

*SPIN 2012, Dubna, Russia,
September 17-22, 2012*

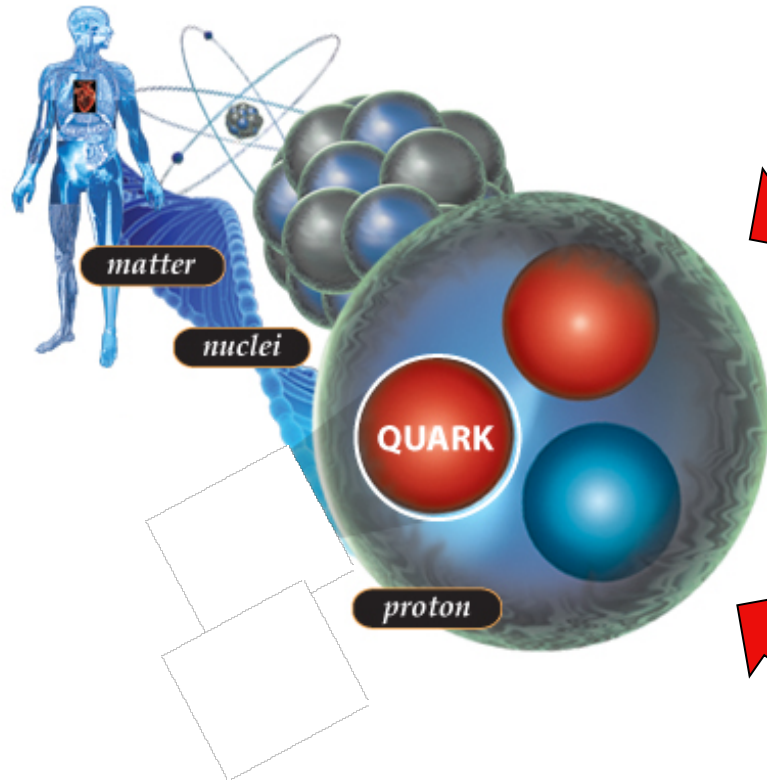


QCD and Spin Effects

Alexei Prokudin



Exploring the nucleon: a fundamental quest



Know what we
are made of !

Understand the
strong force:
"QCD"

Use protons as tool
for discovery
(e.g. LHC)

Nucleon landscape

Nucleon is a many body dynamical system of quarks and gluons

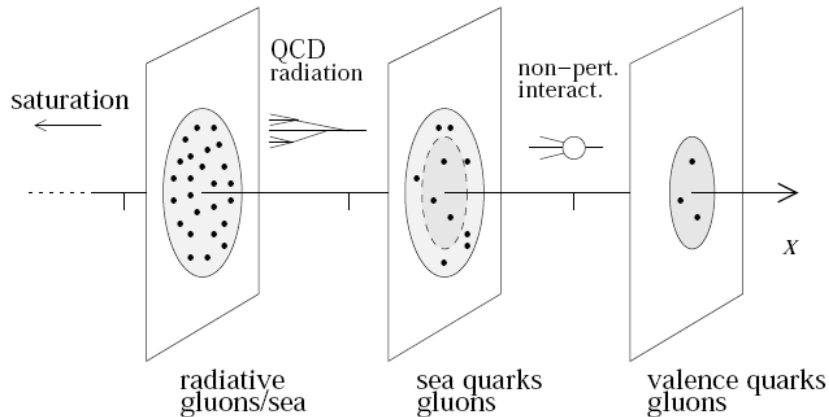
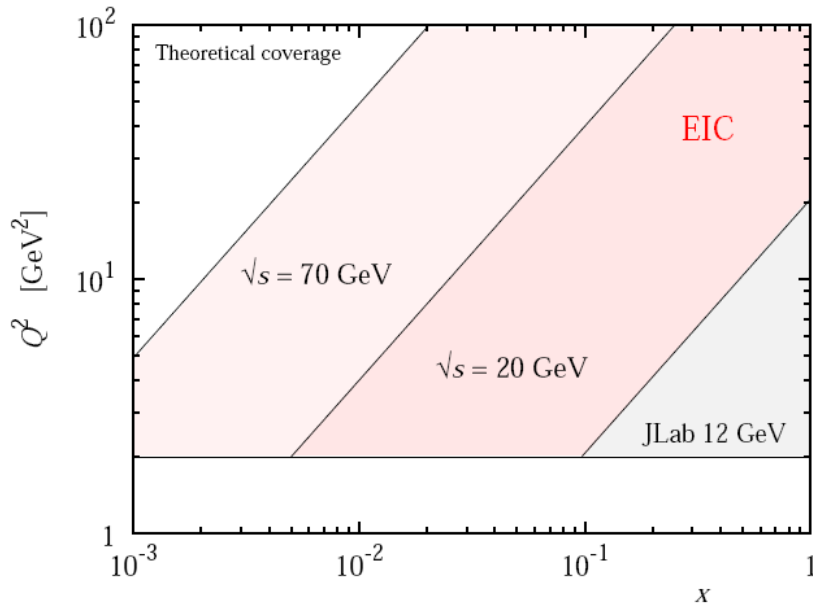
Changing x we probe different aspects of nucleon wave function

How partons move and how they are distributed in space is one of the future directions of development of nuclear physics

Technically such information is encoded into Generalised Parton Distributions and Transverse Momentum Dependent distributions

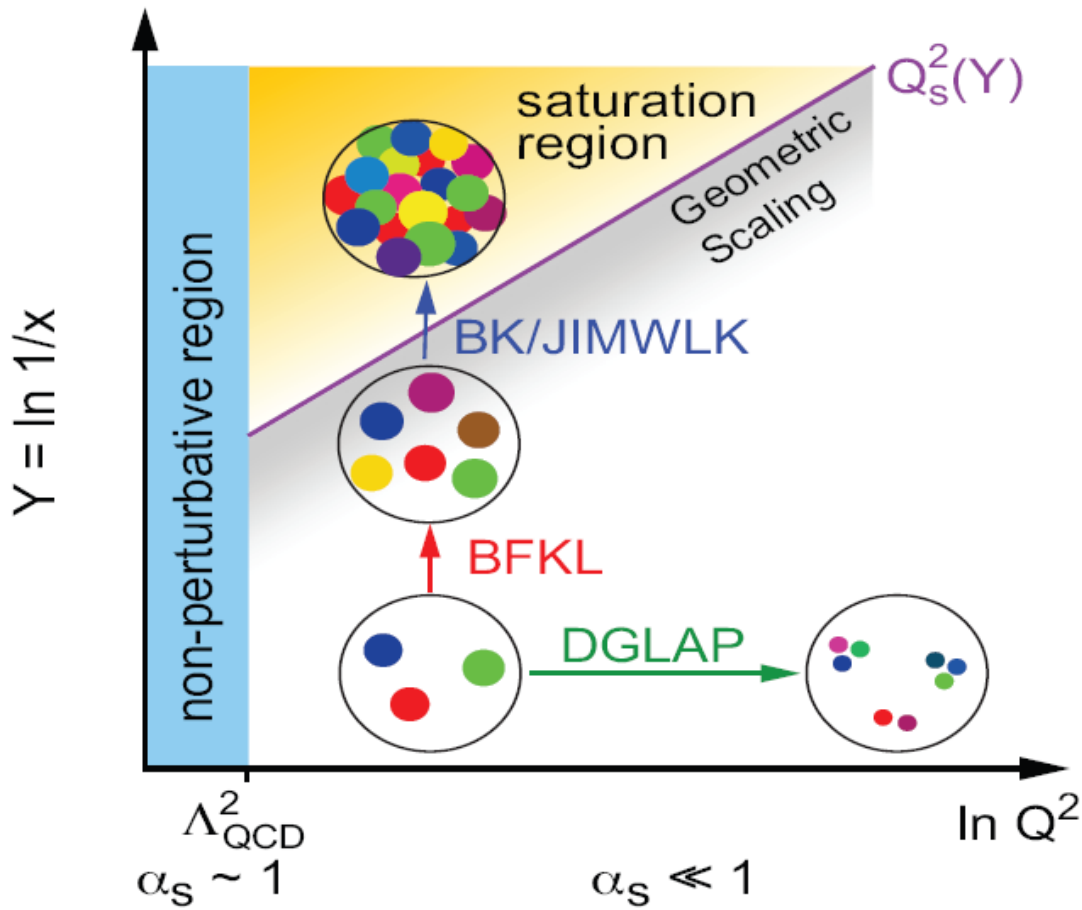
[See talk by Peter Kroll](#)

These distributions are also referred to as 3D (three-dimensional) distributions



Plot courtesy of Christian Weiss

Nucleon landscape

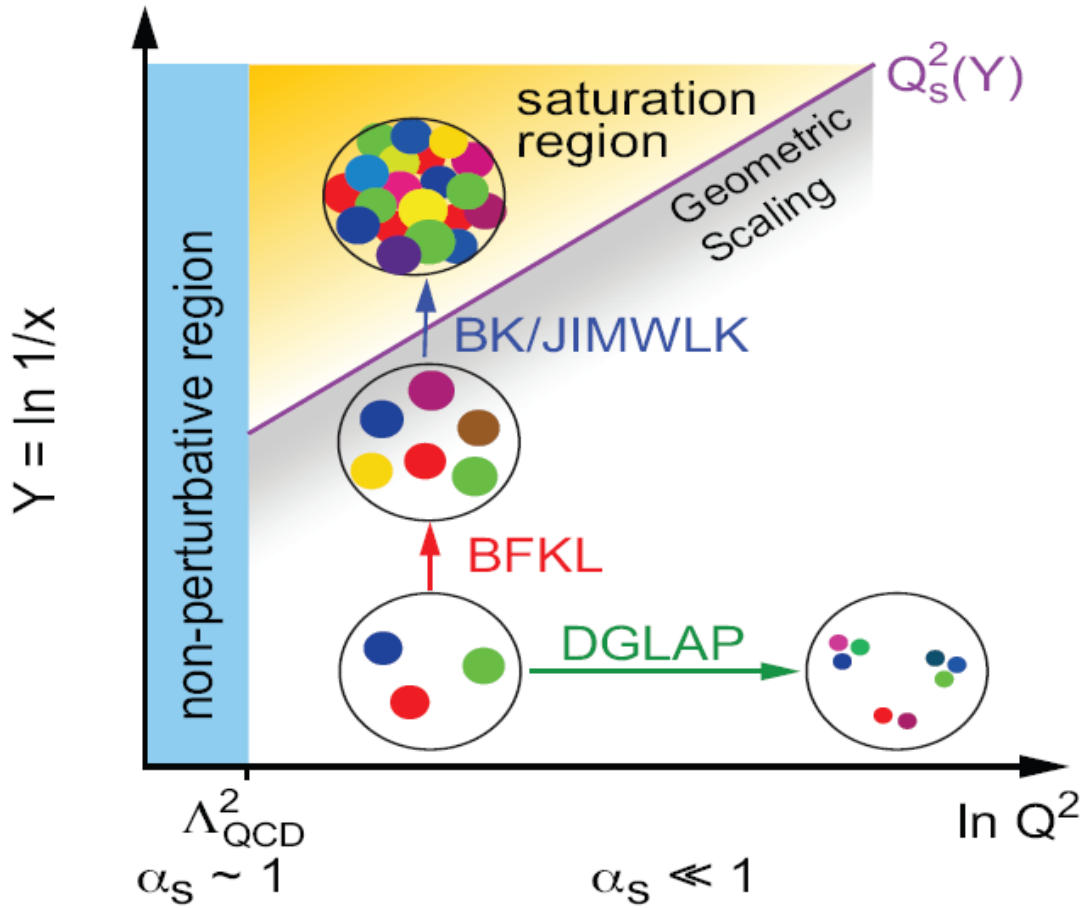


Virtual photon serves as a microscopic probe of the nucleon:

Larger Q^2 probe smaller distances - DGLAP evolution

Plot from EIC whitepaper

Nucleon landscape

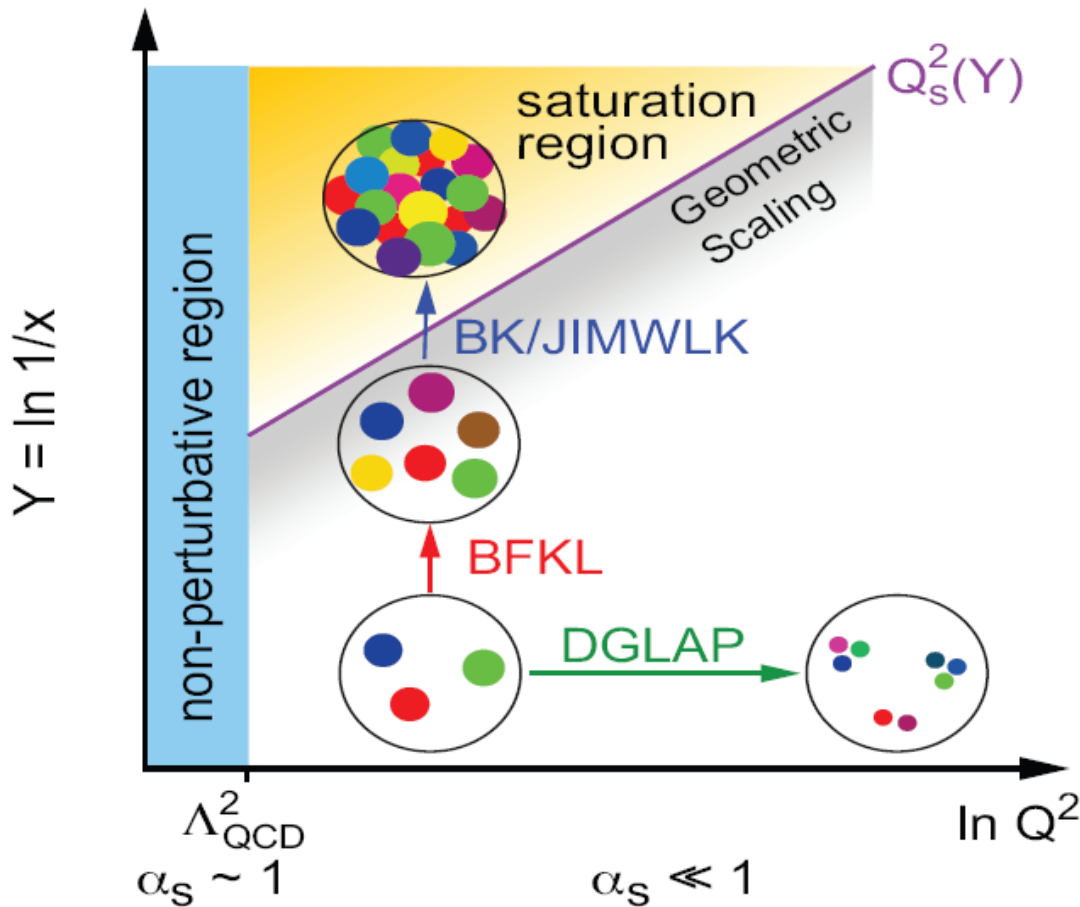


Virtual photon serves as a microscopic probe of the nucleon:

Fixing Q^2 and changing the energy we probe BFKL evolution

Plot from EIC whitepaper

Nucleon landscape



Virtual photon serves as a microscopic probe of the nucleon:

Recombination of gluons leads to non linear effects - BK/JIMWLK evolution and phenomenon of saturation. Dilute vs dense regime of QCD.

Plot from EIC whitepaper

Spin and QCD

"Experiments with spin have killed more theories than any other single physical parameter"

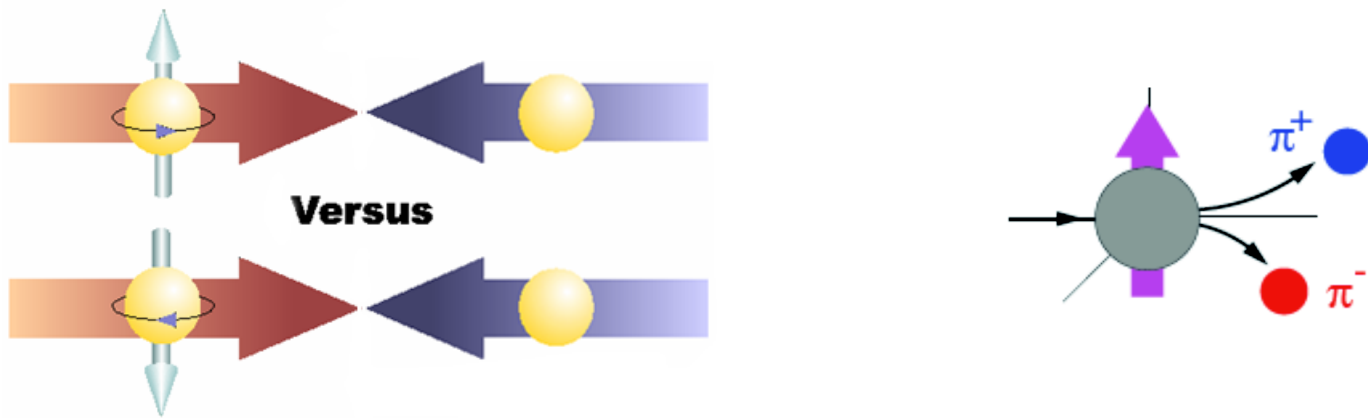
Elliot Leader, Spin in Particle Physics, Cambridge U. Press (2001)

"Polarisation data has often been the graveyard of fashionable theories. If theorists had their way they might well ban such measurements altogether out of self-protection."

J. D. Bjorken, Proc. Adv. Research Workshop on QCD Hadronic Processes, St. Croix, Virgin Islands (1987).

Spin and QCD

Consider A_N in hadron hadron collision:

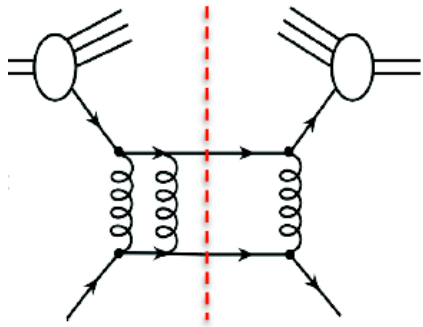


$$A_N = \frac{\sigma^\uparrow - \sigma^\downarrow}{\sigma^\uparrow + \sigma^\downarrow}$$

Spin and QCD

QCD had a very simple prediction:

Helicity flip is proportional to the small mass of the quark, thus



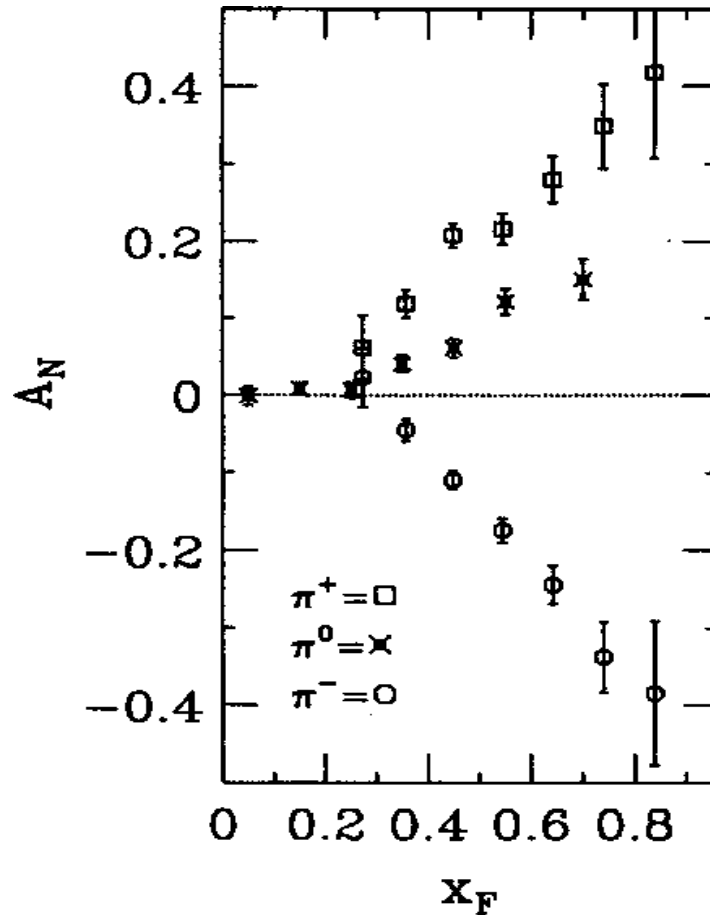
$$\propto \alpha_s \frac{m_q}{p_T}$$

$$A_N \simeq 0.001$$

Kane, Pumplin and Repko (1978)

Spin and QCD

Experiment proved this prediction wrong



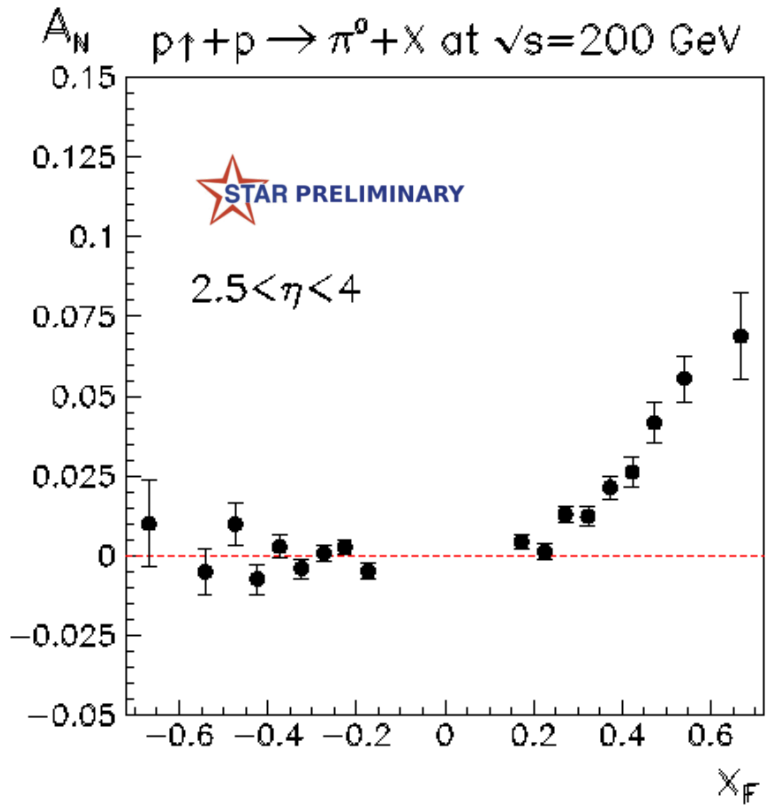
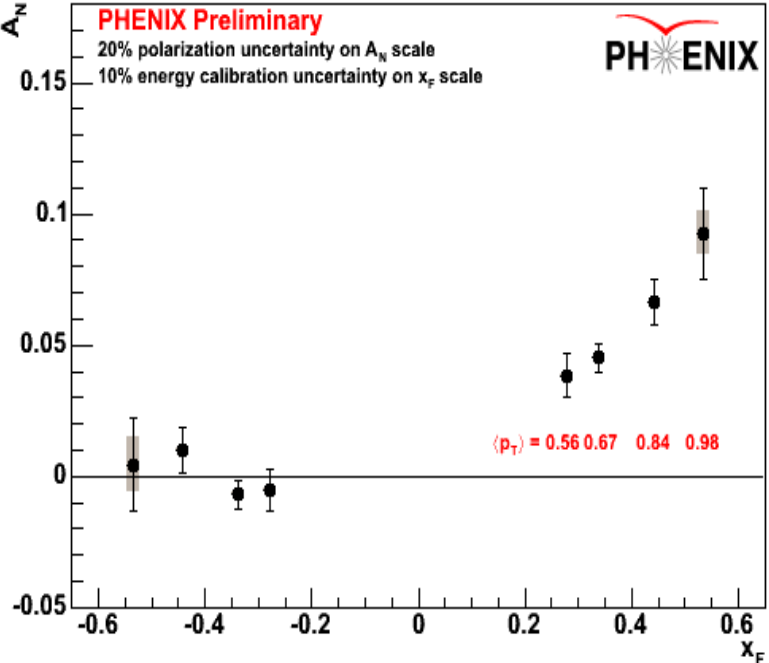
$$A_N \simeq 40\%$$

E704 (1991)

Spin and QCD

Asymmetry survives with energy

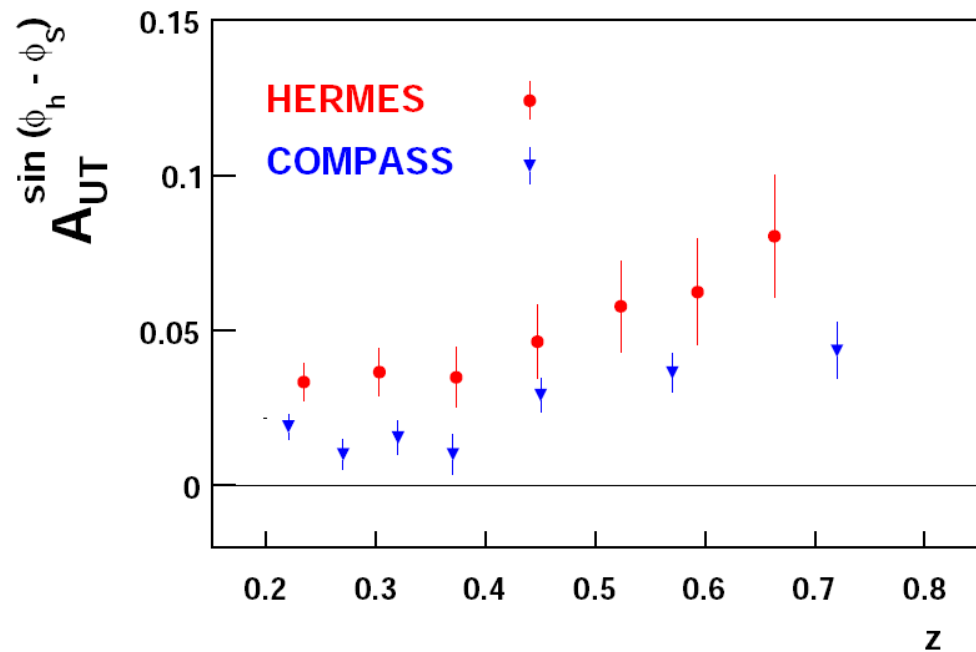
$\sqrt{s} = 62 \text{ GeV}$



RHIC: STAR, BRAHMS and PHENIX

Spin and QCD

Asymmetry survives with energy



HERMES and COMPASS

Failure of QCD?



Not at all: better understanding of QCD



Better understanding of QCD

$$\sigma(Q, \vec{s}) \propto \left| \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \dots \end{array} \right|^2$$

The diagrams show a series of Feynman diagrams for a scattering process. The first diagram shows a hard vertex (represented by a vertical oval) with two incoming lines (double lines) and two outgoing lines (triple lines). A gluon line (represented by a wavy line) connects the hard vertex to a soft vertex (represented by a horizontal oval). The soft vertex has two incoming lines (triple lines) and two outgoing lines (triple lines). The momentum of the hard vertex is labeled p, \vec{s} , the momentum of the gluon is k , and the momentum of the soft vertex is $t \sim 1/Q$. The subsequent diagrams show higher-order corrections involving multiple gluon lines connecting the hard and soft vertices.

Multy parton correlations contribute to the cross section.

These correlations are called [Efremov-Teryaev-Qiu-Sterman](#) matrix elements, They appear at twist-3 level in cross section.

$$\begin{aligned} \sigma &= \sigma^{LT} + \frac{Q_s}{Q} \sigma^{HT} + \dots \\ &= H^{LT} \otimes f_2 \otimes f_2 + \frac{Q_s}{Q} H^{HT} \otimes f_3 \otimes f_2 + \dots \end{aligned}$$

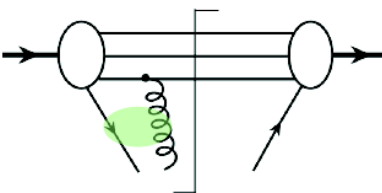
Better understanding of QCD

If only one large scale is present in the process, then

$$\begin{aligned} A_N &\propto \sigma(p_T, S_\perp) - \sigma(p_T, -S_\perp) \\ &\propto T^{(3)}(x, x, S_\perp) \otimes \hat{\sigma}_T \otimes D(z) + \delta q(x, S_\perp) \otimes \hat{\sigma}_D \otimes D^{(3)}(z, z) + \dots \end{aligned}$$

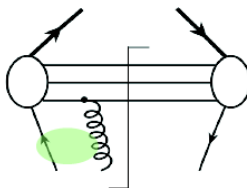
Leading power cancels

Twist-3 parton correlation functions

$$T^{(3)}(x, x, S_\perp) \propto$$


Qiu-Sterman 1991

Twist-3 parton fragmentation functions

$$D^{(3)}(z, z) \propto$$


Kang, Yuan, Zhou 2010

No probability interpretation!

Evolution of twist-3 matrix elements

One starts from factorization

$$\Delta\sigma = 1/QH(Q/\mu_F, \alpha_s) f_2(\mu_F) f_3(\mu_F)$$

- Calculate directly

Kang, Qiu 2009
Yuan, Zhou 2009

- Calculate from scale dependence of hard part

Yuan, Vogelsang 2009

- Renormalization of twist-3 operators

Braun et al 2009

Good agreement of results!

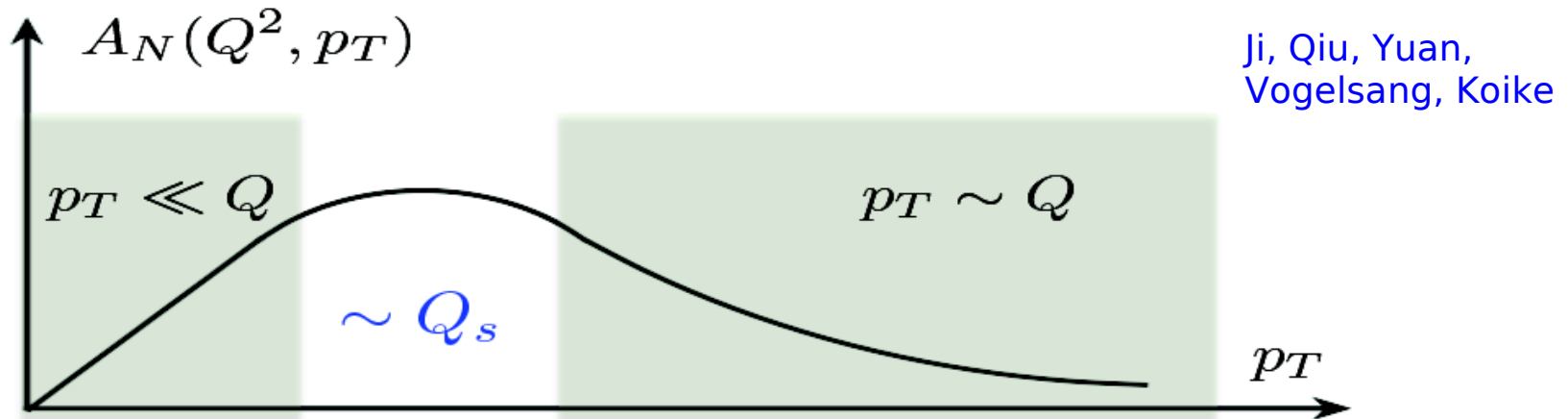
Collinear vs TMD factorization

We can consider two different kinematical regions

$$Q_1, Q_2, \dots \gg \Lambda_{QCD} \quad \text{Collinear}$$

$$Q_1 \gg Q_2 > \Lambda_{QCD} \quad \text{TMD}$$

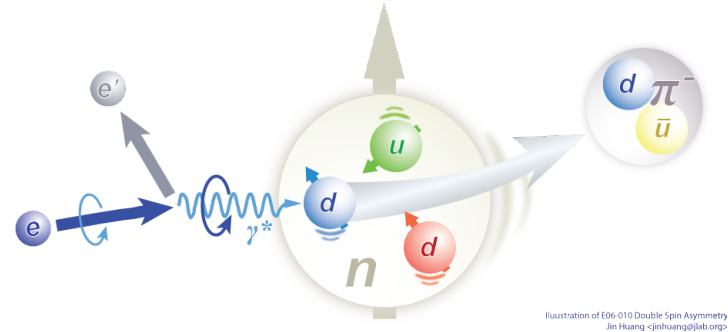
- Twist-3 - integration over parton momenta
- TMD - direct information on partonic transverse motion



Consistent in the overlap region!

Transverse Momentum Dependent distributions

SIDIS



$$\mathbf{l} + \mathbf{P} \rightarrow \mathbf{l}' + \mathbf{h} + \mathbf{X}$$

If produced hadron has low transverse momentum $P_{hT} \sim \Lambda_{QCD} \ll Q$

it will be sensitive to quark transverse momentum k_{\perp}

TMD factorization proven in QCD

Ji, Ma, Yuan (2002)

Collins (2011)

For processes with two hadrons:

SIDIS, DY, $e+e-$

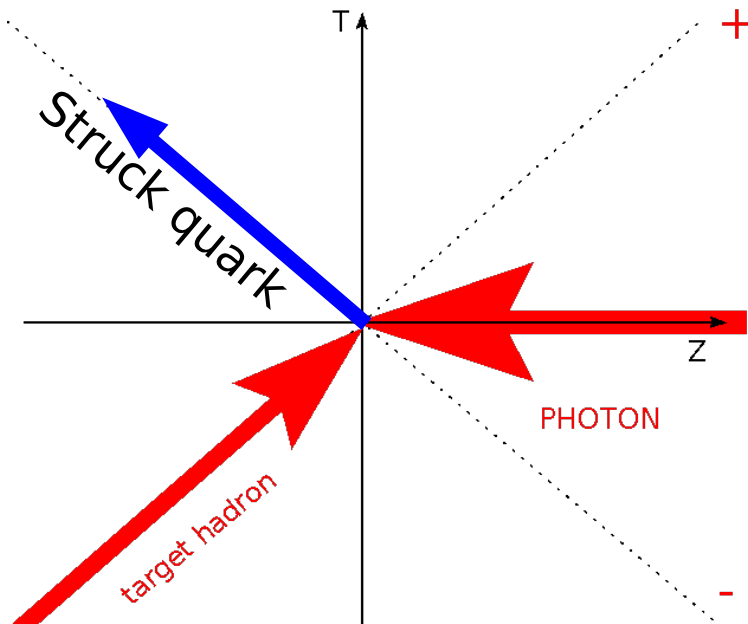
GAUGE INVARIANT

$$\Phi_{ij}(x, \mathbf{k}_{\perp}) = \int \frac{d\xi^{-}}{(2\pi)} \frac{d^2\xi_{\perp}}{(2\pi)^2} e^{ixP^{+}\xi^{-} - i\mathbf{k}_{\perp}\xi_{\perp}} \langle P, S_P | \bar{\psi}_j(0) \mathcal{U}(\mathbf{0}, \xi) \psi_i(\xi) | P, S_P \rangle$$

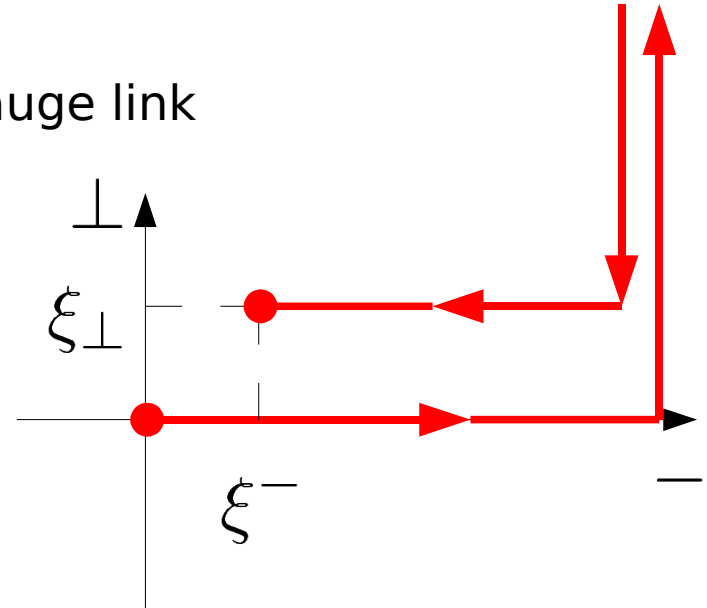
Transverse Momentum Dependent distributions

$$\Phi_{ij}(x, \mathbf{k}_\perp) = \int \frac{d\xi^-}{(2\pi)} \frac{d^2\xi_\perp}{(2\pi)^2} e^{ixP^+\xi^- - i\mathbf{k}_\perp\xi_\perp} \langle P, S_P | \bar{\psi}_j(0) \mathcal{U}(\mathbf{0}, \xi) \psi_i(\xi) | P, S_P \rangle |_{\xi^+=0}$$

SIDIS in IMF:




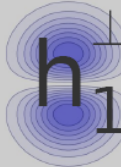


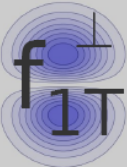

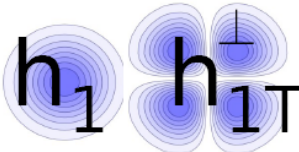
Gauge link



$$\mathcal{U}(a, b; n) = e^{-ig \int_a^b d\lambda n \cdot A_\alpha(\lambda n) t_\alpha}$$

Ensures gauge invariance of the distribution, cannot be canceled by gauge choice

TMDs

$N \backslash q$	U	L	T
U			
L			
T			

8 functions in total (at leading Twist)

Each represents different aspects of partonic structure

Each function is to be studied

Kotzinian (1995), Mulders, Tangerman (1995), Boer, Mulders (1998)

Correlation of transverse quark motion and the nucleon spin - Sivers function

$$f(x, \mathbf{k}_T, \mathbf{S}_T) = f_1(x, \mathbf{k}_T^2) - f_{1T}^\perp(x, \mathbf{k}_T^2) \frac{\mathbf{k}_x}{M}$$

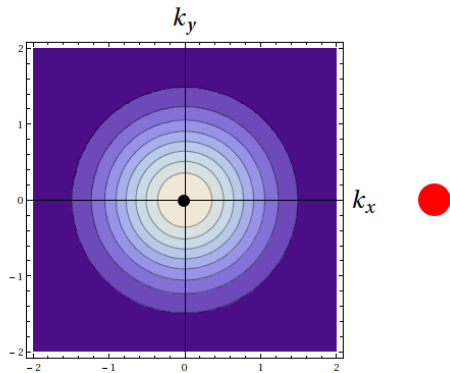
This function gives access to 3D imaging

Spin-orbit correlation

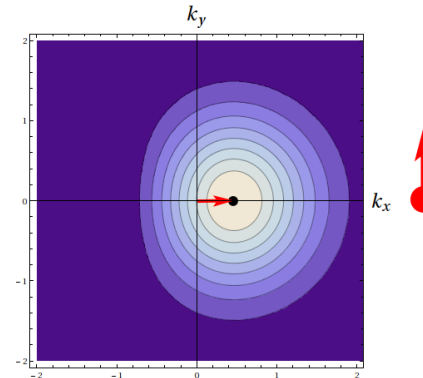
Physics of gauge links is represented

Requires Orbital Angular Momentum

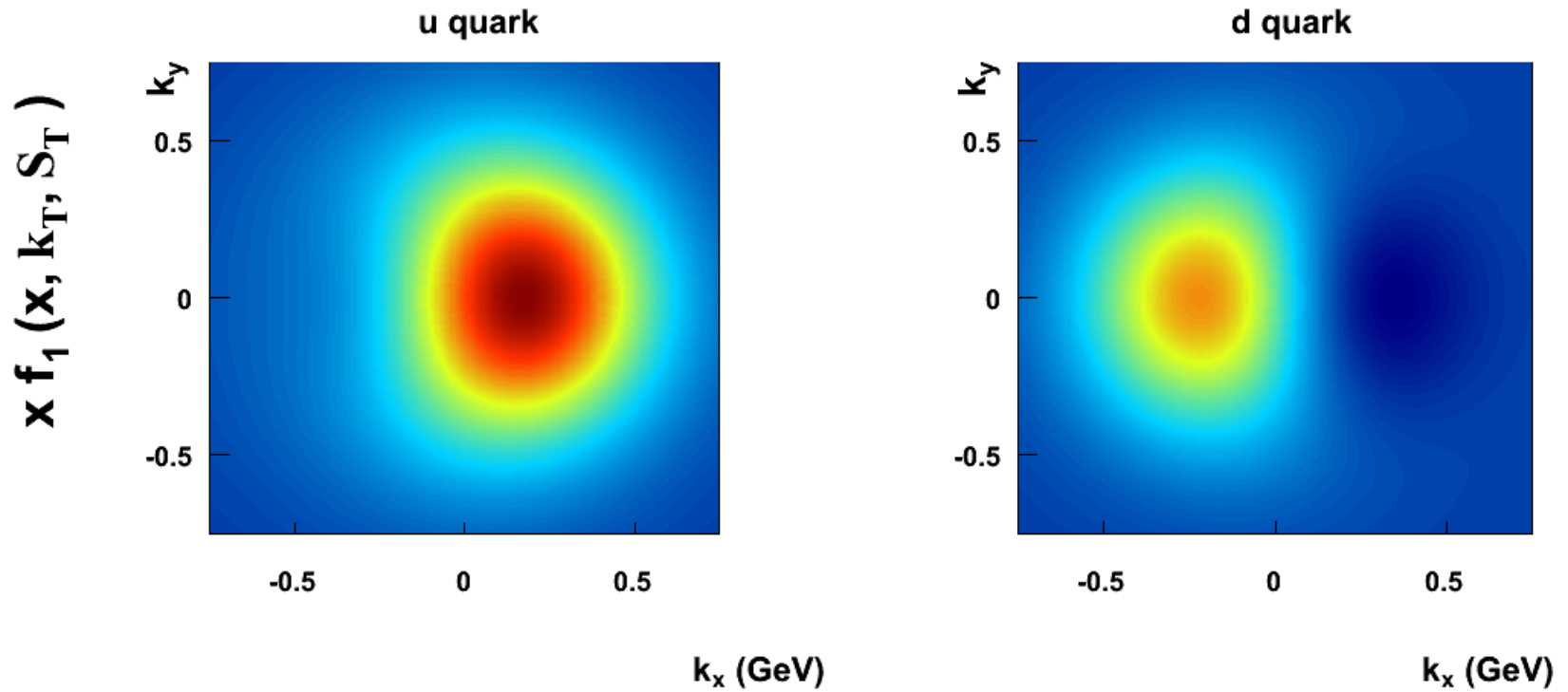
No polarisation:



Polarisation:



Tomographic scan of the nucleon



Anselmino et al 2009

Wigner distribution

Transverse
Momentum
Dependent
distributions

$$W(x, \mathbf{k}_\perp, \mathbf{b}_\perp)$$

Generalized
Parton
Distributions

$$\int d^2 \mathbf{b}_\perp$$

$$\int d^2 \mathbf{k}_\perp$$

$$f(x, \mathbf{k}_\perp)$$

$$H(x, \xi, t)$$

$$\int d^2 \mathbf{k}_\perp$$

Parton
Distribution
Functions

Form
Factors

$$\int dx$$

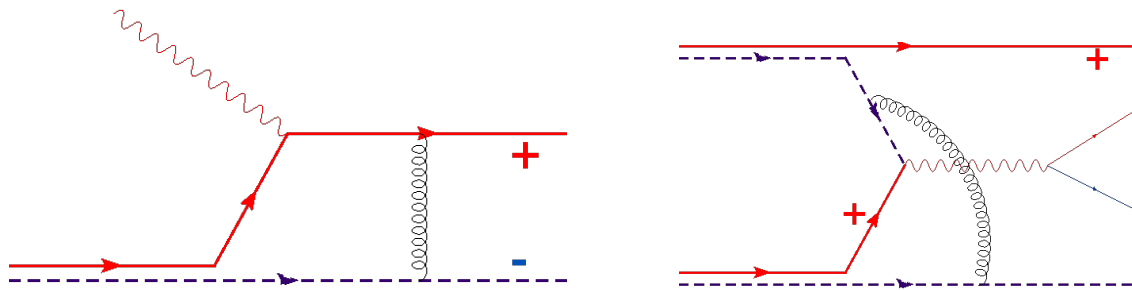
$$f(x)$$

$$F(Q^2)$$

Physics of gauge links

Colored objects are surrounded by gluons, profound consequence of gauge invariance.

Sivers function has opposite sign when gluon couple after quark scatters (SIDIS) or before quark annihilates (Drell-Yan)



Brodsky, Hwang,
Schmidt
Belitsky, Ji, Yuan
Collins
Boer, Mulders, Pijlman,
etc

$$f_{1T}^{\perp \text{SIDIS}} = -f_{1T}^{\perp \text{DY}}$$

See talk by Michela Chiosso

One of the main goals is to verify this relation.
It goes beyond “just” check of TMD factorization.
Motivates Drell-Yan experiments

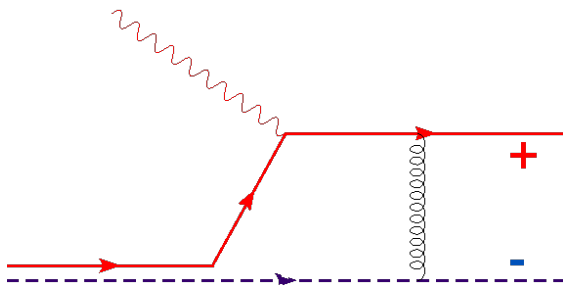
AnDY, COMPASS, JPARC, PAX etc

Barone et al., Anselmino et al., Yuan, Vogelsang, Schlegel et al., Kang, Qiu, Metz, Zhou

Physics of gauge links

Colored objects are surrounded by gluons, profound consequence of gauge invariance.

Sivers function has opposite sign when gluon couple after quark scatters (SIDIS) or before quark annihilates (Drell-Yan)



$$f_{1T}^{\perp \text{SIDIS}} = -$$

Drell-Yan is at much different resolution scale Q .
 EIC will operate at higher Q .
 What do we know about evolution of TMDs?

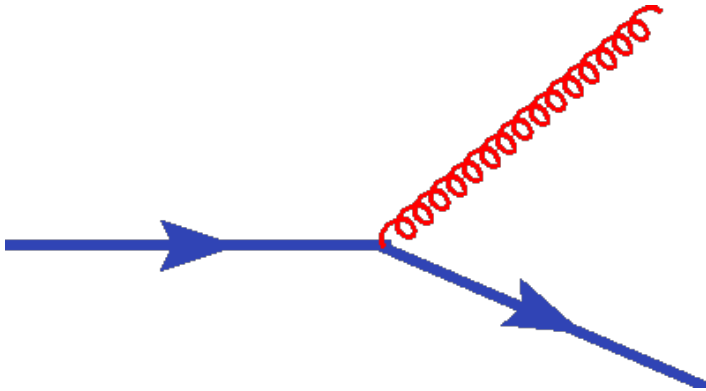
One of the main goals is to verify evolution.
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AnDY, COMPASS, JPARC, PAX etc

Barone et al., Anselmino et al., Yuan, Vogelsang, Schlegel et al., Kang, Qiu, Metz, Zhou

SIDIS and parton model

“QCD improved” parton model:



Radiation of gluons create transverse momenta

Terms like this appear

$$(\alpha_s)^n \left(\ln \frac{Q^2}{P_T^2} \right)^m$$

Result needs to be resummed for small P_T

Dokshitzer, Dyakonov, Troyan 1980
Parizi, Petronzio 1979
Collins, Soper 1982
Collins, Soper, Sterman 1985
Koike, Nagashima, Vogelsang
Kang, Xiao, Yuan

} Implementation of resummation
In QCD

Resummation

Dokshitzer, Dyakonov, Troyan 1980

Parizi, Petronzio 1979

Collins, Soper 1982

Collins, Soper, Sterman 1985

Koike, Nagashima, Vogelsang

Kang, Xiao, Yuan

Resummation (CSS) is in configuration space
Fourier transform is needed for observables

For Drell-Yan

$$\frac{d\sigma}{dq_T} \sim \int d^2 b_T e^{iq_T \cdot b_T} \hat{W}(x_1, x_2, b_T) e^{-S(b_T, Q)} + Y(q_T, Q)$$

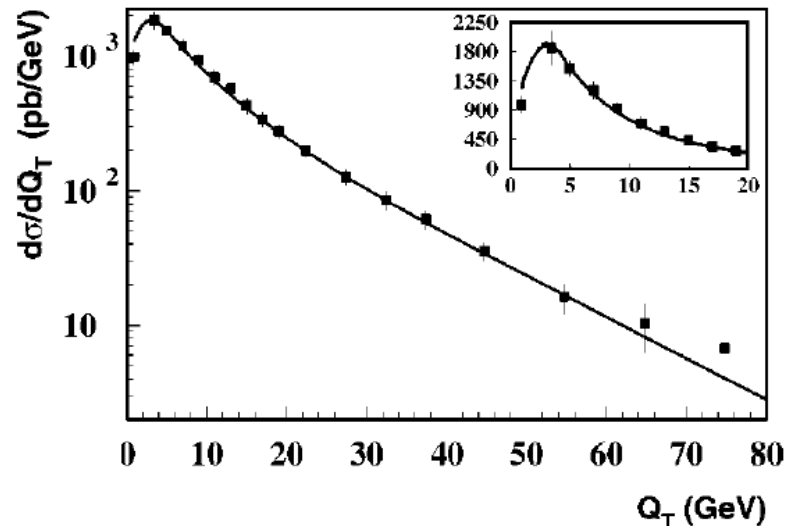
A lot of phenomenology done. Energies from 20 GeV to 2 TeV.

Brock, Landry, Nadolsky, Yuan 2003

Qiu, Zhang 2001

Drawbacks:

- Process dependent fits
- No direct connection to TMDs
- Designed for large energies



Evolution of TMDs

One needs a unique definition of TMDs

Foundations of perturbative QCD
Collins 2011

$$W^{\mu\nu} = \sum_f |H_f(Q^2, \mu)|^{\mu\nu} \\ \times \int d^2\mathbf{k}_{1T} d^2\mathbf{k}_{2T} F_{f/P_1}(x_1, \mathbf{k}_{1T}; \mu, \zeta_F) F_{\bar{f}/P_1}(x_2, \mathbf{k}_{2T}; \mu, \zeta_F) \\ \times \delta^{(2)}(\mathbf{k}_{1T} + \mathbf{k}_{2T} - \mathbf{q}_T) + Y(\mathbf{q}_T, Q)$$

$F_{f/P_1}(x_1, \mathbf{k}_{1T}; \mu, \zeta_F)$ **TMD distribution of partons in a hadron**

Renorm group (RG) renormalization  Rapidity divergence regulator

Evolution of TMDs

Evolution of TMDs is done in coordinate space \mathbf{b}_T

$$F_{f/P}(x, \mathbf{k}_T; \mu, \zeta_F) = \frac{1}{(2\pi)^2} \int d^2\mathbf{b}_T e^{i\mathbf{k}_T \cdot \mathbf{b}_T} \tilde{F}_{f/P}(x, \mathbf{b}_T; \mu, \zeta_F)$$

Colins Soper 1982

Foundations of perturbative QCD Collins 2011

Why coordinate space?

Convolutions become simple products:

$$W^{\mu\nu} = \sum_f |H_f(Q^2, \mu)|^{\mu\nu} \times \int d^2\mathbf{b}_T e^{i\mathbf{b}_T \mathbf{q}_T} \tilde{F}_{f/P_1}(x_1, \mathbf{b}_T; \mu, \zeta_F) \tilde{F}_{\bar{f}/P_1}(x_2, \mathbf{b}_T; \mu, \zeta_F)$$

Collins, Soper 1982

Collins, Soper, Sterman 1985

Idilbi, Ji, Ma, Yuan 2004

Boer, Gamberg, Musch, AP 2011

In principle experimental study of functions in coordinate space
Is possible

Boer, Gamberg, Musch, AP 2011

TMD evolution

Energy evolution

$$\frac{\partial \ln \tilde{F}(x, b_T, \mu, \zeta)}{\partial \ln \sqrt{\zeta}} = \tilde{K}(b_T, \mu) \longrightarrow \text{Collins-Soper kernel in coordinate space}$$

Renormalization group equations

TMD:
Collins 2011
Rogers, Aybat 2011
Aybat, Collins, Qiu, Rogers 2011

$$\frac{d\tilde{K}(b_T, \mu)}{d \ln \mu} = -\gamma_K(g(\mu))$$

$$\frac{d \ln \tilde{F}(x, b_T, \mu, \zeta)}{d \ln \mu} = -\gamma_F(g(\mu), \zeta)$$

TMD evolution

Energy evolution

$$\frac{\partial \ln \tilde{F}(x, b_T, \mu, \zeta)}{\partial \ln \sqrt{\zeta}} = \tilde{K}(b_T, \mu) \longrightarrow \text{Collins-Soper kernel in coordinate space}$$

At small \mathbf{b}_T perturbative treatment is possible

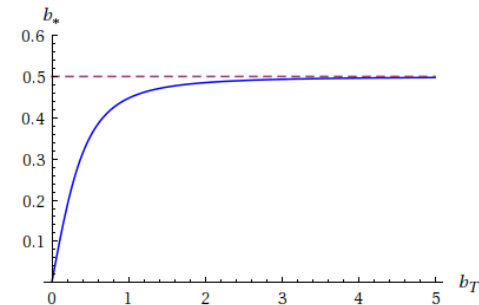
TMD:
Collins 2011
Rogers, Aybat 2011
Aybat, Collins, Qiu, Rogers 2011

$$\tilde{K}(b_T, \mu) = -\frac{\alpha_s C_F}{\pi} \left(\ln(\mu^2 b_T^2) - \ln 4 + 2\gamma_E \right) + \mathcal{O}(\alpha_s^2)$$

Large \mathbf{b}_T nonperturbative - matching via \mathbf{b}_* Collins Soper 1982

$$b_*(b_T) = \frac{b_T}{\sqrt{1 + b_T^2/b_{max}^2}}$$

Brock, Landry, Nadolsky, Yuan 2003



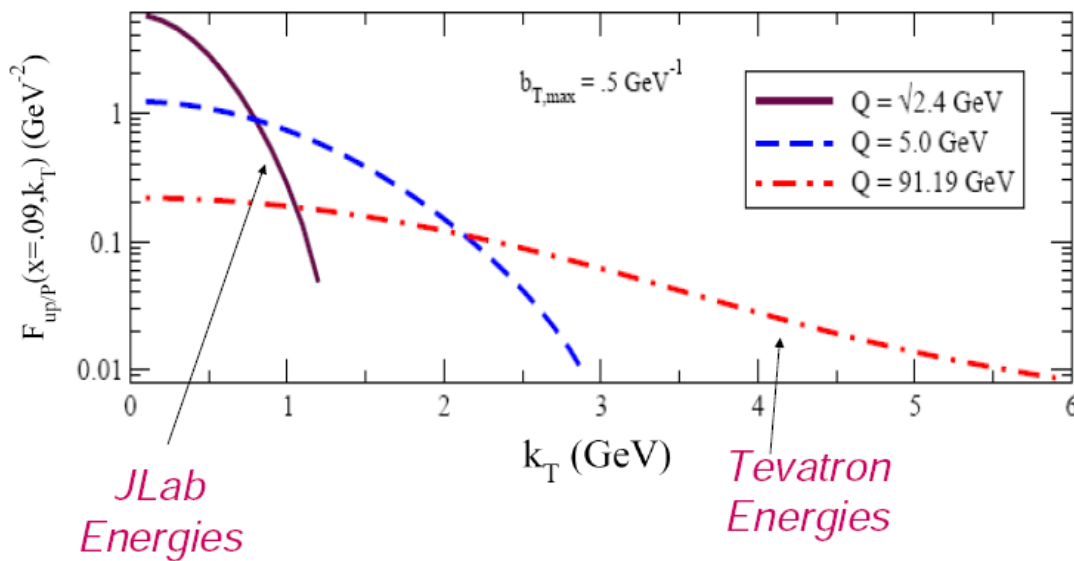
Other methods of matching are available i.e. Vogelsang et al

TMD evolution

Solution [Rogers, Aybat 2011](#)
[Aybat, Collins, Qiu, Rogers 2011](#)

$$\tilde{F}_{f/P}(x, b_T; Q, \zeta_F) = F_{f/P}(x; Q_0) \exp \left(- \underbrace{\left[\frac{\langle k_T^2 \rangle}{4} + \frac{g_2}{2} \ln \frac{Q}{Q_0} \right]}_{\text{Non perturbative}} b_T^2 \right)$$

Non perturbative



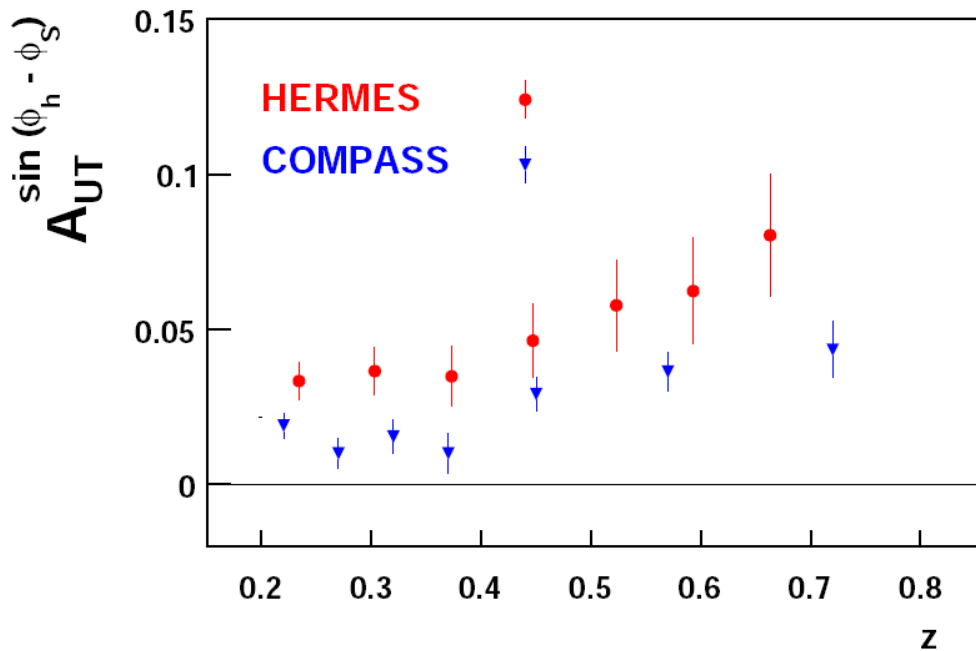
Gaussian behaviour is appropriate only in a limited range

TMDs change with energy and resolution scale

TMD evolution

Can we see signs of evolution in the experimental data?

Aybat, AP, Rogers 2011



COMPASS data is at

$$\langle Q^2 \rangle \simeq 3.6 \text{ (GeV}^2\text{)}$$

HERMES data is at

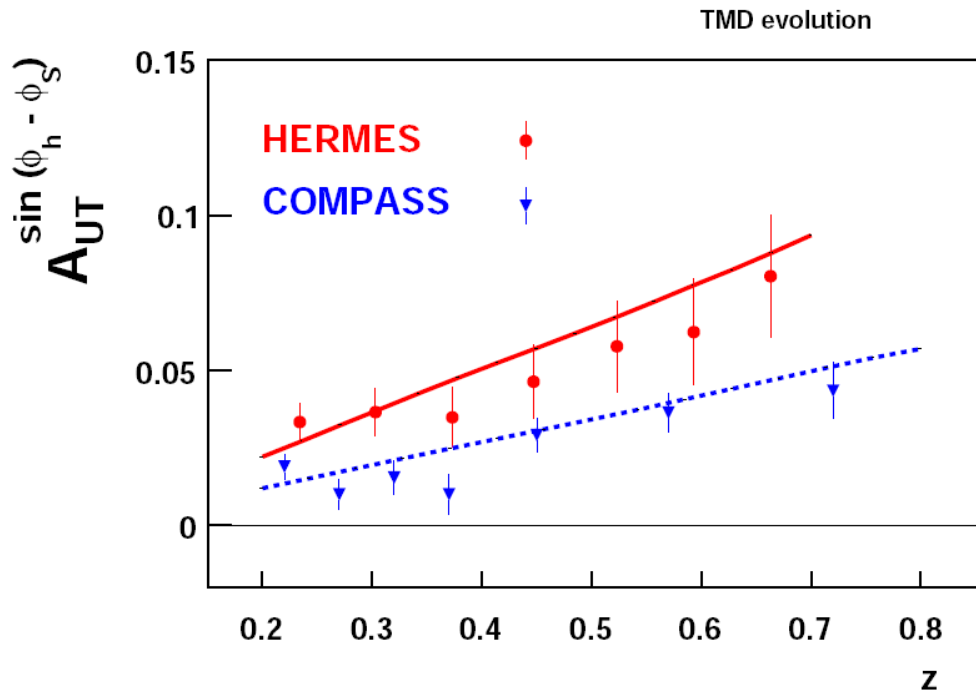
$$\langle Q^2 \rangle \simeq 2.4 \text{ (GeV}^2\text{)}$$

TMD evolution

Can we **explain** the experimental data?

Full TMD evolution is needed!

Aybat, AP, Rogers 2011



COMPASS dashed line

$$\langle Q^2 \rangle \simeq 3.6 \text{ (GeV}^2\text{)}$$

HERMES solid line

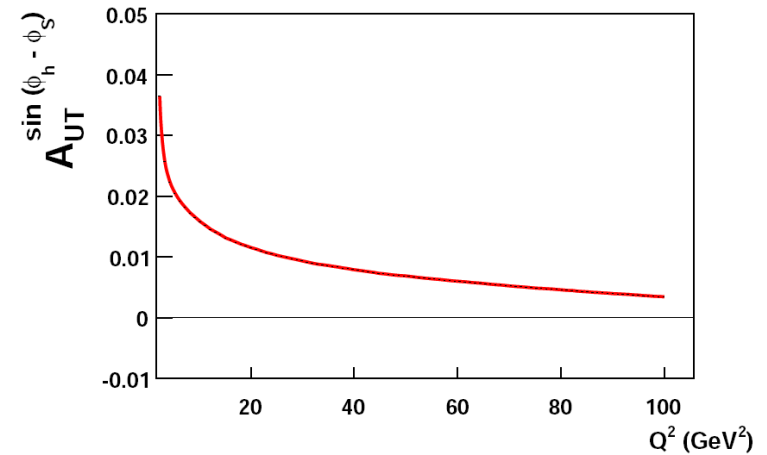
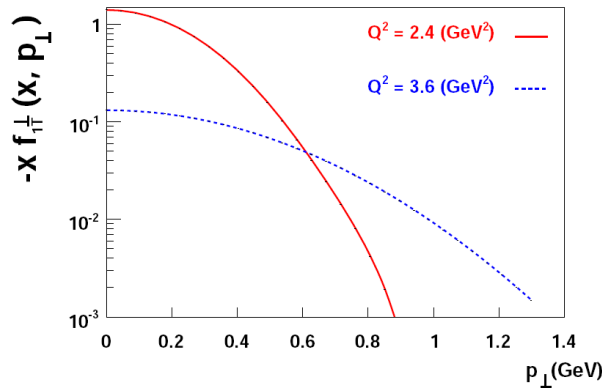
$$\langle Q^2 \rangle \simeq 2.4 \text{ (GeV}^2\text{)}$$

TMD evolution

This is the first implementation of TMD evolution for observable

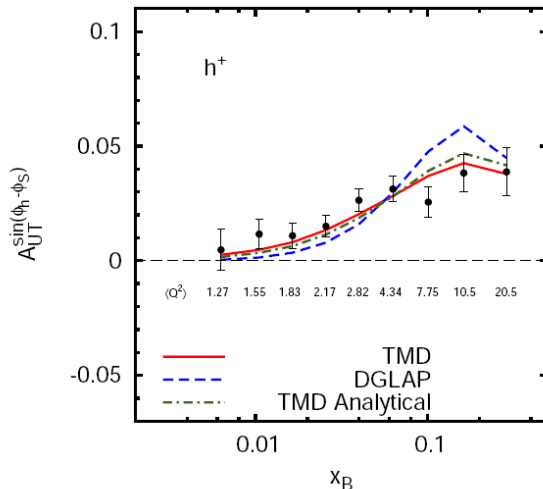
Aybat, AP, Rogers 2011

Asymmetry changes with Q^2



Functions change with energy

COMPASS PROTON



Anselmino, Boglione, Melis 2012

Solid line - TMD evolution fit
Dashed line - DGLAP fit

Phenomenological analysis with evolution is now possible

Helicity structure

$$\Delta q(x) = \left| \left. \begin{array}{c} P, + \\ \Rightarrow \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right\} X \right|^2 - \left| \left. \begin{array}{c} P, + \\ \Rightarrow \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right\} X \right|^2$$

$$\Delta g(x) = \left| \left. \begin{array}{c} P, + \\ \Rightarrow \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right\} X \right|^2 - \left| \left. \begin{array}{c} P, + \\ \Rightarrow \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right\} X \right|^2$$

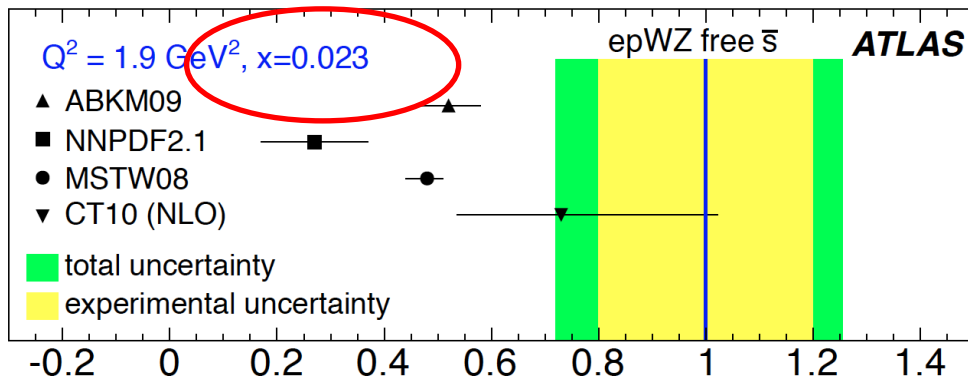
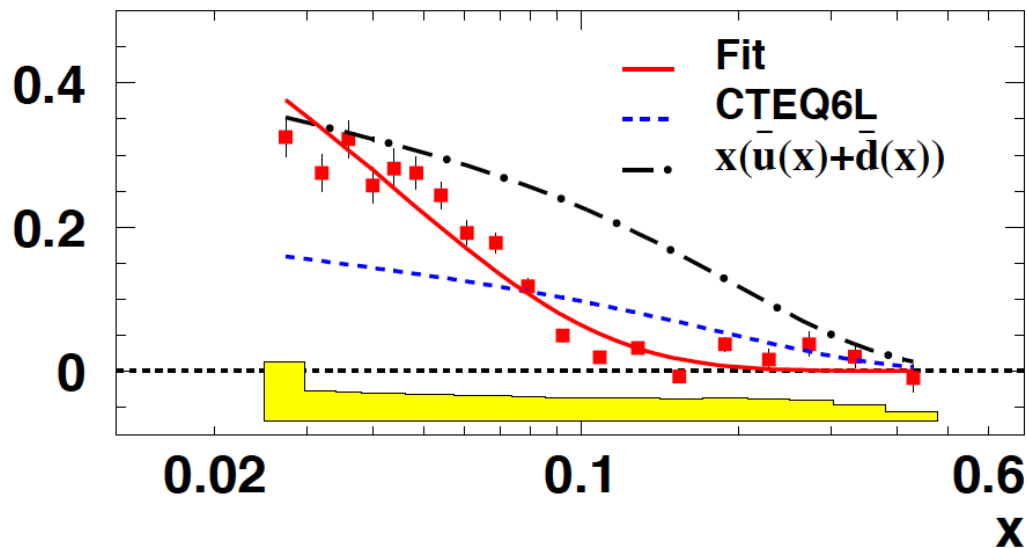
$$\frac{1}{2} = \frac{1}{2} \Delta \Sigma + L_q + \Delta G + L_g$$

$$\Delta \Sigma(Q^2) = \int_0^1 dx \left[\Delta u + \Delta \bar{u} + \Delta d + \Delta \bar{d} + \Delta s + \Delta \bar{s} \right] (x, Q^2)$$

$$\Delta G(Q^2) = \int_0^1 dx \Delta g(x, Q^2)$$

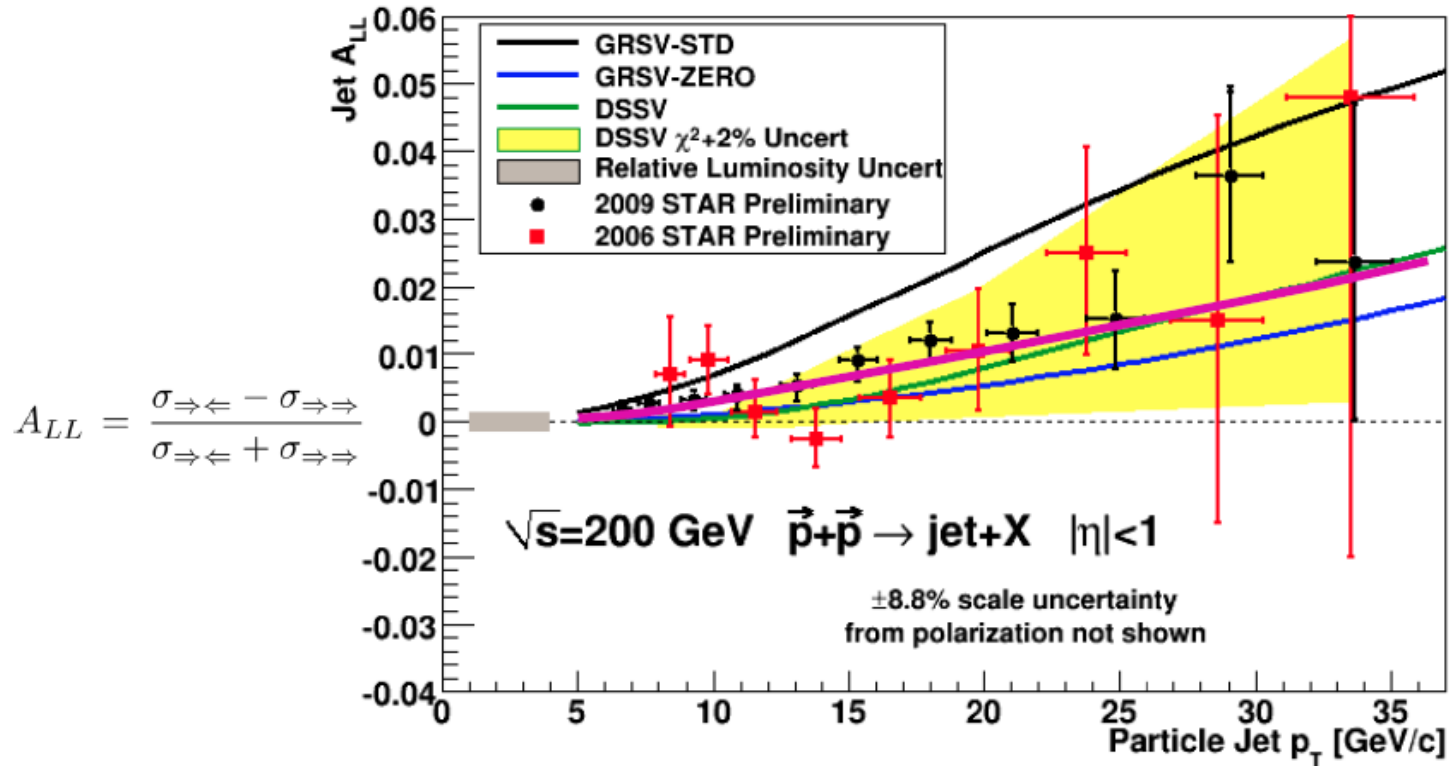
We still have a lot to learn about strangeness:

$$x(s + \bar{s})$$



$$r_s = 0.5 \frac{s + \bar{s}}{\bar{d}}$$

New developments on Δg : de Florian, Sassot, Stratmann, Vogelsang



gives gluon with $\int_{0.05}^{0.2} dx \Delta g \approx 0.1$

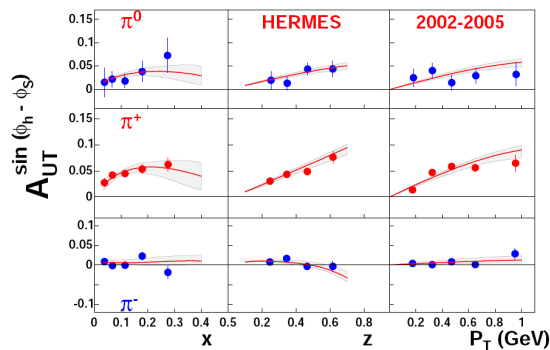
TMDs and twist-3 are related

At operator level:

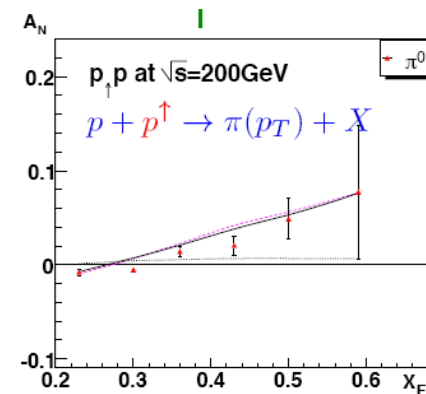
$$T_F(x, x) = - \int d^2 \vec{k}_\perp \frac{k_\perp^2}{M} (f_{1T}^\perp(x, k_\perp))_{SIDIS}$$

Boer, Mulders, Pijlman,
 Ji, Qiu, Vogelsang, Yuan,
 Koike, Vogelsang, Yuan
 Zhou, Yuan, Liang
 Bacchetta, Boer, Diehl, Mulders

We can compare results from SIDIS and PP:



TMD analysis:
 Anselmino et al (2008)

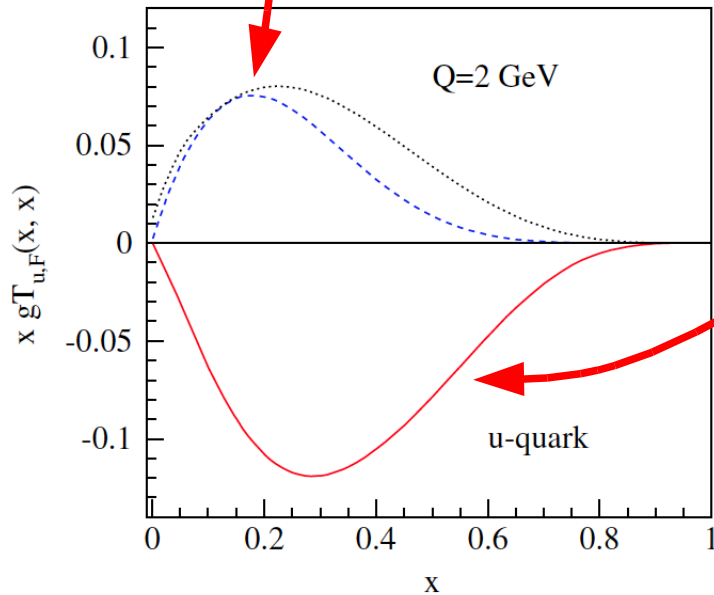


Collinear analysis: Kouvaris, Qiu,
 Vogelsang, Yuan (2006)

→ a sign puzzle

Kang, Qiu, Vogelsang, Yuan (2011)

$$\text{from } - \int d^2 \vec{k}_\perp \frac{k_\perp^2}{M} (f_{1T}^\perp(x, k_\perp))_{SIDIS}$$



$T_F(x, x)$ from PP

Kang, Qiu, Vogelsang, Yuan (2011)

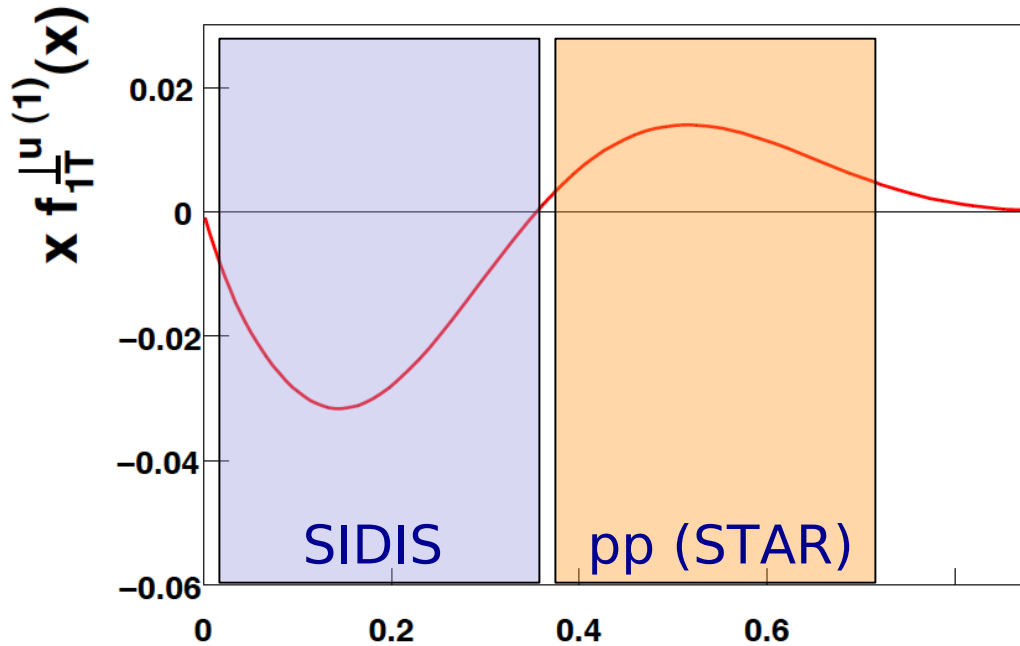
Conclusion?

Inconsistency in QCD formalism for single-spin ?

Collins-type effect dominant in $pp \rightarrow \pi X$?

Can one describes it with nodes in x or kt ?

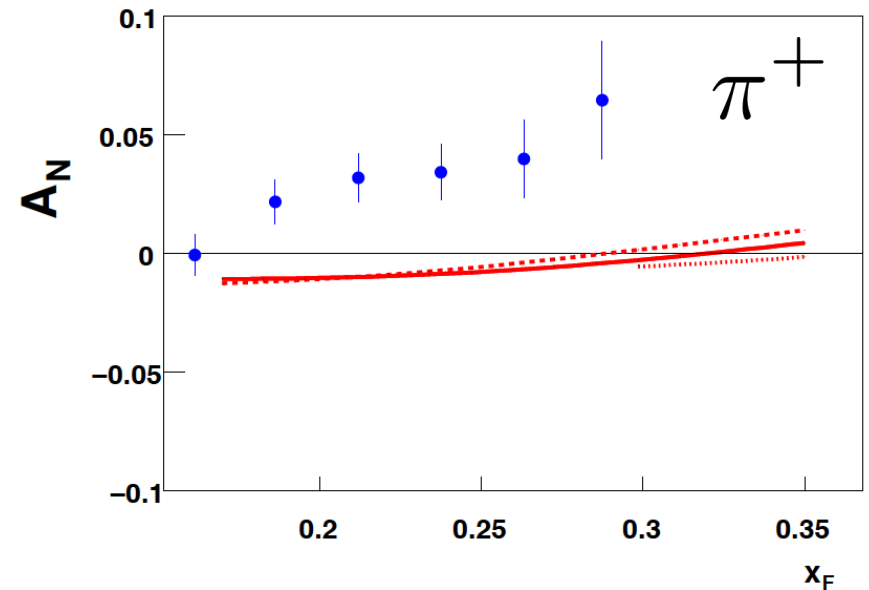
Joint fit to SIDIS and pp data:



Kang, AP (2011)

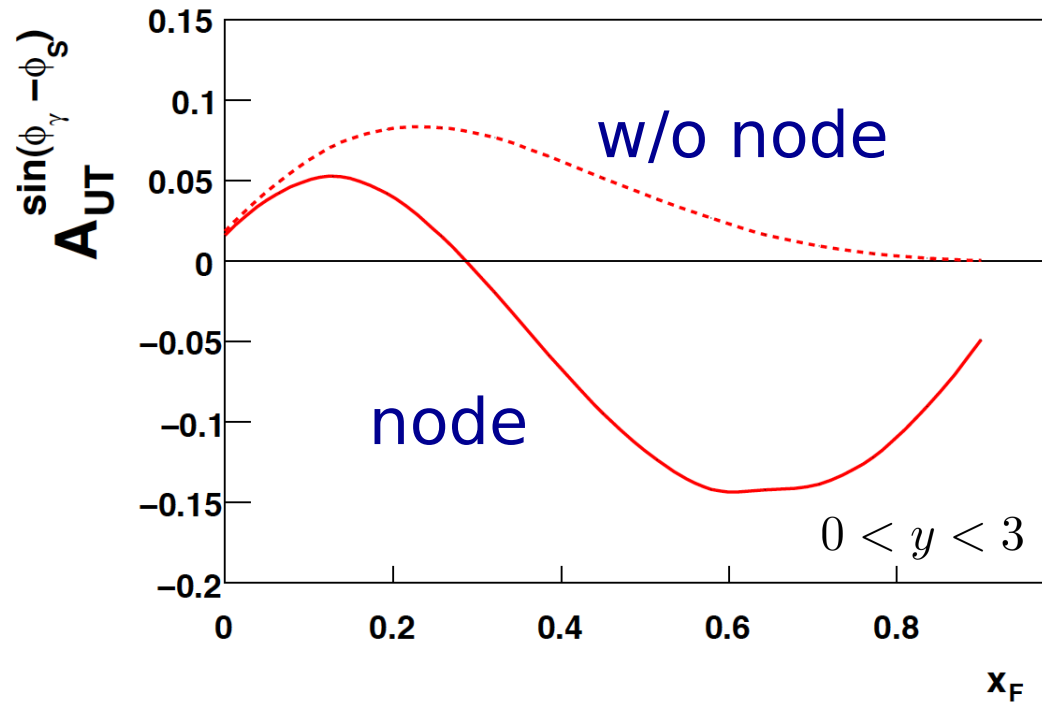
Conclusion?

Works (reasonably) well for SIDIS and STAR, but fails for BRAHMS!



Has ramifications for DY spin asymmetry:

Kang,AP (2011)



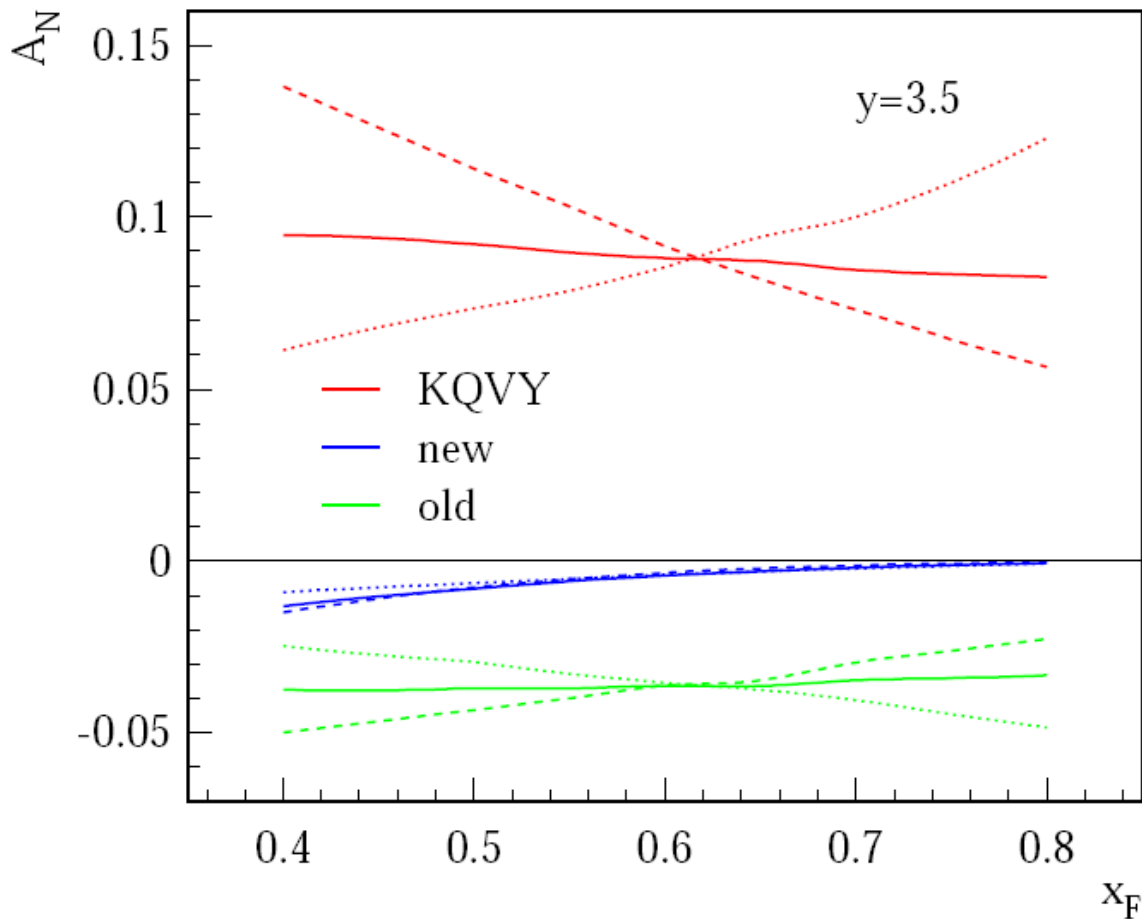
Strengthens case for study of DY “sign change” !

AnDY, COMPASS, E906,W bosons at RHIC

Direct (prompt) photon A_N



Kang, Gamberg (2012)
Kanazawa, Koike (2012)



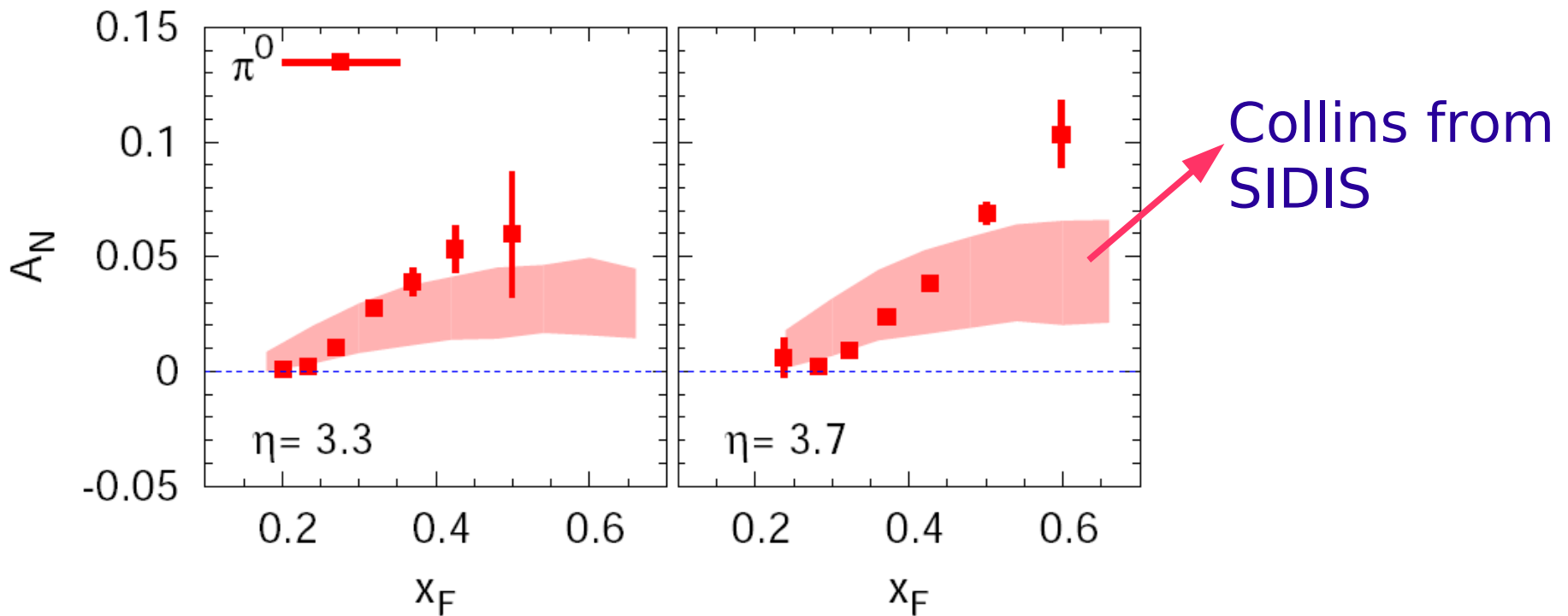
} $T_F(x, x)$ from pp

} $-\int d^2\vec{k}_\perp \frac{k_\perp^2}{M} (f_{1T}^\perp(x, k_\perp))_{SIDIS}$

Sensitivity to Sivers type of contribution only.

Collins contribution

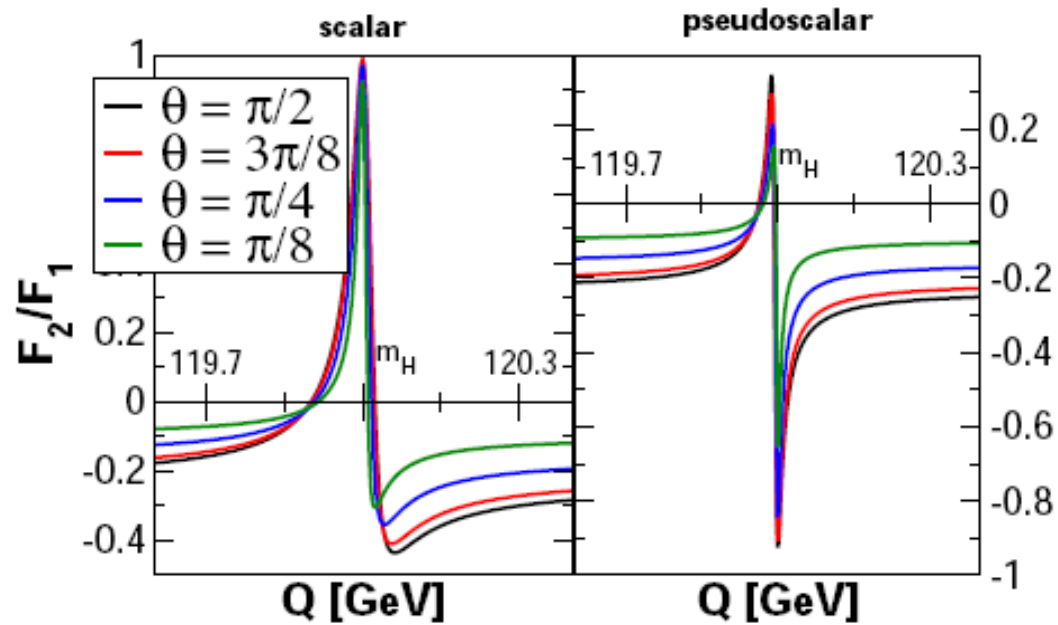
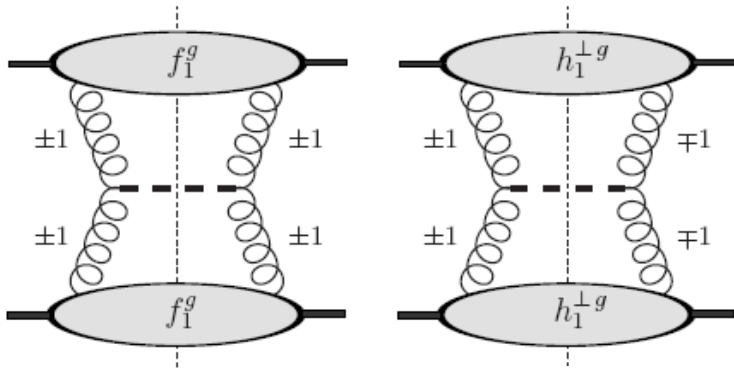
Anselmino, Boglione, D'Alesio, Leader (5), Melis, Murgia, AP (2012)



Collins contribution can be substantial

TMDs for LHC

Boer, den Dunnen, Pisano, Schlegel, Vogelsang (2012)



Parity of the Higgs!

CONCLUSIONS

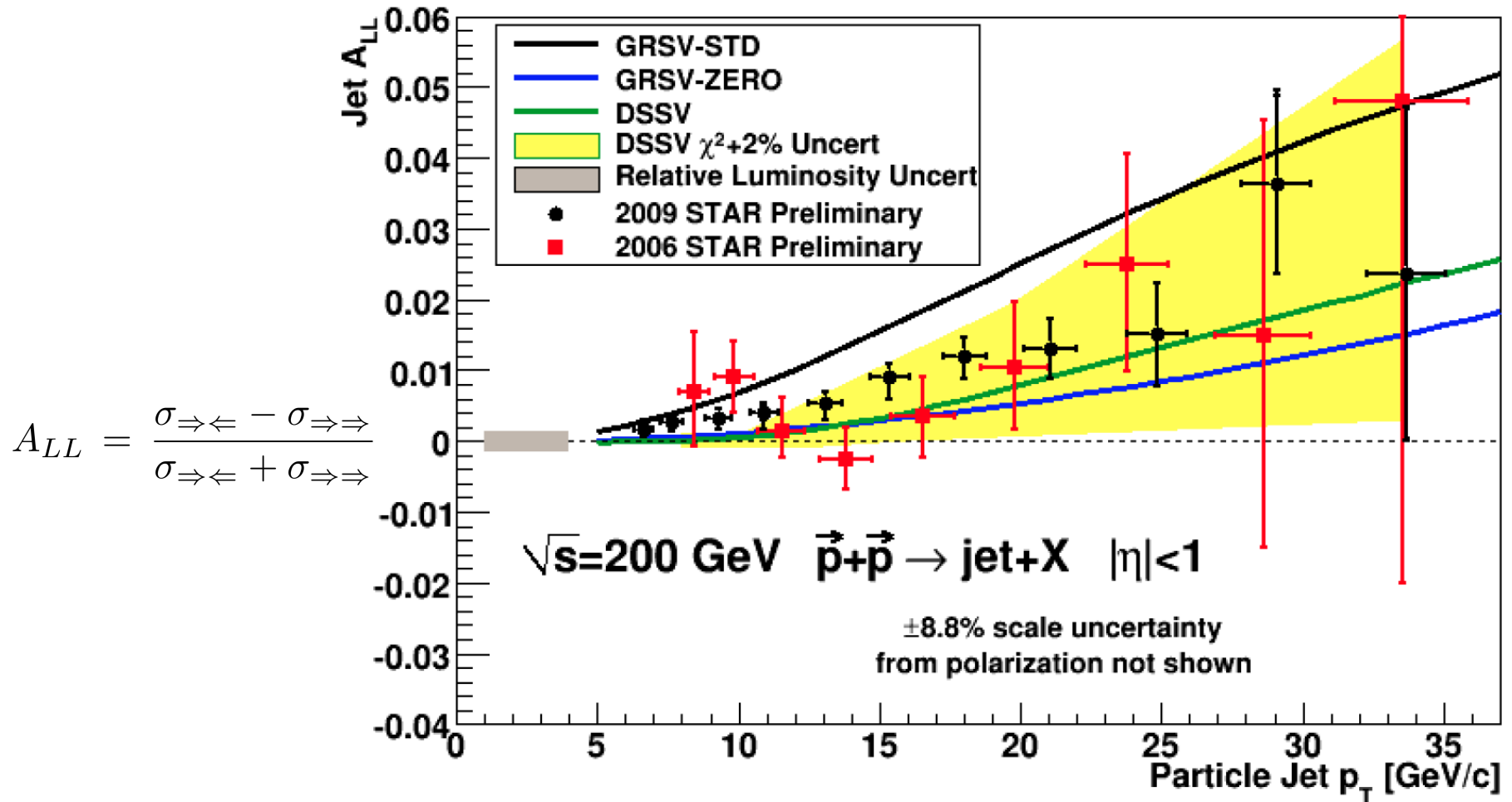
CONCLUSIONS

A lot of new results will be presented this week!

Enjoy SPIN 2012!

Spares to follow

New developments on Δg : de Florian, Sassot, Stratmann, Vogelsang

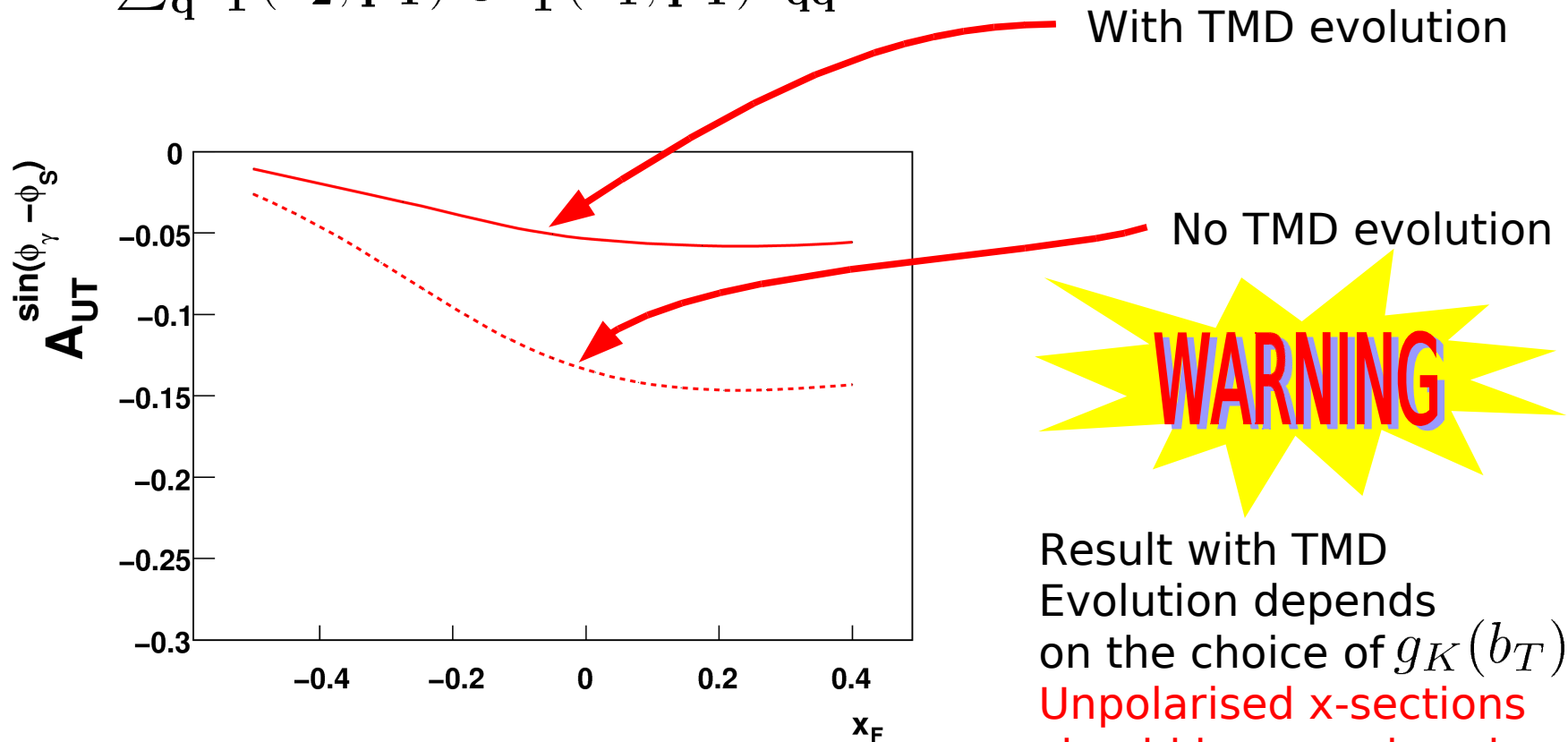


gives gluon with $\int_{0.05}^{0.2} dx \Delta g \approx 0.1$

Drell Yan

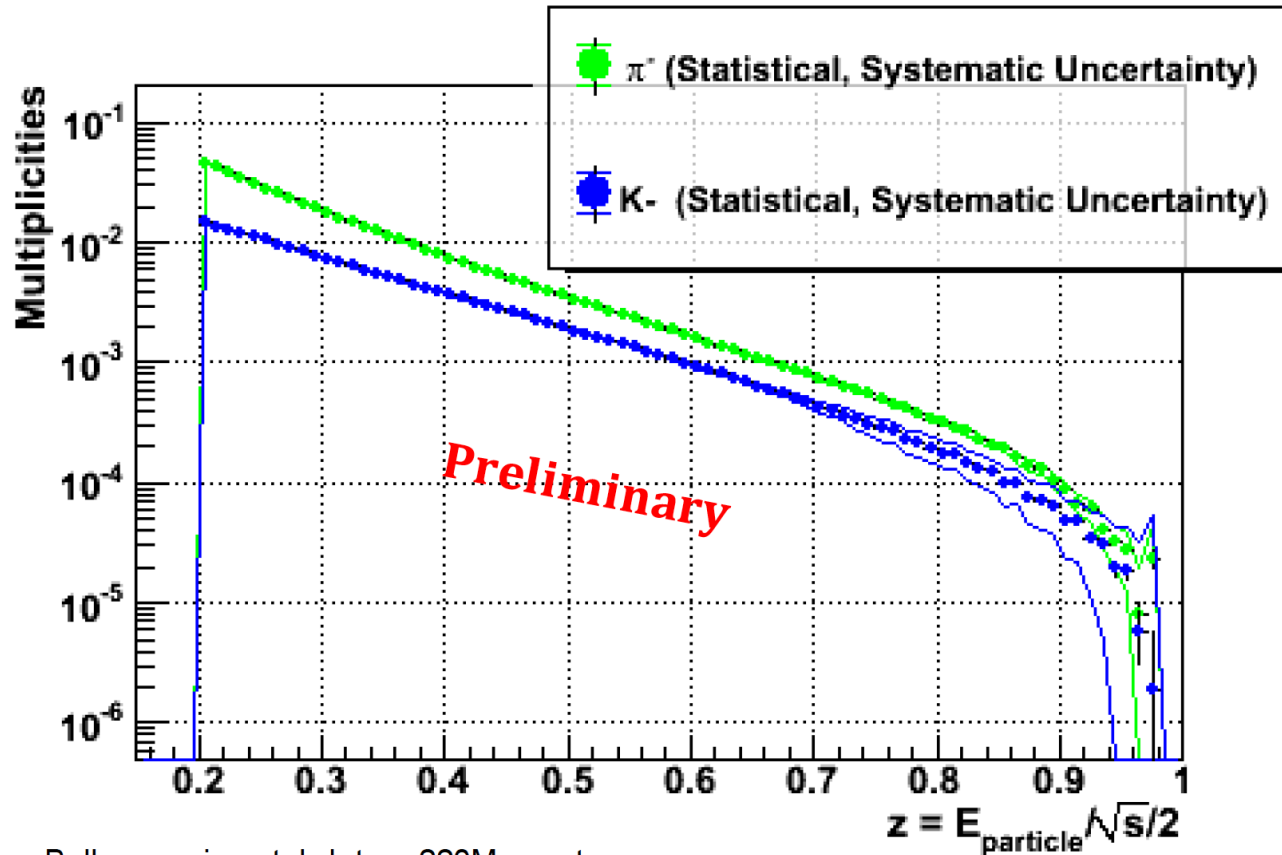
$$A_N = \frac{\sum_q f_{1T}^{\perp q}(\mathbf{x}_2, \mathbf{p}_T) \otimes f_1^{\bar{q}}(\mathbf{x}_1, \mathbf{p}_T) \sigma_{q\bar{q}}}{\sum_q f_1^q(\mathbf{x}_2, \mathbf{p}_T) \otimes f_1^{\bar{q}}(\mathbf{x}_1, \mathbf{p}_T) \sigma_{q\bar{q}}}$$

Analysis in hadronic cm frame



Asymmetry is suppressed with respect to LO analysis

Fragmentation FF: BELLE (M. Leitgab at DIS 2012)



Belle experimental data, ~220M events

- marks new era of precision fragmentation functions.
Evolution $\rightarrow D_g^h$ e^+e^- vs. RHIC(pp)

TMD evolution

Relation to collinear treatment:

$$\tilde{F}_f(x, b_T, \mu, \zeta) = \sum_j \int_x^1 \frac{d\hat{x}}{\hat{x}} \tilde{C}_{j/f}\left(\frac{x}{\hat{x}}, b_T, \mu, \zeta\right) f_j(\hat{x}, \mu) + \mathcal{O}(\Lambda_{QCD} b_T)$$

Collins Soper 1982

Valid at small \mathbf{b}_T , lowest order:

$$\tilde{C}_{j/f}\left(\frac{x}{\hat{x}}, b_T, \mu, \zeta\right) = \delta_{jf} \delta\left(\frac{x}{\hat{x}} - 1\right) + \mathcal{O}(\alpha_s)$$

Higher order for TMD PDFs

Aybat Rogers 2011

Higher order for Sivers function

Kang, Xiao, Yuan 2011

TMD evolution

Solution [Rogers, Aybat 2011](#)
[Aybat, Collins, Qiu, Rogers 2011](#)

$$\begin{aligned}
 \tilde{F}_{f/P}(x, b_T; Q, \zeta_F) &= \tilde{F}_{f/P}(x, b_T; Q_0, Q_0^2) \left. \vphantom{\tilde{F}_{f/P}} \right\} \text{Non perturbative} \\
 &\times \exp \left[-g_K(b_T) \ln \frac{Q}{Q_0} \right] \\
 &\times \exp \left[\ln \frac{Q}{Q_0} \tilde{K}(b_*; \mu_b) + \int_{Q_0}^Q \frac{d\mu'}{\mu'} \left[\gamma_F(g(\mu'); 1) - \ln \frac{Q}{\mu'} \gamma_K(g(\mu')) \right] \right] \left. \vphantom{\tilde{F}_{f/P}} \right\} \\
 &+ \int_{Q_0}^{\mu_b} \frac{d\mu'}{\mu'} \ln \frac{Q}{Q_0} \gamma_K(g(\mu')) \left. \vphantom{\tilde{F}_{f/P}} \right\} \text{Perturbative}
 \end{aligned}$$

Typically for TMDs:

$$\tilde{F}_{f/P}(x, b_T; Q_0, Q_0^2) = F_{f/P}(x; Q_0) \exp \left(-\frac{\langle k_T^2 \rangle}{4} b_T^2 \right)$$