Manifestly conformal descriptions and higher symmetries of bosonic singletons

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Based on:

Singletons:

Conformal scalar and spinor \textit{(Dirac, 1963)}

More generally, conformal fields originating from unitary Poincaré irreps \textit{(Siegel, 1989)}

Spin-$s$ singletons:
these are Lorentz fields in $d$ -dimensions ($d$ even) whose curvatures are described by

\[
\begin{array}{cccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & s \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
\frac{d}{2} & \cdot & \cdot & \cdot & \cdot & \\
\end{array}
\]

+ (anti)self-duality in each column.
Scalar singleton underlies the structure of higher spin interactions in $d + 1$-dim AdS space. In particular:

- Flato–Fronsdal theorem: tensor product of two singletons equals a direct sum of AdS HS fields \((Flato-Fronsdal, 1978)\)

- Underling the AdS HS field interactions is a higher spin (HS) algebra (certain enveloping of the conformal algebra) \((Vasileiv, 1990, 2003)\)

- In terms of conformal fields, HS algebra is an algebra of higher symmetries of the equations of motion for the conformal scalar. This was identified in \(Eastwood, 2002\) as enveloping of \(o(d, 2)\)

- In $d = 4$ algebras of symmetries were identified in some cases in \(Fushchich, Nikitin 1987\) and for all massless fields in $d = 4$ in \(Pohjanpelto, Anco 2003, 2008\) and \(Vasiliev 2002\)

- From the 1st-quantized point of view: singleton is a quantized massless scalar particle. The respective UIR is its physical subspace while HS algebra is an algebra of its quantum observables.
It is natural to expect that extending this picture to general singletons would lead to understanding interactions of mixed-symmetry fields on $AdS$.

Two aims:

1. To construct explicitly-local and explicitly $o(d, 2)$-invariant formulation of bosonic singletons.

2. To describe higher symmetries of singleton equations of motion using the constructed formulation.
Plan

• To explicitly illustrate the construction in the known case of a scalar singleton

• To show how the construction generalizes to the case $s > 0$

• Rederive the Eastwood classification of higher symmetries of the scalar singleton.

• To extend the classification to $s > 0$

• To describe the structure of invariants
Ambient space description of the conformal geometry

Conformal space can be seen as a quotient of the hypercone \((\text{Dirac},1936)\)

\[ Y \cdot Y = 0, \quad Y \sim \lambda Y \quad \lambda \neq 0 \]

in \(\mathbb{R}^{d,2}\). Scalar product is defined using the ambient space metric

\[ \eta_{AB} = diag - - + + \ldots + \]

e.g. \(Y^2 = Y \cdot Y = Y^A Y^B \eta_{AB}\).
In terms of constrained dynamics the phase space corresponding to the conformal space (cotangent bundle over the conformal space) is determined by the following first class constraints

\[ Y \cdot Y = 0, \quad Y \cdot P = 0, \]

For \( Y \) variables:
\( Y \cdot Y = 0 \) restricts to the hypercone while \( Y \cdot P = 0 \) implements the identification \( X \sim \lambda X \) as a gauge symmetry.

For the momenta \( P \):
\( Y \cdot P = 0 \) and the gauge symmetry generated by \( Y \cdot Y = 0 \) restrict the momentum space accordingly.
Scalar particle on the projective hypercone

In order to describe scalar particle one adds the mass-shell constraint $P^2 = 0$. The entire set of constraints:

$$Y \cdot Y = 0, \quad \frac{1}{2} (Y \cdot P + P \cdot Y) = 0, \quad P \cdot P = 0$$

These form $sp(2)$ algebra w.r.t. the Poisson bracket. This constrained system was considered by many authors: *Marnelius, Bars, Vasileiv, …*

After quantization: space of states – functions in $Y$. Constraints:

$$Y^2, \quad Y \cdot \frac{\partial}{\partial Y} + \frac{d + 2}{2}, \quad \frac{\partial}{\partial Y} \cdot \frac{\partial}{\partial Y}$$

Representation of $sp(2)$ on functions in $Y$.

In the appropriate functional space these constraints single out usual conformal scalar in $d$-dimensions *Marnelius, 1979*.

**Apparent disadvantage:** locality is not explicit because $d$-dimensional field theory is described in terms of $d + 2$ dimensional fields.
Introducing notation $Z^A_a = (Y^A, P^A)$ generators of $sp(2)$ and $o(d-1,1)$ can be written as follows:

$$t_{ab} = Z^A_a Z^B_b \eta_{AB} \quad J^{AB} = \epsilon^{ab} Z^A_a Z^B_b$$

These algebras form a dual pair in the sense of *Howe duality*.

More precisely, coordinates $Z^A_a$ are naturally considered as those on the tensor product of 2-dim. symplectic space and d+2-dim. pseudo-Euclidean one.
Maintaining locality: ambient space as a fiber

The way out is to consider fields that are $o(d - 1, 1)$ tensors defined on the true conformal space. The well-known examples (Stelle–West, 1979) for AdS gravity and Vasileiv, 2001 for AdS HS fields takes into account AdS geometry through AdS connection and the compensator field i.e., use Cartan description. Analogous approach to conformal fields is also known (Preitschopf, Vasiliev 1998).

What we need here is a more refined description that is analogous to the Vasiliev unfolded description. From the first quantized point of view it can be seen as an analog of Fedosov quantization.

To maintain locality: putting ambient space as the fiber over the true space-time

The construction is best illustrated in the first quantized terms and using BRST formalism.

1st step:
Scalar particle BRST operator:

$$\Omega_0 = c_+ P^2 + \frac{1}{2} c_0 (Y \cdot P + P \cdot Y) + c_- Y^2 + \text{ghosts}$$

2nd step:
Introduce new pure gauge variables $\bar{P}_A, X^B$ along with new constraints $P_A - \bar{P}_A = 0$ and new ghost variables $\Theta^A$. The equivalent BRST operator

$$\Omega = \Theta^A (P_A - \bar{P}_A) + \bar{\Omega},$$

$$\bar{\Omega} = c_+ P^2 + \frac{1}{2} c_0 ((X+Y) \cdot P + P \cdot (X+Y)) + c_- (X+Y)^2 + \text{ghosts}$$
3rd step: Identifying $dX^A \equiv \Theta^A$ it is easy to see that one can use general coordinates $X^A$ and general local frame:

$$\Omega = -\Theta^A \bar{P}_A + \Theta^A E_A^B P_B + \omega^C_{AB} Y^B P_C + \bar{\Omega}$$

where $E$ and $W$ are compatible vielbein and flat connection satisfying

$$\nabla^2 = 0, \quad \nabla X(X) = E$$

4th step: One can eliminate extra coordinates $X$ and respective $\theta$ because they are pure gauge. Using coordinate representation for the remaining “intrinsic” coordinates $x^\mu, \theta^\mu$ the equivalent BRST operators takes the form:

$$\Omega = \theta^\mu \frac{\partial}{\partial x^\mu} - \theta^\mu e_A^\mu \frac{\partial}{\partial Y_A} - \omega^B_{\mu A}(x) Y^A \frac{\partial}{\partial Y_B} + \bar{\Omega}$$

The cartesian coordinate $X^A$ expressed through the intrinsic coordinates on the conformal space are components conformal compensator field $V$, i.e. $V^A(x) = X^A(x)$.

Note that $V \cdot V = 0$ and $V$ is naturally defined up to rescaling

$$V \to \lambda V \quad \lambda \neq 0$$
In the frame where $V^A = const$ the BRST operator takes the following form

$$\Omega = \nabla + \bar{\Omega}, \quad \nabla = d + \omega_A^B (Y^A + V^A) \frac{\partial}{\partial Y^B},$$

$$\bar{\Omega} = c_+ P^2 + \frac{1}{2} c_0 ((Y + V) \cdot P + P \cdot (Y + V)) + c_- (Y + V)^2 + \text{ghosts}$$

$\nabla - o(d, 2)$- flat covariant derivative, $\Omega - sp(2)$ BRST operator.

Equations of motion, gauge symmetries, etc.:

$$\Omega \Psi^{(0)} = 0, \quad \delta \Psi^{(0)} = \Omega \Psi^{(-1)}, \ldots$$

Familiar form of the generating BRST formulation of various fields:

Poincaré invariant fields: Barnich, M.G., Semikhatov, Tipunin, 2004,
Alkalaev, M.G., Tipunin, 2008


More generally, can be constructed taking as a fiber the space equipped with the action of two Howe dual algebra and an appropriate space-time manifold.
Reduction to Lorentz fields

Adapted frame:

\[ V^+ = 1, \quad V^- = 0, \quad V^a = 0, \]
\[ \eta_{+-} = 1, \quad \eta_{a+} = \eta_{a-} = 0, \quad \eta_{ab} = diag(- + + \ldots +), \]

The system can be reduced to the Lorentz description: Recall:

\[
\Omega = \nabla + \bar{\Omega} = \nabla + c_+ \frac{\partial}{\partial Y} \cdot \frac{\partial}{\partial Y} + c_0 \left( \frac{\partial}{\partial Y^+} + Y \cdot \frac{\partial}{\partial Y} - \frac{d - 2}{2} \right) + (Y \cdot Y + 2Y^-) \frac{\partial}{\partial b_-} + \text{ghosts}
\]

Two steps of the reduction:

1st step: eliminate \( Y^- \) and \( b_- \) by reducing to cohomology of \( Y^- \frac{\partial}{\partial b_-} \)

2nd step: eliminate \( Y^+ \) and \( c_0 \) by reducing to cohomology of \( c_0 \frac{\partial}{\partial Y^+} \).
Finally:

\[ \Omega^{\text{reduced}} = \nabla + c + \frac{\partial}{\partial Y^a} \frac{\partial}{\partial Y_a} \]

In the Cartesian coordinates where \( e^a_{\mu} = \delta^a_{\mu} \) and \( \omega = 0 \) this is equivalent to

\[ \Omega^{\text{standard}} = c + \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^a} \]

leading to a usual massless Klein-Gordon equations of motion.

Explicitly \( o(d, 2) \)-invariant and explicitly-local description of the conformal scalar
spin-s singletons

Representation space:

\[(\text{functions in } Y) \rightarrow (\text{functions in } Y) \otimes (\text{Grassmann algebra in } \vartheta^A_i)\]

\(\vartheta^A_i\)–usual fermionic oscillators.

Constraint algebra:

\[\text{sp}(2) – \text{algebra} \rightarrow \text{osp}(2s|2) – \text{superalgebra}\]

Besides \(\text{sp}(2)\) constraints \(Y^2, Y \cdot \frac{\partial}{\partial Y} + \frac{d+2}{2}, \frac{\partial}{\partial Y} \cdot \frac{\partial}{\partial Y}\) one has “algebraic constraints”

\[\vartheta_i \cdot \frac{\partial}{\partial \vartheta_j} - \delta^j_i \frac{d+2}{2}, \quad \vartheta_i \cdot \vartheta_j, \quad \frac{\partial}{\partial \vartheta_i} \cdot \frac{\partial}{\partial \vartheta_j}\]

forming \(o(2s)\) algebra. Together with \(4s\) fermionic constraints

\[X \cdot \vartheta_i, \quad X \cdot \frac{\partial}{\partial \vartheta_i}, \quad \frac{\partial}{\partial X} \cdot \vartheta_i, \quad \frac{\partial}{\partial X} \cdot \frac{\partial}{\partial \vartheta_i}\]

all these form \(\text{osp}(2s|2)\) superalgebra.
In the appropriate functional space these determine spin-$s$ singleton. 
(Siegel 1988, Arvidsson, Marnelius 2006).

These constraints originate from extended supersymmetric spinning particle model Howe, Penati, Pernici,Townsend 1988

Algebraic structure: Howe dual $o(d, 2)$ and $osp(2s|2)$ algebras. Introducing $Z^A_\alpha = \{Y^A, P_A, \vartheta^A_i, \pi^i_A\}$ the generators are

$$t_{\alpha\beta} = Z^A_\alpha Z^B_\beta \eta_{AB} \quad J^{AB} = \epsilon^{\alpha\beta} Z^A_\alpha Z^B_\beta$$

Just like in the conformal particle case locality can be maintained by passing to the generating BRST description

$$\Omega = \nabla + \bar{\Omega}, \quad \bar{\Omega} = C^I t_I + \frac{1}{2} U^K_{IJ} \dot{C}^I \dot{C}^J \mathcal{P}_K,$$

where $t_I = \{t_{\alpha\beta}\} - osp(2s|2)$-generators. In contrast some auxiliary off-shell constraints to eliminate doubling of states.

Explicitly $o(d, 2)$-invariant and explicitly-local description of spin $s$-singletons
Higher symmetries

Equations of motion:

\[ T_I \Psi = 0, \quad T_I \text{ – differential operators} \]

Symmetry: operator \( K \) that maps solutions to solutions, i.e.

\[ T_I K \Psi = 0 \quad \implies \quad [T_I, K] = U_I^J T_J \]

Equivalence:

\[ K \sim K + W^J T_J \]

This is nothing but the Dirac definition of the observable for the constrained system with 1st class constraints \( T_I \)

inequivalent symmetries \( \simeq \) first-quantized observables

Using BRST formalism: \( \Omega = C^I T_I + \ldots, \quad C^I, P_J \) – ghost variables

\[ H^0([\Omega, \cdot], \text{operators}) \simeq \text{inequivalent symmetries} \]

More precisely, this is true if \( \Omega \) is proper. Otherwise there can be additional classes.
Inequivalent symmetries is naturally an associative algebra. The product is the operator product induced in the cohomology. From the 1st-quantized point of view – algebra of quantum observables.

What we need to compute is the cohomology of \( \frac{1}{\hbar} [\Omega, \cdot] \) at zeroth ghost number.

For \( \Omega \) of the form \( \Omega = \nabla + \widetilde{\Omega} \) the cohomology is isomorphic to that of \( \frac{1}{\hbar} [\widetilde{\Omega}, \cdot] \) (if De Rham cohomology do not contribute)

Work in terms of Weyl symbols. Operator multiplication → Weyl \(*\)-product. For

\[
\Phi = C^I T_I + \frac{1}{2} U^K_{IJ} C^I C^J P_K
\]

\[
D = T_I \frac{\partial}{\partial P_I} + C^I [T_I, \cdot]_* + \text{ghosts} + O(\hbar^2)
\]

Cohomology computation: first cohomology of \( \delta = T_I \frac{\partial}{\partial P_I} \) (Kozule differential). If no \( P \)-dependent classes then all cohomology can be assumed totally traceless (e.g. in scalar case \( T_I = \{ P^2, Y \cdot P, Y^2 \} \), i.e. all possible traces) one ends up with \( [T_I, \phi]_* = 0 \) i.e. invariants of \( osp(2s|2) \) in this representation.
In this case: no $\mathcal{P}$-dependent cohomology classes of $\delta = T_I \frac{\partial}{\partial P_I}$ (Barnich, M.G., Semikhatov, Tipunin, 2004).

Higher symmetries are then given by totally traceless $sp(2)$-invariant polynomials in $Y, P$. These are generated by $Y^A P^B - Y^B P^A$, i.e. form an enveloping algebra of $o(d, 2)$ in this representation (Eastwood, 2002)

$s > 0$

In this case $\delta = T_I \frac{\partial}{\partial P_I}$ in general can have $\mathcal{P}$-dependent classes. However, these originate from relations between constraints and are to be eliminated through additional terms in the BRST operator

Koszule differential $\rightarrow$ Koszule–Tate differential

Hence these classes can not contribute to nontrivial symmetries. One ends up with invariants of $osp(2s|2)$ in

traceless polynomials in $Y^A, P_A$ and $\vartheta^A_i, \pi^i_A$
Structure of invariants

$osp(2s|2)$-invariants in representations of these type are known in the math literature *(Sergeev, 1992, 1998)*. More precisely,

$$\mathcal{U} = \mathcal{U}_0 \oplus \mathcal{U}_1$$

$\rho(g)\mathcal{O} = \mathcal{O}$ for $\mathcal{O} \in \mathcal{U}_0$, $\rho(g)\mathcal{O} = \det(g)\mathcal{O}$ for $\mathcal{O} \in \mathcal{U}_1$, for all $g \in O(2s)$ (for a tensor representation the action of $o(2s) \subset osp(2s|2)$ can be integrated to the group).

- $\mathcal{U}_0$ – enveloping of $o(d, 2)$
- $\mathcal{U}_1$ – “chiral” symmetries

Recall, that irreducible singletons are also self-dual $*_{k} \phi = \phi$

$k$-Hodge conjugation in $k$-th column defined such that $*_{k} *_{k} = 1$.

It turns out that chiral symmetries are not compatible, e.g.

$$\sigma_{k}(\mathcal{O}) = *_{k} \mathcal{O} *_{k}$$

Finally, for an irreducible singleton the algebra of higher symmetries is an enveloping of $o(d, 2)$. 
Conclusions

- Provides a higher analog of the usual higher spin algebra and gives a 1st-quantized interpretation to HS algebras.
- First step towards generalization of the Vasileiv nonlinear equations to mixed-symmetry AdS fields
- Provides a framework to study a spin-$s$ version of the Flato-Fronsdal theorem
- Together with agamous description for AdS (gauge) fields can be used as an explicitly covariant and gauge invariant framework to analyze bulk/boundary correspondence ((partially)gauge fixed versions (Metsaev, 1999, . . . ))