is the only non-vanishing diagonal electromagnetic moment of Majorana neutrinos. ${ }^{14}$

As we discussed above, Dirac neutrinos can in general have both diagonal and transition dipole moments, whereas for Majorana neutrinos only transition dipole moments are allowed. For neutrinos of both types transition magnetic and electric dipole moments will cause radiative decays of heavier neutrinos into lighter ones, $\nu_{i} \rightarrow \nu_{k}+\gamma$. Although the rates of the radiative decay of Dirac and Majorana neutrinos are in general different, because of large uncertainties in the involved neutrino parameters it is impossible to establish the nature of neutrinos by measuring their radiative decay widths. However, the circular polarizations of the produced photons are very different in the Dirac and Majorana cases, and this is completely independent of the neutrino unknowns [40]. In addition, for polarized parent neutrinos, the angular distributions of the emitted photons are different for neutrinos of the two types [40, 44]. Thus, at least in principle, one could distinguish between the Dirac and Majorana neutrinos by measuring the polarization or angular distribution of the photons produced in radiative neutrino decay.

Unfortunately, it is rather unlikely that neutrino radiative decays will ever be observed, as they are doubly suppressed by the smallness of the neutrino magnetic moments (which implies small transition amplitude) and of neutrino mass (which means very small phase

[^12]space volume of the decay). There is, however, some chance to observe radiative neutrino transitions if relatively heavy sterile neutrinos exist.

Neutrino diagonal and transition dipole moments can in principle manifest themselves differently - through neutrino spin precession in strong electromagnetic fields. Such a process can be caused by the interaction of neutrino magnetic [43] or electric [45] dipole moments with external fields. Transition dipole moments can give rise to spin-flavour precession, in which neutrino spin and flavour are flipped simultaneously [36,46]. This process can be resonantly enhanced when neutrinos propagate in matter [47, 48].

Let us compare spin and spin-flavour precessions of Dirac and Majorana neutrinos. It is convenient to introduce the matrix of neutrino electromagnetic moments [46]

$$
\begin{equation*}
\tilde{\mu}=\mu+i \epsilon \tag{102}
\end{equation*}
$$

where $\mu$ and $\epsilon$ are the hermitian matrices of neutrino magnetic dipole and electric dipole moments, respectively. In the flavour eigenstate basis the dipole moment couplings of neutrinos to an external electromagnetic field are described by the effective operators

$$
\begin{align*}
& \frac{1}{2}\left[\bar{\nu}_{\beta} \sigma^{\mu \nu}\left(\mu-i \epsilon \gamma_{5}\right)_{\beta \alpha} \nu_{\alpha}\right] F_{\mu \nu}+h . c .=\frac{\tilde{\mu}_{\beta \alpha}}{2} \overline{\nu_{\beta R}} \sigma^{\mu \nu} \nu_{\alpha L} F_{\mu \nu}+\text { h.c. } \quad \text { (Dirac); }  \tag{103}\\
& \frac{1}{2}\left[\bar{\nu}_{\beta} \sigma^{\mu \nu}\left(\mu-i \epsilon \gamma_{5}\right)_{\beta \alpha} \nu_{\alpha}\right] F_{\mu \nu}+h . c .=\frac{\tilde{\mu}_{\beta \alpha}}{2} \overline{\left(\nu_{\beta L}\right)^{c}} \sigma^{\mu \nu} \nu_{\alpha L} F_{\mu \nu}+\text { h.c. } \quad \text { (Majorana), } \tag{104}
\end{align*}
$$

where $F_{\mu \nu}=\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}$ is the electromagnetic field tensor. Note that there is no extra factor $1 / 2$ in eq. (104) because in the Majorana case the matrix $\tilde{\mu}$ is antisymmetric.

It is instructive to look at the right-hand sides of eqs. (103) and (104), which reveal the nature of the involved neutrinos. Eq. (103) means that in a transverse ${ }^{15}$ external magnetic field e.g. a left-handed (active) electron neutrino $\nu_{e L}$ can be converted into a right-handed sterile neutrino of the same or different flavour. At the same time, in the Majorana case only flavour-off-diagonal transitions are allowed. For instance, for $\alpha=e$ and $\beta=\mu$ the interaction in eq. (104) describes the transformation of an active left-handed electron neutrino $\nu_{e L}$ to the active right-handed muon neutrino state $\bar{\nu}_{\mu R}=\left(\nu_{\mu L}\right)^{c}$ which is usually called the muon antineutrino. From this discussion it is clear that neutrino spin and/or spin-flavour precession lead to physically very different final states for Dirac and Majorana neutrinos and therefore could in principle be used to discriminate between them.

Although the neutrino dipole moments are expected to be very small, the neutrino spin precession and spin-flavour precession can still occur with sizeable probabilities in extremely strong magnetic fields which may be present in astrophysical objects. In particular, in the

[^13]Majorana neutrino case strong magnetic fields present in supernovae during the explosion stage may cause the resonantly enhanced conversion $\nu_{e} \rightarrow \bar{\nu}_{\mu}$. The resulting muon antineutrinos will then experience the usual flavour transitions on their way from the supernova to the Earth, converting them to electron antineutrinos. As a result, the overall neutrino transmutation chain $\nu_{e} \rightarrow \bar{\nu}_{\mu} \rightarrow \bar{\nu}_{e}$ will transform electron neutrinos into electron antineutrinos. Such a conversion of supernova $\nu_{e}$ 's would have a very clear signature in the terrestrial detectors [49,50], provided that the supernova event occurs in our galaxy and that the transition magnetic moments of Majorana neutrinos $\mu \gtrsim 10^{-14} \mu_{B}{ }^{16}$ While such relatively large values of $\mu$ are not easily achieved, they are predicted in some models and are not excluded by the current data and observations. At the same time, the $\nu_{e} \rightarrow \bar{\nu}_{e}$ conversion (which is a $\Delta L=2$ process) cannot occur if neutrinos are Dirac particles. Thus, future supernova neutrino experiments may shed some light onto the Dirac vs Majorana nature of neutrinos. Yet, the most practical means of disentangling these two neutrino types is probably neutrinoless double $\beta$-decay, which will be discussed in sec. 15.7.

### 15.6 Majorana particles in SUSY

In supersymmetric (SUSY) theories each boson has a supersymmetric partner which is a fermion and each fermion has a bosonic superpartner. Such theories predict the existence of a plentitude of Majorana fermions, which are supersymmetric partners of neutral bosons [8,14, 15]. These include the photino, as well as the gluino, zino and neutral higgsinos (the SUSY partners of the photon, gluon, $Z^{0}$-boson and of the neutral Higgs scalars, respectively). More precisely, since these particles can mix, what actually makes the Majorana fermions are the so-called neutralinos - the linear superpositions of the above-mentioned particles that have definite masses. ${ }^{17}$ In addition, if the spontaneous breaking of global supersymmetry occurs, there should exist the goldstino - a massless neutral Goldstone fermion. In supergravity the goldstino is absorbed, through a supersymmetric analogue of the Higgs mechanism, into the gravitino, which is a massive spin-3/2 Majorana fermion (the SUSY partner of the graviton). In SUSY versions of the models where the so-called strong CP problem is solved through the existence of a light neutral pseudoscalar particle - the axion - there is yet another Majorana fermion, the axino.

In SUSY models with conserved $R$-parity the lightest supersymmetric particle is stable. If it is neutral, it can be the so-called WIMP (weakly interacting massive particle) and play a role of the dark matter particle, i.e. account for the missing matter of the Universe [51-53].

[^14]The lightest supersymmetric particle is then the lightest neutralino, the gravitino or the axino. ${ }^{18}$ Thus the dark matter problem, which is one of the most important problems of modern cosmology, may have its solution through the existence of a Majorana fermion.

### 15.7 Experimental searches for Majorana neutrinos and other Majorana particles

### 15.7.1 Neutrinoless $2 \beta$ decay and related processes

As was mentioned above, the most practical way of discriminating between Dirac and Majorana neutrinos seems to be by looking for neutrinoless double beta decay (see refs. [55-58] for reviews). The usual double beta decay is the process in which a nucleus $A(Z, N)$ decays into an isobar with the electric charge differing by two units:

$$
\begin{equation*}
A(Z, N) \rightarrow A(Z \pm 2, N \mp 2)+2 e^{\mp}+2 \bar{\nu}_{e}\left(2 \nu_{e}\right) \tag{105}
\end{equation*}
$$

In such decays two neutrons of the nucleus are simultaneously converted into two protons, or vice versa. At the fundamental (quark) level, these are transitions of two $d$ quarks into two $u$ quarks or vice versa (see fig. 2a). Double beta decay is the process of the second order in weak interaction, and the corresponding decay rates are very low: typical lifetimes of the nuclei with respect to the $2 \beta$ decay are $T \gtrsim 10^{19}$ years. The processes (105) are called $2 \nu \beta \beta$ decays. Two-neutrino double beta decays with the emission of two electrons $\left(2 \beta^{-}\right)$were experimentally observed for a number of isotopes with the half-lives in the range $\sim 10^{19}-10^{24}$ years [58]; there are few candidate nuclei for $2 \beta^{+}$decay, and the experimental observation of this process is difficult because of the very small energy release ( $Q$ values).

If neutrinos are Majorana particles, the lepton number is not conserved, and the neutrino emitted in one of the elementary beta decay processes forming the $2 \beta$ decay can be absorbed in another (fig. 2b), leading to the neutrinoless double beta ( $0 \nu \beta \beta$ ) decay [59]:

$$
\begin{equation*}
A(Z, N) \rightarrow A(Z \pm 2, N \mp 2)+2 e^{\mp} \tag{106}
\end{equation*}
$$

Such processes would have a very clear experimental signature: since the recoil energy of a daughter nucleus is negligibly small, the sum of the energies of the two electrons or positrons in the final state should be equal to the total energy release, i.e. should be represented by a discrete energy line. Therefore $0 \nu \beta \beta$ decays could serve as a sensitive probe of the lepton number violation and Majorana nature of neutrinos. In some extensions of the standard model exotic modes of $0 \nu \beta \beta$ decay are possible, e.g. decays with a Majoron emission [55,57].

[^15]In this case the sum of the energies of two electrons or positrons is not a discrete line, but the $2 \beta$ energy spectra (as well as the single $\beta$-particle spectra) are expected to be different from those in the case of $2 \nu \beta \beta$ decay.


Figure 2: Some Feynman diagrams for the amplitudes of $2 \beta$ decay.

Neutrinoless $2 \beta$ decays break not only the lepton number; since the absorbed $\nu_{e}$ or $\bar{\nu}_{e}$ has a 'wrong' chirality, $0 \nu \beta \beta$ decays also break chirality conservation. Therefore, if $0 \nu \beta \beta$ decay is mediated by the standard weak interactions and exchange of light neutrinos, the amplitude of the process must be proportional to the neutrino mass. More precisely, as follows from fig. 2b, it is proportional to the ee-entry of the neutrino Majorana mass matrix, whose modulus is usually called $\left\langle m_{\beta \beta}\right\rangle$ :

$$
\begin{equation*}
A(0 \nu \beta \beta) \propto\left|\sum_{i} U_{e i}^{2} m_{i}\right| \equiv\left\langle m_{\beta \beta}\right\rangle . \tag{107}
\end{equation*}
$$

Notice that this expression contains $U_{e i}^{2}$ rather than $\left|U_{e i}\right|^{2}$. If CP is conserved in the leptonic sector, the mixing matrix $U_{a i}$ can always be made real; however in this case the mass parameters $m_{i}$ in (107) (the eigenvalues of the neutrino mass matrix) can be of either sign, their relative signs being related to the relative CP parities of neutrinos. This means that in general significant cancellations between various contributions to the sum in (107) are possible. As we discussed in sec. 15.3.3, a pair of Majorana neutrinos with equal physical masses $\left|m_{i}\right|$, opposite CP parities and maximal mixing is equivalent to a Dirac neutrino. It is easy to see that such a pair does not contribute to the amplitude in (107) - the contributions of the two components of the pair cancel exactly. Analogously, the contribution of a pseudoDirac neutrino to (107) would be strongly suppressed. Partial cancellations of contributions of different neutrino mass eigenstates to $\left\langle m_{\beta \beta}\right\rangle$ are also possible, of course, when CP is violated, i.e. when the leptonic mixing matrix is complex (with the convention that all neutrino masses are non-negative). In the case of just three usual light neutrino species
eqs. (107) and (61) yield

$$
\begin{equation*}
\left\langle m_{\beta \beta}\right\rangle=\left|c_{13}^{2} c_{12}^{2} m_{1}+c_{13}^{2} s_{12}^{2} e^{2 i \varphi_{1}} m_{2}+s_{13}^{2} e^{2 i\left(\varphi_{2}-\delta_{C P}\right)} m_{3}\right| \tag{108}
\end{equation*}
$$

By now, the neutrino oscillation experiments measured rather accurately the leptonic mixing angles $\theta_{12}, \theta_{23}, \theta_{13}$ and the neutrino mass squared differences $\Delta m_{21}^{2}$ and $\left|\Delta m_{31}^{2}\right|$. Global analyses of the data [60-62] yield

$$
\begin{array}{r}
\Delta m_{21}^{2} \simeq 7.5 \times 10^{-5} \mathrm{eV}^{2}, \quad\left|\Delta m_{31}^{2}\right| \simeq 2.4 \times 10^{-3} \mathrm{eV}^{2} \\
\theta_{12} \simeq 33^{\circ}, \quad \theta_{23} \simeq 40^{\circ} \text { or } 50^{\circ}, \quad \theta_{13} \simeq 9^{\circ} . \tag{110}
\end{array}
$$

At the same time, at present there is essentially no information on the CP-violating phases and the neutrino mass ordering (the sign of $\Delta m_{31}^{2}$ ), while for the absolute scale of the neutrino masses only upper limits exist: direct neutrino mass measurements in nuclear $\beta$ decay experiments and cosmology yield $m_{i} \lesssim \mathcal{O}(1) \mathrm{eV}$. With these data, it follows from eq. (108) that sizeable cancellations between the contributions of the different neutrino mass eigenstates to $\left\langle m_{\beta \beta}\right\rangle$ are possible only in the case of the so-called normal neutrino mass hierarchy, $m_{1}, m_{2} \ll m_{3}$.

If, in addition to the usual three light neutrino species, there exist heavy neutrinos $N_{i}$, the active flavour-eigenstate neutrinos are linear superpositions of the left-handed components of both light and heavy neutrino mass eigenstates (see eq. (52)). Since the chiralityflipping part of the fermion propagator $m /\left(p^{2}-m^{2}\right) \simeq-1 / m$ for $m^{2} \gg p^{2}$, the contribution of the diagram 2 b with exchanges of heavy Majorana neutrinos to the amplitude of $0 \nu \beta \beta$ decay is proportional to

$$
\begin{equation*}
\left\langle m_{N}^{-1}\right\rangle \equiv\left|\sum_{i=4}^{n} U_{e i}^{2} m_{i}^{-1}\right| \tag{111}
\end{equation*}
$$

Thus, one should distinguish effects of light and heavy Majorana neutrino exchanges. The latter requires the existence of extra neutrino species and can be considered as one of nonstandard mechanisms of $0 \nu \beta \beta$ decay. The effect of Majorana neutrino exchanges on lepton number violating processes is expected to be maximal when the mass of the exchanged neutrino is of the order of the characteristic energy of the process. This applies not only to $0 \nu \beta \beta$ decay but to all $\Delta L=2$ processes, including those considered in sec. 15.7.2 below.

In extensions of the standard model, such as the left-right symmetric, SUSY or grand unification models, additional mechanisms of $0 \nu \beta \beta$ decay are possible, in which the process is mediated by right-handed currents, SUSY particles or leptoquarks (see, e.g., [57]). One of the diagrams contributing to the amplitude of $0 \nu \beta \beta$ decay in left-right symmetric theories is shown in fig. 2c. It may appear that no Majorana mass $m_{L}$ of $\nu_{L}$ is necessary in such models, i.e. $0 \nu \beta \beta$ decay can occur even if $m_{L}=0$ and the neutrinos are Dirac particles, or
even if they are massless. This is, however, incorrect: in all models in which $0 \nu \beta \beta$ decay occurs, the Majorana masses of $\nu_{L}$ must be different from zero. An elegant 'black box' proof of this statement was presented in [63]. In fig. 3 the black box represents an unspecified mechanism by which two $d$-quarks can be converted to two $u$-quarks and two electrons, with no accompanying neutrinos. Next, we make use of the crossing symmetry to transform the initial-state $d$-quarks to the final-state $\bar{d}$-quarks, join the $u \bar{d}$ lines to produce $W$-bosons and then attach the other ends of the $W$-boson lines to the electron lines to produce neutrinos. This yields the diagram corresponding to the effective operator $\bar{\nu}_{e L}\left(\nu_{e L}\right)^{c}$ describing the $\bar{\nu}_{e R} \rightarrow \nu_{e L}$ transition, i.e. the Majorana mass term for $\nu_{e}$. Thus, no matter what mechanism


Figure 3: The black box argument for Majorana neutrino mass [63].
causes $0 \nu \beta \beta$ decay, observation of this process would constitute an unambiguous proof that neutrinos are Majorana particles.

If neutrinos are of Majorana nature, then in addition to the usual $0 \nu \beta \beta$ decay (105), some related processes should occur, such as e.g. $e_{B}+A(Z, N) \rightarrow A(Z-2, N+2)+e^{+}$ (neutrinoless electron capture) or $2 e_{B}+A(Z, N) \rightarrow A(Z-2, N+2)^{*} \rightarrow A(Z-2, N+2)+X$ (neutrinoless double electron capture) [64]. Here $e_{B}$ stands for a bound atomic electron and $A(Z-2, N+2)^{*}$ denotes the excited state of the $A(Z-2, N+2)$ atom with two holes in the atomic $1 S$ orbit, which then de-excites with the emission of atomic $X$-rays, Auger electrons, etc.. These processes are always energetically allowed if the usual $2 \beta^{+}$decay of $A(Z, N)$ is allowed, and in some cases may also be allowed even if $A(Z, N)$ is stable. The neutrinoless double electron capture may be resonantly enhanced provided that the total energy release is very close to the excitation energy of the $A(Z-2, N+2)^{*}$ atomic state. The search for isotopes with suitable atomic mass differences and excitation energies of the daughter atoms is currently under way [65].

Other processes related to $0 \nu \beta \beta$ decay have been discussed, such as muon conversion on nuclei, $\mu^{-}+A(Z, N) \rightarrow A(Z-2, N+2)+e^{+}$or $\mu^{-}+A(Z, N) \rightarrow A(Z-2, N+2)+\mu^{+}$. Just like the $0 \nu \beta \beta$ decay, such conversion processes break the total lepton number $L=L_{e}+L_{\mu}+L_{\tau}$ by two units (note, however, that the $\left(\mu^{-}, e^{+}\right)$conversion conserves $\left.L_{e}-L_{\mu}\right)$. If these processes are mediated by exchanges of light or heavy Majorana neutrinos, their expected rates are too small to render them observable in a foreseeable future $[66,67]$. Thus, the
muon conversion processes may be of interest only if they are dominated by non-standard mechanisms.

It is customary to discuss the available data of $0 \nu \beta \beta$ decay experiments as well as expected sensitivities of the ongoing and future experiments in terms of the limits on the half-lives of the parent nuclei $T_{1 / 2}^{0 \nu}$ and interpret them in terms of the effective mass parameter $\left\langle m_{\beta \beta}\right\rangle$ defined in (107). It should, however, be remembered that such an interpretation makes sense only when the standard diagram of fig. 2a with exchange of light Majorana neutrinos is the sole (or the dominant) contribution to the amplitude of the process, in which case $T_{1 / 2}^{0 \nu} \propto 1 /\left\langle m_{\beta \beta}\right\rangle^{2}$. Otherwise, there is no or little connection between these two quantities, and the masses of light neutrinos cannot be directly probed in experiments on $0 \nu \beta \beta$ decay. ${ }^{19}$ If $0 \nu \beta \beta$ decay is dominated by non-standard mechanisms, experiments can only give upper limit on the parameter $\left\langle m_{\beta \beta}\right\rangle$ and therefore on the Majorana masses of the neutrino mass eigenstates. However, as was stressed above, even if the Majorana neutrino mass gives negligible contribution to $0 \nu \beta \beta$ decay, an observation of this process would be an unambiguous proof of the Majorana nature of neutrinos. Experimentally, one can in principle distinguish between different mechanisms of $0 \nu \beta \beta$ decay by studying the properties of the decay products, e.g. angular correlations of the two produced $\beta$-particles, or by looking for processes related to $0 \nu \beta \beta$ decay (see e.g. discussion in sec. 6 of ref. [57]).

Neutrinoless double beta decay has been actively searched for but up to now it has not been experimentally discovered. ${ }^{20}$ The available data allow to put upper bounds on the effective Majorana neutrino mass $\left\langle m_{\beta \beta}\right\rangle$. The best current limits come from the EXO-200 experiment on $0 \nu \beta \beta$ decay of ${ }^{136} \mathrm{Xe}$ [69] and GERDA experiment with ${ }^{76} \mathrm{Ge}[70]$ ( $90 \%$ C.L.):

$$
\begin{equation*}
\left\langle m_{\beta \beta}\right\rangle<0.14-0.38 \mathrm{eV} \quad(\text { EXO-200 }) ; \quad\left\langle m_{\beta \beta}\right\rangle<0.2-0.4 \mathrm{eV} \quad(\text { GERDA }) \tag{112}
\end{equation*}
$$

where the ranges are due to the uncertainties in the values of the nuclear matrix elements. The most promising current and forthcoming experiments are expected to be sensitive to the values $\left\langle m_{\beta \beta}\right\rangle \gtrsim 0.03-0.1 \mathrm{eV}[58]$; as follows from (109) and (110), they will be able to explore the Majorana neutrino mass only if neutrinos are quasi-degenerate in mass ( $m_{1} \sim m_{2} \sim m_{3}$ )

[^16]or have the inverted mass hierarchy ( $m_{3} \ll m_{1}, m_{2}$ ).
How does it work? As an example, assume that from independent experiments (e.g. neutrino oscillations plus cosmology) we know that neutrino masses obey the inverted hierarchy, i.e. $m_{3} \ll m_{1}, m_{2}$. From eq. (108) it then follows that the quantity $\left\langle m_{\beta \beta}\right\rangle$ cannot be smaller than $\left\langle m_{\beta \beta}\right\rangle_{\min } \approx c_{13}^{2} \cos 2 \theta_{12} \sqrt{\left|\Delta m_{31}^{2}\right|}(\sim 0.02 \mathrm{eV})$. If from a $0 \nu \beta \beta$ experiment an upper limit on $\left\langle m_{\beta \beta}\right\rangle$ is inferred which is smaller than $\left\langle m_{\beta \beta}\right\rangle_{\text {min }}$, this would rule out Majorana nature of neutrinos, barring destructive interference with some non-standard $0 \nu \beta \beta$ decay mechanisms.

If the neutrino masses obey the normal mass hierarchy, probing the Majorana neutrino mass through $0 \nu \beta \beta$ decay will be problematic and will in any case require multiton-scale detectors and very low backgrounds. It would be extremely difficult to uncover Majorana vs Dirac nature of neutrinos in this case, unless an efficient non-standard mechanism of $0 \nu \beta \beta$ decay is at play.

### 15.7.2 Other lepton number violating processes

The Majorana nature of neutrinos can be revealed not only through neutrinoless double $\beta$-decay, but also through other lepton number violating processes. One of such processes - neutrino spin-flavour precession in strong external magnetic fields - was discussed in sec. 15.5. Here we shall briefly discuss two of the other types of $\Delta L=2$ processes, rare particle decays and like-sign dilepton production at accelerators.

If neutrinos are Majorana particles, they should mediate rare $\Delta L=2$ particle decays, such as $K^{+} \rightarrow \pi^{-} e^{+} e^{+}, K^{+} \rightarrow \pi^{-} \mu^{+} \mu^{+}$, and similar decays of charged $B$ and $D$ mesons. The typical Feynman diagrams for such processes are essentially the same as the one in fig. 2b, except that some other quarks may be involved. However, the number of the parent particles in the case of rare meson decays is suppressed by a huge factor of order of the Avogadro number $N_{A}$ as compared to those in $2 \beta$-decay experiments. Therefore, rare meson decays cannot compete with neutrinoless $2 \beta$-decay in unraveling the neutrino nature when the exchanged Majorana neutrinos are very light or very heavy. Still, rare decays can provide tighter limits on the Majorana neutrino masses in the region of the order of the energy release of the corresponding process (for rare kaon decays, a few hundred MeV ).

In accelerator-based experiments the Majorana nature of neutrinos can be tested in the processes with like-sign dilepton production as well as in related reactions. The basic $\Delta L=2$ processes are in this case $W^{ \pm} W^{ \pm} \leftrightarrow \ell_{1}^{ \pm} \ell_{2}^{ \pm}$, where $\ell_{1,2}=e, \mu$ or $\tau$. A typical reaction that can be studied at hadron colliders (first discussed in [71] in the context of right-handed currents) is $p p \rightarrow \ell_{1}^{ \pm} \ell_{2}^{ \pm} X$. Majorana neutrinos contribute to the amplitude of this process through the diagrams of fig. 4 and similar diagrams with other quarks in the final state.
Here the left diagram is similar to diagram 2 b that gives the standard contribution to


Figure 4: Representative Majorana neutrino exchange diagrams contributing to like-sign dilepton production. Left: $W$-boson fusion with $t$-channel $N$ exchange. Right: $s$-channel $W$-exchange diagram with production and subsequent decay of $N$.
the amplitude of $0 \nu \beta \beta$ decay, whereas the right diagram corresponds to production and subsequent decay of a virtual or real Majorana neutrino $N$. If real Majorana neutrinos are kinematically accessible, the dilepton production process can be resonantly enhanced.

Lepton number violating rare decays and like-sign dilepton production processes have been actively looked for experimentally at accelerators, but no signals have been found so far. This allowed one to put important constraints on the properties of Majorana neutrinos. For a detailed and comprehensive discussion of these (and other) $\Delta L=2$ processes as well as of the effects of Majorana neutrinos on electroweak precision observables we refer the reader to ref. [72] (see also [57] and references therein).

There have also been extensive searches for the SUSY Majorana particles both at accelerators and in dark matter detectors. For recent discussions of these experiments see e.g. [73-75]. Unfortunately, up to now no unambiguous evidence for such particles has been obtained.

### 15.8 Baryogenesis through leptogenesis and Majorana neutrinos

In addition to providing a simple and natural way of explaining the observed smallness of the neutrino mass, the seesaw mechanism of the neutrino mass generation brings with it a free bonus: It furnishes a very simple and attractive mechanism of producing the observed baryon asymmetry of the Universe (BAU). The term baryon asymmetry simply reflects the fact that the observed Universe is made predominantly of matter rather than of equal amount of matter and antimatter (there are very stringent constraints on the antimatter abundance in the Universe [76]). The ratio of the net baryon number to photon number in
the Universe is now measured very accurately [77]:

$$
\begin{equation*}
\eta \equiv \frac{N_{B}-N_{\bar{B}}}{N_{\gamma}}=(6.04 \pm 0.08) \times 10^{-10} \tag{113}
\end{equation*}
$$

where $N_{B}, N_{\bar{B}}$ and $N_{\gamma}$ are the number densities of baryons, antibaryons and photons at the present epoch. The observed BAU could not have resulted from an initial state of the Universe with $B \neq 0$, as any such pre-existing asymmetry would have been diluted to an absolutely negligible level during the stage of the accelerated expansion of the Universe predicted by the cosmic inflation [78] (which is the standard paradigm now). Thus, the BAU should have been generated dynamically in the post-inflationary epoch.

Under what conditions such a dynamical generation of the BAU can occur? These conditions were actually formulated by Sakharov in 1967 [79]: (i) baryon number violation; (ii) C- and CP-violation; (iii) deviation from thermal equilibrium. The first two conditions are necessary for the baryon asymmetry to be produced in the first place; the last condition ensures that the BAU produced in some processes is not destroyed by the inverse processes.

> As an illustration, consider a process $X \rightarrow Y+b$, where $X$ denotes an initial state with zero baryon number, $Y$ stands for a set of final-state particles with vanishing net baryon number and $b$ represents the produced excess baryons. Then, if condition (i) is not met, the process $X \rightarrow Y+b$ just does not take place. If either C or CP is conserved, the processes $X \rightarrow Y+b$ and $\bar{X} \rightarrow \bar{Y}+\bar{b}$ occur at the same rate, and no net baryon number is produced (provided that the initial state of the system contained equal numbers of $X$ and $\bar{X}$ or that $X=\bar{X}$ ). If the system is in thermal equilibrium, the processes $X \rightarrow Y+b$ and $Y+b \rightarrow X$ occur at the same rate (which is also true, of course, for $\bar{X} \rightarrow \bar{Y}+\bar{b}$ and $\bar{Y}+\bar{b} \rightarrow \bar{X}$ ), and the baryon asymmetry produced in direct processes is washed out by the inverse ones.

All these conditions are actually satisfied in the standard model of particle physics, though the amount of CP violation in this model is insufficient and, most importantly, the deviation from thermal equilibrium is far too small to account for the measured value of the BAU (113) [80]. The Sakharov's conditions are fulfilled and the successful generation of the BAU is possible in many extensions of the standard model, such as grand unification theories and SUSY models $[80,81]$. Here we concentrate on the so-called baryogenesis via leptogenesis [82], which is built-in in the seesaw mechanism of the neutrino mass generation and does not require any new physics besides the existence of heavy Majorana neutrinos. We just outline the mechanism here; for details and ramifications we refer the reader to a comprehensive review [83].

To produce the phenomenologically acceptable mass spectrum of the usual light neutrinos, the seesaw mechanism should include at least two heavy electroweak-singlet (i.e. sterile) Majorana neutrinos $N_{i}$. The same number of $N_{i}$ 's turns out to be sufficient for the generation of the BAU. The mechanism works as follows. First, a lepton number $L_{0}$
is produced through the out-of-equilibrium $L$ - and ( $B-L$ )-violating decays of the $N_{i}$ 's. The produced lepton number is then reprocessed into the baryon number by the so-called sphaleron processes (hence the name baryogenesis through leptogenesis).

Let us consider this in more detail. The singlet neutrinos $N_{i}$ are actually not completely sterile: they cannot have gauge interactions in the standard model, but can have the usual Yukawa couplings $h_{\alpha i}^{*} \bar{\ell}_{\alpha} N_{i R} H+h . c .^{21}$ with the lepton doublets $\ell_{\alpha}=\left(\nu_{\alpha L}, e_{\alpha L}\right)^{T}$ and the Higgs field $H=\left(H^{0}, H^{-}\right)^{T}$, which are allowed by the electroweak gauge symmetry. The Yukawa couplings result in decays of $N_{i}$ into the usual leptons and the Higgs particles. Since the $N_{i}$ 's are Majorana particles, their decay proceeds in a lepton number violating way, i.e. they can decay both via $N_{i} \rightarrow \ell_{\alpha} \bar{H}$ and through the CP-conjugate channel $N_{i} \rightarrow \bar{\ell}_{\alpha} H$. If the Yukawa couplings $h_{\alpha i}$ are complex, CP is not conserved in the leptonic sector, and the rates of the above decay modes are in general different: $\Gamma\left(N_{i} \rightarrow \ell_{\alpha} \bar{H}\right) \neq \Gamma\left(N_{i} \rightarrow \bar{\ell}_{\alpha} H\right)$. This leads to the production of a non-zero net lepton number. Note that CP-violation manifests itself through the interference between the tree-level and 1-loop Feynman diagrams describing the $N_{i}$ decay; at the tree level the decay rates are proportional to $\left|h_{\alpha i}\right|^{2}$, and the complexity of the Yukawa constants does not reveal itself. The parameter that describes the generation of the lepton asymmetry in the decay of $N_{i}$ is

$$
\begin{equation*}
\epsilon_{i}=\sum_{\alpha} \frac{\Gamma\left(N_{i} \rightarrow \ell_{\alpha} \bar{H}\right)-\Gamma\left(N_{i} \rightarrow \bar{\ell}_{\alpha} H\right)}{\Gamma\left(N_{i} \rightarrow \ell_{\alpha} \bar{H}\right)+\Gamma\left(N_{i} \rightarrow \bar{\ell}_{\alpha} H\right)}=\frac{1}{8 \pi\left(h^{\dagger} h\right)_{i i}} \sum_{j \neq i} \operatorname{Im}\left[\left(h^{\dagger} h\right)_{i j}^{2}\right] g\left(x_{j}\right), \tag{114}
\end{equation*}
$$

where $x_{j} \equiv M_{j}^{2} / M_{i}^{2}$ and $g(x)$ is model dependent. In the standard model

$$
\begin{equation*}
g(x)=\sqrt{x}\left[\frac{2-x}{1-x}-(1+x) \ln \left(\frac{1+x}{x}\right)\right] . \tag{115}
\end{equation*}
$$

In (114) we for simplicity summed over the flavours of final-state leptons (note that flavour effects may actually be important, see the discussion in sec. 9 of [83]). Eq. (114) is valid only when the mass differences of the heavy singlet Majorana neutrinos are large compared with their decay widths, $\left|M_{j}-M_{i}\right| \gg \Gamma_{i}+\Gamma_{j}$; the opposite case, which leads to resonant leptogenesis [84], requires a special consideration.

The deviation from thermal equilibrium is provided by the expansion of the Universe, the rate of which is given by the Hubble parameter

$$
\begin{equation*}
H(T)=1.66 \sqrt{g_{*}} T^{2} / M_{\mathrm{Pl}} . \tag{116}
\end{equation*}
$$

Here $T$ is the temperature of the Universe, $M_{\mathrm{Pl}}=1.2 \times 10^{19} \mathrm{GeV}$ is the Planck mass and $g_{*}$ is the number of relativistic degrees of freedom in the thermal bath (in the standard model

[^17]$\left.g_{*}=106.75\right)$. If the processes that create and destroy some particles are fast compared with the Hubble expansion rate $H(T)$, they equilibrate particle distributions, otherwise the thermal equilibrium is not achieved. For the singlet neutrino $N_{i}$ the condition of deviation from thermal equilibrium requires that at the time of the $N_{i}$ decay $\left(T \sim M_{i}\right)$ the decay rate $\Gamma_{i}=\frac{\left(h^{\dagger} h\right)_{i i}}{8 \pi} M_{i}$ be smaller than the Hubble rate: $\left.\frac{\left(h^{\dagger} h\right)_{i i}}{8 \pi} M_{i} \lesssim 1.66 \sqrt{g_{*}} \frac{T^{2}}{M_{\mathrm{Pl}}}\right|_{T \sim M_{i}}$. Thus, the lightest singlet neutrino $N_{i}$ is typically the last one to go out of equilibrium in the course of the expansion and cooling of the Universe. Therefore the lepton asymmetry produced in decays of heavier Majorana neutrinos is washed out by the processes involving $N_{1}$, and the net lepton asymmetry of the Universe is produced in the decays of $N_{1} .{ }^{22}$ The out-ofequilibrium condition for these decays can be rewritten as
\[

$$
\begin{equation*}
\tilde{m}_{1} \equiv \frac{\left(h^{\dagger} h\right)_{11} v^{2}}{M_{1}} \lesssim 8 \pi \cdot 1.66 \sqrt{g_{*}} \frac{v^{2}}{M_{\mathrm{Pl}}} \simeq 1.1 \times 10^{-3} \mathrm{eV} \tag{117}
\end{equation*}
$$

\]

where $v=174 \mathrm{GeV}$ is the Higgs VEV. In the opposite case the produced lepton asymmetry is strongly washed out. It is interesting to note that, since the Dirac-type neutrino mass matrix $m_{D}=h^{T} v$, the left-hand side of (117) is roughly of the same order of magnitude as the masses of the light active neutrinos predicted by the seesaw mechanism, ${ }^{23}$ whereas the right-hand side is close to the light neutrino mass scale that follows from the oscillation experiments assuming that neutrino masses are hierarchical. Typically, one expects the left-hand side of (117) to slightly exceed its right-hand side, leading to a moderate washout of the lepton asymmetry.

How can the produced lepton number be converted into the baryon one? In the standard model the baryon and lepton numbers are conserved at the tree level but are violated at 1-loop level by the so-called chiral anomalies. The anomalies of the baryon and lepton number currents are the same, and so $B-L$ is exactly conserved, but the sum $B+L$ is not. Although the $(B+L)$-violating processes are strongly suppressed at zero temperature, the situation at high temperatures is different [85]. The standard model predicts the existence of topologically non-trivial configurations of the gauge and Higgs fields (called sphalerons) which violate $B+L$ with the rate $\Gamma_{\text {sph }}$ that exceeds the Hubble rate for $100 \mathrm{GeV} \lesssim T \lesssim$ $10^{12} \mathrm{GeV}$. In this temperature interval the sphaleron processes can efficiently wash out $B+L$ and thus reprocess the lepton asymmetry into the baryon one. In a somewhat simplified way, the reprocessing mechanism can be described as follows.

Assume that at a time $t=0$ a net lepton number $L_{0}$ is produced, while the initial baryon number $B_{0}=0$. Noting that $B$ and $L$ can be represented as linear combinations of

[^18]$B-L$ and $B+L$ and that $B+L$ is exponentially suppressed with time by the sphaleron processes, for the values of $L$ and $B$ at a time $t \geq 0$ we have
\[

$$
\begin{align*}
L(t) & =-\frac{1}{2}(B-L)_{0}+\frac{1}{2}(B+L)_{0} e^{-\Gamma_{\mathrm{sph}} t} \\
B(t) & =\frac{1}{2}(B-L)_{0}+\frac{1}{2}(B+L)_{0} e^{-\Gamma_{\mathrm{sph}} t} . \tag{118}
\end{align*}
$$
\]

Thus, at the times $t \gg \Gamma_{\mathrm{sph}}^{-1}$ we have $L \simeq L_{0} / 2, B \simeq-L_{0} / 2$, i.e. we end up with non-zero baryon number. A realistic calculation, which takes into account that only lefthanded quarks and leptons are coupled to the $W$-boson field, yields (in the standard model) $B=-(29 / 78) L_{0}$ rather than $-L_{0} / 2$. In addition, one should carefully take into account the processes that wash out the lepton asymmetry, such as inverse $N_{1}$ decays and $2 \rightarrow 2$ scattering processes. This is usually done by solving a system of Boltzmann equations or quantum kinetic equations. As a result, one finds that for hierarchical masses of heavy singlet neutrinos the observed value of the BAU can be generated provided that the mass of the lightest among the heavy Majorana neutrinos $N_{1}$ satisfies $M_{1} \gtrsim 10^{8} \mathrm{GeV}$. For quasidegenerate in mass heavy neutrinos the viable baryon asymmetry can be achieved, through the resonant leptogenesis, even for $M_{i}$ as small as $\sim 1 \mathrm{TeV}$ [84].

In the discussed baryogenesis mechanism all three Sakharov's conditions are satisfied. The baryon number violation is provided by the combination of $L$ (and $B-L$ ) violation in decays of heavy Majorana neutrinos and $B+L$ violation by the sphaleron processes. C-violation follows from the chiral nature of the Yukawa couplings, while CP-violation is a consequence of the complexity of the corresponding coupling constants. Finally, the condition of deviation from thermal equilibrium is met because in certain ranges of the parameters the rates of decay and inverse decay of the heavy Majorana neutrinos (as well as the rates of other $L$-violating processes) do not exceed significantly the Hubble expansion rate.

What we discussed above was baryogenesis via leptogenesis in type I seesaw. Similar mechanisms work in the case of type II and type III seesaw [83]. There also exists an alternative leptogenesis mechanism [86,87], in which the lepton asymmetry is generated in CP-violating oscillations of the heavy Majorana neutrinos $N_{i}$ rather than in their decays. The produced asymmetry is then communicated from the $N_{i}$ 's to the usual leptons through their Yukawa couplings, and the reprocessing of the lepton number to the baryon number proceeds through the sphaleron processes in the usual way.

In all the discussed versions of baryogenesis through leptogenesis the Majorana nature of the singlet neutrinos plays a crucial role. There also exist leptogenesis scenarios with Dirac neutrinos (see sec. 10.4 of [83] and references therein), but they are based on more complicated and less economical models.

### 15.9 Miscellaneous

Here we collect a few assorted remarks on Majorana particles.
It is usually said that in e.g. nuclear $\beta^{-}$decay an electron antineutrino $\bar{\nu}_{e}$ is emitted, while positron production in $\beta^{+}$decay is accompanied by the emission of an electron neutrino $\nu_{e}$, and we know that these are distinct particles. Does that mean that we have already established that neutrinos are Dirac particles and $\nu_{e} \neq\left(\nu_{e}\right)^{c}$ ? Not really. The point is that the charged-current weak interactions are chiral, so that only left-handed particles and their right-handed $\hat{C}$-conjugates can be emitted or absorbed. In the Dirac case, this means that only the left-handed component of the Dirac field $\nu_{e}=\nu_{e L}+\nu_{e R}$ (as well as the right-handed component $\left(\nu_{e L}\right)^{c} \equiv \bar{\nu}_{e}$ of $\left.\left(\nu_{e}\right)^{c}=\left(\nu_{e L}\right)^{c}+\left(\nu_{e R}\right)^{c}\right)$ take part in the interactions, while $\nu_{e}$ and $\left(\nu_{e}\right)^{c}$ are indeed different particles. In the Majorana case we have $\nu_{e}=\nu_{e L}+\left(\nu_{e L}\right)^{c}$, and both chiral components of the field participate in weak interactions. What we call $\nu_{e}$ and $\bar{\nu}_{e}$ are in this case merely the left-handed and right-handed components of the same Majorana field of the electron neutrino. They are to a very good accuracy distinct because neutrinos we deal with are always highly relativistic, and the transitions between their left-handed and right-handed components are suppressed by the factor $\left(m_{\nu} / E\right)^{2} \ll 1$. Thus, the role of the lepton number, which is conserved in the Dirac case, is played for relativistic Majorana neutrinos by chirality, which is nearly conserved. This illustrates once again the point we have already made more than once - the smallness of the neutrino mass makes it very difficult to discriminate between Dirac and Majorana neutrinos.

Can one still tell these two neutrino types apart by studying neutrino propagation under extreme conditions, such as e.g. very high densities and/or strong magnetic fields which are expected to be present in stellar environments? This question was studied in [88], and the answer unfortunately turns out to be essentially negative.

It is a well known but not yet completely understood fact that electric charge is quantized. Possible explanations include the existence of the magnetic monopole and grand unification of particles and forces. It turns out, however, that electric charge quantization can be understood even outside these frameworks if neutrinos are Majorana particles [89]. In the minimal standard model with no right-handed (singlet) neutrinos $\nu_{R}$, charge quantization is a consequence of the hypercharge assignment of the particles that follows from the requirement of the anomaly cancellation. The cancellation of anomalies, in turn, is necessary for internal consistency of the theory. However, the minimal standard model is not realistic in the sense that neutrinos are massless in it. It therefore in any case has to be amended by a neutrino mass generating mechanism. If one adds right-handed singlet neutrinos to the standard model, there are essentially two possibilities. First, one imposes a lepton number conservation which allows only Dirac mass terms for neutrinos. In this case
the anomaly cancellation condition no longer leads to electric charge quantization. If no lepton number conservation is imposed, massive neutrinos turn out to be Majorana particles. In this case anomaly cancellation always results in electric charge quantization [89]. The authors of [89] have also studied a wide class of non-grand-unified extensions of the standard model which allow massive neutrinos, and found that in virtually all cases the Majorana nature of neutrinos led to electric charge quantization, whereas for Dirac neutrinos no such quantization occurred. Thus, the observed quantization of electric charge in Nature may have its explanation through the existence of Majorana neutrinos.

It is conceivable that our (3+1)-dimensional world is actually embedded in a space-time of higher dimensionality; in particular, higher-dimensional space-times appear in KaluzaKlein, supergravity and superstring models. From the point of view of applications to condensed-matter physics, it may also be interesting to consider space-times of lower dimensionality. Can Majorana particles live in such unconventional space-times? The answer is yes, but not in all of them. For $d$-dimensional space-times with $d-1$ space-like and one time-like dimensions, massive Majorana fermions can exist only if $d=2,3$ and $4 \bmod 8$ (see e.g. sec. 2 of [90] and references therein). Massless self-conjugate fermions can live in the space-times of the same dimensionality, and in addition in $d=8$ and $9 \bmod 8$ dimensions. ${ }^{24}$ These results can also be extended to the case of $n>1$ time-like dimensions.

### 15.10 Summary and conclusions

The possibility of existence of fermions which are their own antiparticles is certainly the most famous and arguably the most important result obtained by Ettore Majorana. Extensions of the standard model typically predict neutrinos to be massive Majorana particles. There are some experimental hints in favour of possible existence of extra neutrino species (on top of the already known $\nu_{e}, \nu_{\mu}$ and $\nu_{\tau}$ ); if exist, they are very likely Majorana particles. The Majorana nature of neutrinos would imply lepton number violation - a very interesting phenomenon which is now being intensely searched for experimentally.

Possible existence of heavy electroweak-singlet Majorana neutrinos provides us, through the seesaw mechanism, with a natural and elegant explanation of the smallness of the masses of the usual neutrinos. Heavy (or relatively heavy) Majorana neutrinos furnish very simple and attractive mechanisms for generating the observed baryon asymmetry of the Universe. Majorana particles are abundant in SUSY models. Majorana fermions can play a role of the dark matter particles and thus provide a solution of one of the most important problems

[^19]of modern cosmology. Majorana neutrinos may hold a clue to the understanding of electric charge quantization observed in Nature.

If Majorana particles exist, they should have special properties with respect to C-, CPand CPT-transformations and possess very peculiar electromagnetic properties. By studying them we may be able to learn a great deal about the fundamental properties of particles and their interactions.

Particle-like excitations of Majorana nature have been found in condensed-matter systems (see chapter 14 of this book). However, very active direct and indirect searches for Majorana neutrinos and other fundamental Majorana particles in many laboratories in the world have up to now brought no fruit. This should not discourage us too much - just remember that it took us over 40 years to discover neutrino oscillations after their possibility had first been proposed! After all, the idea of Majorana fermions is so elegant and attractive that Nature just could not have missed the opportunity to create them.

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[^0]:    *Also at the National Research Centre Kurchatov Institute, Moscow, Russia
    ${ }^{\dagger}$ email: akhmedov@mpi-hd.mpg.de

[^1]:    ${ }^{1}$ On could have also argued that neutron and antineutron are distinguished by their baryon number $(+1$ and -1 , respectively), but conservation of baryon number is not an exact symmetry of Nature.

[^2]:    ${ }^{2}$ We use the natural units $\hbar=c=1$ and assume summation over repeated indices in this chapter.

[^3]:    ${ }^{3}$ Note that the kinetic term of the Lagrangian (17) is decomposed as $\bar{\psi} i \gamma^{\mu} \partial_{\mu} \psi=\bar{\psi}_{L} i \gamma^{\mu} \partial_{\mu} \psi_{L}+$ $\bar{\psi}_{R} i \gamma^{\mu} \partial_{\mu} \psi_{R}$. In other words, for each chiral component the kinetic term can be written separately and therefore it does not require the existence of both components.

[^4]:    ${ }^{4}$ Expansions (26) and (27) are sometimes defined with a phase factor $\lambda$ in front of the creation operators. This factor, however, enters physical observables only together with other phase factors, discussed in sec. 15.2 , i.e. it is not separately observable. We therefore choose $\lambda=1$ throughout this chapter.

[^5]:    ${ }^{5}$ Indeed, $\bar{\psi} i \gamma^{\mu} \partial_{\mu} \psi=\overline{\psi_{L}} i \gamma^{\mu} \partial_{\mu} \psi_{L}+\overline{\left(\psi_{L}\right)^{c}} i \gamma^{\mu} \partial_{\mu}\left(\psi_{L}\right)^{c}$, and using eqs. (9) and (10) one can rewrite the last term as $\overline{\left(\psi_{L}\right)^{c}} i \gamma^{\mu} \partial_{\mu}\left(\psi_{L}\right)^{c}=-\partial_{\mu}\left[\bar{\psi}_{L} i \gamma^{\mu} \psi_{L}\right]+\bar{\psi}_{L} i \gamma^{\mu} \partial_{\mu} \psi_{L}$. Thus, we have $\bar{\psi}_{L} i \gamma^{\mu} \partial_{\mu} \psi_{L}=(1 / 2) \bar{\psi} i \gamma^{\mu} \partial_{\mu} \psi+$ total derivative term.
    ${ }^{6}$ Note, however, that formally one can also write the Majorana mass term at the classical level if one assumes that $\psi(x)$ is an anticommuting classical field, i.e. a field that takes as values Grassmann numbers.

[^6]:    ${ }^{7}$ Here we mostly follow ref. [13], though some of our phase conventions are different.

[^7]:    ${ }^{8}$ This can be most easily seen if we rewrite the kinetic term of the Lagrangian $\mathcal{L}_{k}=\bar{\psi} i \gamma^{\mu} \partial_{\mu} \psi$ as $\mathcal{L}_{k}=(1 / 2)\left[\bar{\psi} i \gamma^{\mu} \partial_{\mu} \psi-\partial_{\mu} \bar{\psi} \cdot i \gamma^{\mu} \psi\right]+$ total derivative term and apply the $\hat{C}$-conjugation (which for massive fermion fields is equivalent to charge conjugation C) to the full Lagrangian of free Majorana particles.
    ${ }^{9}$ An example of such a Majorana fermion is the photino - the supersymmetric partner of the photon $[8,14,15,73]$.

[^8]:    ${ }^{10}$ Recall that even though sterile neutrinos don't have gauge interactions in the standard model, they may possess other interactions, such as the Yukawa ones, or extra gauge interactions in the extensions of the standard model (e.g., $S U(2)_{R}$ gauge interactions in left-right symmetric models).

[^9]:    ${ }^{11}$ There are alternative (but equivalent) forms of $\Lambda^{\mu}(q)$. We prefer the one in (92) because each term in it separately conserves the electromagnetic current. The same applies to eq. (98) below.

[^10]:    ${ }^{12}$ In general, the symmetry or antisymmetry relations $F_{a}\left(q^{2}\right)_{i k}= \pm F_{a}\left(q^{2}\right)_{k i}$ should include an extra phase factor $\eta_{k i}$ related to the CPT-parities of the Majorana neutrinos $\nu_{i}$ and $\nu_{k}$. These CPT-parities, however, are not physically observable (unlike the CP-parities in the case when CP is conserved), and so one can set $\eta_{k i}=1$ without loss of generality.

[^11]:    ${ }^{13}$ There will also be extra diagrams in the Majorana case if the sector of the model responsible for the Majorana mass generation contains charged particles. However, in the limit of vanishing neutrino mass which we consider now such diagrams can be we neglected.

[^12]:    ${ }^{14}$ It should also be noted that there are some difficulties in defining the neutrino charge radius in a gauge-invariant and process-independent way, see discussion in sec. 3.3 of [32].

[^13]:    ${ }^{15}$ Magnetic and electric dipole interactions of relativistic neutrinos with longitudinal (i.e. collinear with the neutrino momentum) magnetic fields are strongly suppressed [36, 43].

[^14]:    ${ }^{16}$ Here we are assuming that at the resonance of spin-flavour conversion the supernova transverse magnetic field $B_{\perp r}$ can be as large as $\sim 10^{9} \mathrm{G}$ [49]. The $\nu_{e} \rightarrow \bar{\nu}_{e}$ conversion efficiency depends on the product $\mu B_{\perp r}$.
    ${ }^{17}$ We assume here that these Majorana particles are non-degenerate in mass. Otherwise two Majorana fermions can merge into a Dirac one, as discussed in sec. 15.3.3.

[^15]:    ${ }^{18}$ The role of a dark matter particle can also be played by a non-SUSY sterile Majorana neutrino, see [52-54] and references therein.

[^16]:    ${ }^{19}$ Similar argument applies to the contributions of heavy Majorana neutrino exchanges to $0 \nu \beta \beta$ decay rates, which depend on the quantity $\left\langle m_{N}^{-1}\right\rangle$ defined in eq. (111), and the possibility to probe the masses of heavy neutrinos.
    ${ }^{20}$ There is one positive claim of observation of $0 \nu \beta \beta$ decay of ${ }^{76}$ Ge by part of the Heidelberg-Moscow Collaboration [68]. However, this result has been subject to criticism (see e.g. [58] and references therein) and is now strongly disfavoured (at $99 \%$ C.L.) by non-observation of $0 \nu \beta \beta$ decay by GERDA [70].

[^17]:    ${ }^{21}$ Following the tradition, for the right-handed components of the singlet neutrinos $N_{i}$ we use here the notation $N_{i R}$ rather than $\nu_{i R}$ that was used in secs. 15.3 and 15.4.

[^18]:    ${ }^{22}$ Under certain circumstances decays of heavier singlet neutrinos can also be important, see sec. 10.2 of [83] and references therein.
    ${ }^{23}$ If the matrix $h$ were real, for hierarchical masses of $N_{i}$ the left-hand side of (117) would have been approximately equal to the trace of the mass matrix of light neutrinos, i.e. to the sum of the light neutrino mass eigenvalues.

[^19]:    ${ }^{24}$ Note that massless spin- $1 / 2$ fermions admit more freedom in the definition of the the particle-antiparticle conjugation operation: the matrix $\mathcal{C}$ that enters eq. (9) may be defined either through the usual relation $\mathcal{C}^{-1} \gamma^{\mu} \mathcal{C}=-\gamma^{\mu T}$ or through $\mathcal{C}^{-1} \gamma^{\mu} \mathcal{C}=+\gamma^{\mu T}$.

