BAKGROUND FIELD METHOD FOR $d = 4$, $\mathcal{N} = 3$ SUPERGAUGE THERIES

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The VIII International Workshop "Supersymmtries and Quantum Symmetries", Dubna, July 29 – August 3,2009

- Formulation of gauge invariant effective action for $d=3$, $\mathcal{N}=3$ supersymmetric gauge theories, formulated in $d=3$, $\mathcal{N}=3$ harmonic superspace.
- Construction of manifestly $\mathcal{N}=3$ supersymmetric effective action in terms of off-shell $d=3$, $\mathcal{N}=3$ superfields.
- $d=3$, $\mathcal{N}=3$ harmonic supergraph calculations.

Talk is based on the results obtained in collaboration with
E. Ivanov, O. Lechtenfeld, N. Pletnev, I. Samsonov, B. Zupnik

- Studing a quantum structure of recently proposed Bagger-Lambert-Gustavsson (BLG) theory (J. Bagger, N. Lambert, Phys. Rev. D75 (2007) 045020; D77 (2008) 065008; JHEP 0802 (2008) 105; A. Gustavsson, JHEP 0804 (2008) 083) and Aharony-Bergman-Jefferis-Maldacena (ABJM) theory (O. Aharony, O. Bergman, D.L. Jafferis, J. Maldacena, JHEP 0810 (2008) 091). Both theories related to low-energy description of M2-branes and contain the Chern-Simons fields and some number of scalar and spinor fields. These theories possess $\mathcal{N}=8$ supersymmetry $\mathcal{N}=6$ supersymmetry resepectively. One can show that BLG and ABJM theories can be formulated in terms of $d=3$, $\mathcal{N}=3$ harmonic superspace (I.L.B, E.A. Ivanov, O. Lechtenfeld, N.G. Pletnev, I.B. Samsonov, B.M. Zupnik, JHEP 03 (2009) 096). This formulation provides 3 manifest (off-shell) supersymmetries and 5 (BLG) or 3 (ABJM) hidden (on-shell) supersymmetries of classical action.
- Developing a general procedure for calculating the effective action for such theories preserving manifest gauge invariance and three manifest supersymmetries of classical action.

Contents

- Gauge and matter theories in $d = 3$, $\mathcal{N} = 3$ harmonic superspace.
- $d = 3$, $\mathcal{N} = 3$ background field method
- Structure of superfield propagators in background field
- $d = 3$, $\mathcal{N} = 3$ non-renormaization theorem
- Summary and prospects

Standard $\mathcal{N}=3$ superspace:

$$\{x^\mu, \theta^{(ij)}_\alpha\}, \qquad i,j=1,2 \text{ (indices of } SU(2)).$$

Flat superspace derivatives:

$$D^{kj}_\alpha = \frac{\partial}{\partial\theta^\alpha_{kj}} + i\theta^{kj\,\beta}\partial_{\alpha\beta}, \qquad \partial_{\alpha\beta} = \frac{\partial}{\partial x^{\alpha\beta}}$$

Gauge covariant derivatives:

$$\nabla^{ij}_\alpha = D^{ij}_\alpha + V^{ij}_\alpha\,, \qquad \nabla_{\alpha\beta} = \partial_{\alpha\beta} + V_{\alpha\beta}\,.$$

Basic superfield strength W^{ij}:

$$\{\nabla^{ij}_\alpha, \nabla^{kl}_\beta\} = i\nabla_{\alpha\beta}(\varepsilon^{ik}\varepsilon^{jl} + \varepsilon^{il}\varepsilon^{jk}) - \frac{1}{2}\varepsilon_{\alpha\beta}(\varepsilon^{ik}W^{jl} + \varepsilon^{il}W^{jk} + \varepsilon^{jl}W^{ik} + \varepsilon^{jk}W^{il})$$

Basic constraint:

$$\nabla^{(ij}_\alpha W^{kl)} = 0$$

Harmonic $\mathcal{N}=3$ superspace:

Coordinates: $\{x^\mu, \theta^{++}_\alpha, \theta^{--}_\alpha, \theta^0_\alpha, u^\pm_i\}$,

Harmonics: $u^\pm_i \in SU(2)$, $\quad u^{+i}u^+_i = 0,\ u^{-i}u^-_i = 0,\ u^{+i}u^-_i = 1.$

Here $\theta^{++}_\alpha, \theta^{--}_\alpha, \theta^0_\alpha$ are harmonic projections of θ^{ij}_α:

$$\theta^{ij}_\alpha \quad \longrightarrow \quad \theta^{++}_\alpha = \theta^{ij}_\alpha u^+_i u^+_j, \quad \theta^{--}_\alpha = \theta^{ij}_\alpha u^-_i u^-_j, \quad \theta^0_\alpha = \theta^{ij}_\alpha u^+_i u^-_j$$

Harmonic derivarives: $\mathcal{D}^{++}, \quad \mathcal{D}^{--}, \quad \mathcal{D}^0 = [\mathcal{D}^{++}, \mathcal{D}^{--}].$

$$\mathcal{D}^{++} = u^+_i \frac{\partial}{\partial u^-_i} + \ldots, \qquad \mathcal{D}^{--} = u^-_i \frac{\partial}{\partial u^+_i} + \ldots,$$

Grassmann derivatives: $D^{++}_\alpha, \quad D^{--}_\alpha, \quad D^0_\alpha.$

Analytic superfields:

$$D^{++}_{\alpha}\Phi_A = 0 \quad \Longrightarrow \quad \Phi_A = \Phi_A(x^{\mu}_A, \theta^{++}_{\alpha}, \theta^{0}_{\alpha}, u),$$

where $x^{\alpha\beta}_A = (\gamma_m)^{\alpha\beta} x^m_A = x^{\alpha\beta} + i(\theta^{\alpha++}\theta^{\beta--} + \theta^{\beta++}\theta^{\alpha--})$.

- q-hypermultiplet: $q^+(x^{\mu}_A, \theta^{++}_{\alpha}, \theta^{0}_{\alpha}, u^{\pm}_i)$

$$q^+ : \quad \{f^i, \bar{f}_i, \psi^i_{\alpha}, \bar{\psi}_{i\alpha}\}, \quad i = 1, 2.$$

- Vector superfield: $V^{++}(x^{\mu}_A, \theta^{++}_{\alpha}, \theta^{0}_{\alpha}, u^{\pm}_i)$

$$V^{++} : \quad \{A_{\mu}, \phi^{(ij)}, \lambda_{\alpha}, \lambda^{(ij)}_{\alpha}\}.$$

$d=3$, $\mathcal{N}=3$ Superfiled actions

- $\mathcal{N}=3$ Chern-Simons action:

$$S_{CS} = \frac{ik}{4\pi}\,\mathrm{tr}\sum_{n=2}^{\infty}\frac{(-1)^n}{n}\int d^3x d^6\theta du_1\ldots du_n \frac{V^{++}(z,u_1)\ldots V^{++}(z,u_n)}{(u_1^+u_2^+)\ldots(u_n^+u_1^+)},$$

- Hypermultiplet action:

$$S_H = \int d\zeta^{(-4)}\bar{q}^+(\mathcal{D}^{++}+V^{++})q^+.$$

- Gauge transformations:

$$\delta_\Lambda V^{++} = -\mathcal{D}^{++}\Lambda - [V^{++},\Lambda].$$

$$\delta_\Lambda q^+ = \Lambda q^+.$$

- Yang-Mills action:

$$S_{SYM} = -\frac{1}{g^2}\,\mathrm{tr}\int d\zeta^{(-4)}\,(W^{++})^2\,, \quad [g]=1/2.$$

- Superfield strength:

$$W^{++} = -\frac{1}{4}D^{++\alpha}D^{++}_{\alpha}V^{--}, \qquad \mathcal{D}^{++}W^{++} + [V^{++}, W^{++}] = 0.$$

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$$V^{--}(z,u) = \sum_{n=1}^{\infty}(-1)^n \int du_1 \dots du_n \frac{V^{++}(z,u_1)V^{++}(z,u_2)\dots V^{++}(z,u_n)}{(u^+u_1^+)(u_1^+u_2^+)\dots(u_n^+u^+)}.$$

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$$\delta_\Lambda W^{++} = [\Lambda, W^{++}]$$

- The background field method is a tool for studying the general structure of effective actions in gauge theories with preservation of classical gauge invariance in quantum theory on all steps of calculations (Yang-Mills theory, quantum gravity, B.S. DeWitt, 1965, 1967).
- Basic idea: splitting the initial fields into classical and quantum fields and fixing the gauge symmetry only for quantum fields.
- Aims: Formulation of the background field method for the $\mathcal{N} = 3$, $d = 3$ Chern-Simons-matter theory with the action

$$S = S_{CS}[V^{++}] + S_q[\bar{q}^+, q^+, V^{++}]$$

- Path integral representation for the effective action, propagators, vertices.

Gauge theory is given by:

- Set of fields Φ^i
- Action $S_0[\Phi]$
- Gauge transformations $\delta\Phi^i = R^i_\alpha[\Phi]\xi^\alpha$

Quantum theory is described by action:

$$S[\Phi, \bar{c}, c] = S_0[\Phi] + S_{GF}[\Phi] + S_{GH}[\Phi, \bar{c}, c]$$

Gauge fixing action S_{GF}: $F^\alpha F^\alpha$
Ghost action S_{GH}: $\bar{c}_\alpha M^\alpha_\beta[\Phi]c^\beta$
Background-quantum splitting: $\Phi \longrightarrow \Phi + \phi$
Gauge fixing function F^α depends both on background and quantum fields, it fixes only quantum field gauge transformation so that the corresponding S_{GF} is invariant under background field gauge transformations.
Effective action constructed in terms of such a gauge fixing function will be gauge invariant under background field gauge transformations.

Background–quantum splitting

$$V^{++} \longrightarrow V^{++} + \kappa v^{++}\,,$$

Realization of initial gauge transformations:

(i) Background transformations

$$\delta V^{++} = -\mathcal{D}^{++}\lambda - [V^{++},\lambda] = -\nabla^{++}\lambda\,, \qquad \delta v^{++} = [\lambda, v^{++}]$$

(ii) Quantum transformations

$$\delta V^{++} = 0\,, \qquad \delta v^{++} = -\frac{1}{\kappa}\nabla^{++}\lambda - [v^{++},\lambda]$$

Covariant harmonic derivative ∇^{++} is constructed on the base of background field V^{++}.

Structure of Chern-Simons action after background-quantum splitting:

$$S_{CS}[V^{++}+v^{++}] = S_{CS}[V^{++}] - \frac{1}{\kappa}\mathrm{tr}\int d\zeta^{(-4)}\, v^{++}W^{++}(V^{++}) + \Delta S_{CS}[V^{++}, v^{++}]$$

Here

$$\Delta S_{CS}[V^{++}, v^{++}] = \mathrm{tr}\sum_{n=2}^{\infty}\frac{(-1)^n\kappa^{n-2}}{n}\int d^9z du_1 \ldots du_n \frac{v_\tau^{++}(z,u_1)\ldots v_\tau^{++}(z,u_n)}{(u_1^+u_2^+)\ldots(u_n^+u_1^+)}$$

Background filed dependent gauge fixing function:

$$\mathcal{F}^{(4)} = \nabla^{++}v^{++}$$

The corresponding Faddev-Popov determinant:

$$\Delta_{FP}[V^{++}, v^{++}] = \mathrm{Det}\,\nabla^{++}(\nabla^{++} + \kappa v^{++})$$

$$e^{i(\Gamma_{CS}[V^{++}]-S_{CS}[V^{++}])} =$$
$$= (\mathrm{Det}^{1/2}_{(4,0)}\hat{\Delta})\int \mathcal{D}v^{++}\mathcal{D}b\mathcal{D}c\mathcal{D}\phi e^{i(S_2[v^{++},b,c,\phi,V^{++}]+S_{int}[v^{++},b,c,V^{++}])}$$

$$S_2[v^{++},b,c,\phi,V^{++}] = \frac{1}{2}\mathrm{tr}\int d\zeta^{(-4)}v^{++}\hat{\Delta}v^{++} + \mathrm{tr}\int d\zeta^{(-4)}b(\nabla^{++})^2 c$$
$$+\frac{1}{2}\mathrm{tr}\int d\zeta^{(-4)}\phi(\nabla^{++})^2\phi,$$

$$S_{int}[v^{++},b,c,V^{++}] = \mathrm{tr}\sum_{n=3}^{\infty}\frac{(-1)^n\kappa^{n-2}}{n}\int d^9z du_1\ldots du_n \frac{v^{++}_\tau(z,u_1)\ldots v^{++}_\tau(z,u_n)}{(u^+_1u^+_2)\ldots(u^+_nu^+_1)}$$
$$-\kappa\mathrm{tr}\int d\zeta^{(-4)}\nabla^{++}b[v^{++},c]$$

$$\hat{\Delta} = \frac{1}{8}(D^{++})^2(\nabla^{--})^2$$

Hypermultiplet action after background-quantum splitting of V^{++}

$$S_q = \int d\zeta^{(-4)} \bar{q}^+ (\nabla^{++} + \kappa v^{++}) q^+ \,,$$

Background-quantum splitting for hypermultiplet

$$q^+ \longrightarrow q^+ + \mathbf{q}^+ \,, \qquad \bar{q}^+ \longrightarrow \bar{q}^+ + \bar{\mathbf{q}}^+ \,.$$

Transformation of hypermultiplet action after background-quantum splitting of both V^{++} and q^+

$$S_q \longrightarrow S_q[\bar{q}^+, q^+, V^{++}] + S_{lin} + S_2 + S_{int}$$

$$S_2 = \int d\zeta^{(-4)} (\bar{\mathbf{q}}^+ \nabla^{++} \mathbf{q}^+ + \bar{\mathbf{q}}^+ \kappa v^{++} q^+ + \bar{q}^+ \kappa v^{++} \mathbf{q}^+)$$

$$S_{int} = \kappa \int d\zeta^{(-4)} \bar{\mathbf{q}}^+ v^{++} \mathbf{q}^+$$

Effective action $\Gamma_q[V^{++}, \bar{q}^+, q^+]$ corresponding to hypermultiplet:

$$e^{i(\Gamma_q[V^{++}+v^{++},\bar{q}^+,q^+]-S_q[V^{++},\bar{q}^+,q^+])} = \int \mathcal{D}\bar{\mathbf{q}}^+ \mathcal{D}\mathbf{q}^+ e^{i(S_2+S_{int})}$$

Total effective action:

$$e^{i(\Gamma[V^{++},\bar{q}^+,q^+]-S[V^{++},\bar{q}^+,q^+])} = \int \mathcal{D}(QF)\mathcal{D}(GH)e^{i(S_2[QF,GH,BF]+S_{int}[QF,GH,BF])}$$

- $S_2[QF, GH, BF]$: Propagators for quantum fields and ghosts in background fields.
- $S_{int}[QF, GH, BF]$: Vertices for quantum fields and ghosts in background fields.
- Total effective action is manifestly gauge invariant and $\mathcal{N} = 3$ supersymmetric by construction
- The loop contributions to effective action are given by harmonic supergraphs, form of which is defined by the above propagators and vertices

Equations of motion for propagators:

$$\langle v^{++}(1)v^{++}(2)\rangle = G^{(2,2)}(1|2): \qquad \hat{\Delta}G^{(2,2)}(1|2) = \delta_A^{(2,2)}(1|2)$$

$$\langle \bar{\mathbf{q}}^{+}(1)\mathbf{q}^{+}(2)\rangle = G^{(1,1)}(1|2): \qquad \nabla^{++}G^{(1,1)}(1|2) = \delta_A^{(3,1)}(1|2)$$

Analytic delta-function:

$$\delta_A^{(4-q,q)}(1|2) = -\frac{1}{4}D^{++\alpha}_{(1)}D^{++}_{(1)\alpha}\delta^9(z_1 - z_2)\delta^{(-q,q)}(u_1, u_2)$$

Solutions to the equations for propagators is given in terms of covariant superfield operators, acting on analytic superfields as follows:

- $\hat{\Delta}^2 = \Box +$ background superstrength and covariant derivative terms.
- $\hat{\Box} = \Box +$ background superstrength and covariant derivative terms.

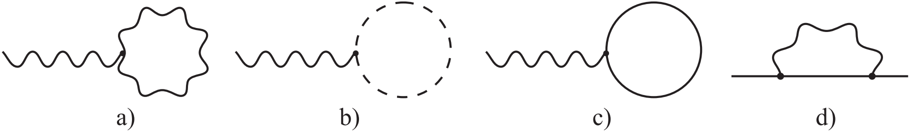
Propagators

- Any loop contribution to effective action is expressed in terms of superstrengths and their covariant derivatives.
- For vanishing background filed V^{++} the propagators are defined in terms of Grassmann delta-functions and spinor derivatives such a way that allows to obtain any loop contribution to effective action in form of integral over full $d=3$, $\mathcal{N}=3$ superspace.
- Free propagators:

$$\begin{aligned} G_0^{(2,2)}(1|2) &= \frac{1}{\Box}(D^0)^2\delta_A^{(2,2)}(1|2)\,, \\ G_0^{(1,1)}(1|2) &= -\frac{1}{16\Box}(D^0_{(1)})^2(D^{++}_{(1)})^2(D^{++}_{(2)})^2\frac{\delta^9(z_1-z_2)}{(u_1^+u_2^+)^3}\,, \end{aligned}$$

Supergraphs

Simple consequence of the $\mathcal{N} = 3$ supergraph technique:



All tadpole one-loop supergraphs as well as one-loop hypermultiplet self energy supergraph vanish only due to propagator structure.

The $\mathcal{N} = 3$, $d = 3$ non-renormalization theorem

General statement: The effective action in the $\mathcal{N} = 3$ Chern-Simons model with arbitrary number of hypermultiplets in some representation of gauge group is completely finite in the sense that there are no any UV quantum divergences in harmonic supergraphs contributing to the effective action.

Divergences in Chern-Simons type theories:

- $\mathcal{N} = 0$ (non-supersymmetric) theories: β-function for Chern-Simons coupling in an arbitrary Chern-Simons-matter theory is trivial (Kapustin, Pronin, 1993, 1994), the divergences may occur only in the sector of matter fields.
- $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric theories: in general case such theories with scale-invariant superpotentials are not free of UV divergences, but the divergence cancellation may occur for some particular superpotentials (Avdeev, Grigoryev, Kazakov, Kondrashuk, 1992, 1993; Gates, Nishino, 1992).
- Aim: Calculation of superficial degree of divergences in $\mathcal{N} = 3$ theories.

Power counting

Consider an arbitrary background filed dependent supergraph G with L loops, P propagators, N_{mat} external matter lines. To evaluate the superficial degree of divergences $\omega(G)$ it is sufficient to know only the free superfield propagators. The result is:

$$\omega(G) = 3L - 2P + (2P - N_{mat} - 3L) - \frac{1}{2}N_D = -N_{mat} - \frac{1}{2}N_{mat}$$

- $3L$ is a contribution of loop momenta
- $-2P$ comes from the factors $\Box^{-1}$ in the propagators
- $+2P$ comes from the factors $(D^{++})^2(D^0)^2$ in hypermultiplet propagators
- $-N_{mat}$ arises due to the fact that one factor $(D^{++})^2$ in each external matter line is used to restore full $\mathcal{N}=3$ measure in hypermultiplet vertices.
- $-3L$ comes since we can apply the identity

  $$\delta^6(\theta_1-\theta_2)(D_1^{++})^2(D_2^{++})^2(D_2^0)^2\delta^9(z_2-z_1) = 16(u_1^+u_2^+)^4\delta^9(z_1-z_2)$$

  in each loop to shrink the loop into dot in θ space.
- $-\frac{1}{2}N_D$ comes from N_D covariant spinor derivatives which can be transfered on the external lines with help of integration by parts

As a result:

- Any supergraph with external matter lines are automatically finite.
- The only dangerous supergraphs are ones with $N_D = 0$.
- Advantage of background field method. Due to manifest gauge invariance of effective action, it means the loop supergraphs for effective action must be expressed in terms of superfield strengths and their covariant derivatives constructed from background superfield V^{++}. Therefore some number of covariant derivatives must be taken from propagators and transfered on external V^{++} lines.
- $N_D > 0$ and hence $\omega(G) < 0$. Any loop supergraph in the theory under consideration is UV finite.

Basic results:

- The background field method for the general $\mathcal{N}=3$ Chern-Simons-matter theory, which allows, in principle, to compute the effective action preserving manifest gauge invariance and $\mathcal{N}=3$ supersymmetry on all steps of quantum calculations.
- Background field dependent propagators and vertices.Possibility to compute any supergraphs.
- Non-renormalization theorem. The theory under consideration is UV finite.
- Only IR singularities can appear in the massless hypermultiplet theory, which can be avoided by using either the massive hypermultiplets or by doing all the calculations with non-zero background field where all the propagators are effectively massive.
- The operator $\hat{\Delta}$

Open problems:

- Structure of low energy effective action in vector multiplet and hypermultiplet sectors.
- Techniques for calculating the one-loop effective action. Problem of $\mathrm{Det}\hat{\Delta}$.
- Applications to the $\mathcal{N} = 6$ and $\mathcal{N} = 8$ supersymmetric ABJM and BLG theories which describe the M2 branes in superstring theory. Study the effective actions in these models and possible relations to the dynamics of M2 branes with the quantum corrections taken into account.
- Study the composite operators for the hypermultiplet superfields in the ABJM theory. Such operators are interesting from the point of view of the AdS/CFT correspondence for the three-dimensional field models.