

**Gauge-invariance and renormalization effects  
in transverse-momentum-dependent  
parton distribution functions**

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Completely **gauge-invariant** definition of the transverse-momentum dependent parton distribution functions is studied taking into account their **renormalization properties** as a starting point. In the **light-cone gauge**, the UV anomalous dimension is calculated in the one-loop order and the consistent treatment of the additional singularities, which produce undesirable contributions in the anomalous dimensions, is discussed. The **generalized renormalization of TMD PDF** based on the renormalization procedure for the Wilson exponentials with obstructions (cusps or intersections) is proposed.

## Plan

- **Integrated PDF's:** definition; gauge invariance; RG properties
- **Unintegrated (TMD) PDF's:** trial (naive) definition; shortcomings (complete gauge invariance, undesirable divergences)
- **Looking for a solution:** saving gauge invariance; non-light-like stuff; factorization; reduction to the integrated case
- Towards the “**completely correct**” TMD PDF's: anomalous dimensions sum rule as a starting point; calculation of AD in the light-cone gauge; generalized renormalization and cancellation of extra divergences
- Conclusions and outlook

**Integrated PDF's:** definition; gauge invariance; RG properties

$$q_{i/h}(x) = \frac{1}{2} \int \frac{d\xi^-}{2\pi} e^{-ik^+\xi^-} \langle h(P) | \bar{\psi}_i(\xi^-, \mathbf{0}_\perp) \gamma^+ \psi_i(0^-, \mathbf{0}_\perp) | h(P) \rangle$$

**Gauge invariance** is saved by the insertion of the **gauge link**:

$$[y, x|\Gamma] = \mathcal{P} \exp \left[ -ig \int_{x[\Gamma]}^y dz_\mu A_a^\mu(z) t_a \right]$$

so that

$$\hat{q}_{i/h}(x) = \frac{1}{2} \int \frac{d\xi^-}{2\pi} e^{-ik^+\xi^-} \langle h(P) | \bar{\psi}_i(\xi^-, \mathbf{0}_\perp) [\xi^-, 0^-] \gamma^+ \psi_i(0^-, \mathbf{0}_\perp) | h(P) \rangle$$

**Renormalization properties** are described by the DGLAP equation.

## Unintegrated (TMD) PDF's

“Naive” definition:

$$f_i(x, \mathbf{k}_\perp) = \frac{1}{2} \int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2} e^{-ik^+\xi^- + ik_\perp \cdot \xi_\perp}.$$

$$\cdot \langle p | \bar{\psi}_i(\xi^-, \xi_\perp) [\xi^-, \xi_\perp; \infty^-, \xi_\perp; ]^\dagger \gamma^+ [\infty^-, \mathbf{0}_\perp; 0^-, \mathbf{0}_\perp] \psi_i(0^-, \mathbf{0}_\perp) | p \rangle \Big|_{\xi^+ = 0}$$

Formally:

$$\int d^2k_\perp f_i(x, \mathbf{k}_\perp) = q_i(x)$$

**However:** this definition suffers from several shortcomings.

- **extra divergences** associated with light-cone gauge, or light-like Wilson lines (*in the integrated case, these divergences cancel*)
- **gauge invariance** is not complete: in the light-cone gauge, dependence on the pole prescription in the gluon propagator still takes place
- reduction to the integrated case: formal integration doesn't produce correct result

## Looking for the solution:

- **gauge invariance** is completely restored by means of the additional transverse Wilson path integral at the light-cone infinity (Belitsky, Ji, Yuan). This gauge link contribute only in the light-cone gauge, and cancel pole-prescription dependence
- **extra divergences** can be avoided by using non-light-like gauge connectors in the covariant gauges (Collins, Soper), or (presumably) non-light-cone axial gauge. Thus, additional variable introduced (rapidity cutoff); calculations become more complicated; problems with factorization.
- **generalized renormalization** for the light-like Wilson lines (Collins, Hautmann): extra divergences cancel by the additional “soft” factor, defined by the vacuum average of special Wilson lines (demonstrated explicitly in the covariant gauge, in the 1-loop order)

## Towards the “completely correct” definition:

- starting from the requirement of the **gauge invariance**, one formulates the anomalous dimensions sum rule:
- calculate the **anomalous dimension** of TMD PDF in the light-cone gauge and identify extra divergences in terms of the defect of anomalous dimension
- perform the **generalized renormalization** of TMD PDF, in analogue to the renormalization of the Wilson contours with cusps or self-intersection



In the **tree** approximation, the TMD PDF reads

$$\begin{aligned}
 f^{(0)}(x, k_{\perp}) &= \\
 &= \frac{1}{2} \int \frac{d\xi^- d^2\xi_{\perp}}{2\pi(2\pi)^2} e^{-ik^+\xi^- + ik_{\perp}\cdot\xi_{\perp}} \langle p | \bar{\psi}(\xi^-, \xi_{\perp}) \gamma^+ \psi(0^-, 0_{\perp}) | p \rangle = \\
 &= \delta(1-x) \delta^{(2)}(\mathbf{k}_{\perp})
 \end{aligned}$$

The one-gluon exchanges, contributing to the UV-divergences, are described by the diagrams:

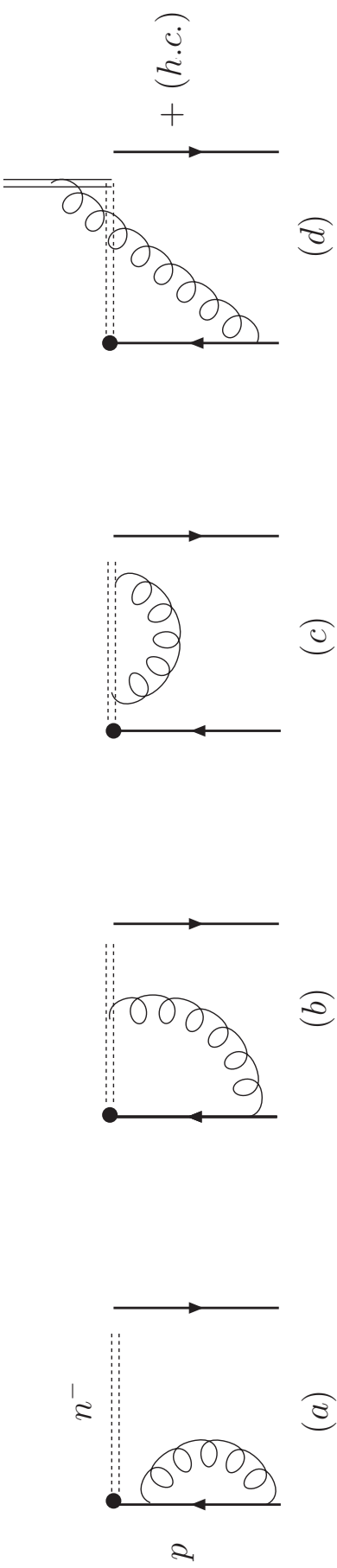


Figure 1: One-gluon exchanges for the TMD PDF: diagrams producing UV divergences. Only (a) and (d) contribute in the light-cone gauge

**Source of the uncertainties and extra divergences:** pole in the gluon propagator

$$D_{\text{LC}}^{\mu\nu}(q) = \frac{1}{q^2} \left[ g^{\mu\nu} - \frac{q^\mu n^{-\nu}}{[q^+]} - \frac{q^\nu n^{-\mu}}{[q^+]} \right]$$

**Possible pole prescription:**

$$d_{\text{PV}}^{\mu\nu}(q) = -(q^\mu n^{-\nu} + q^\nu n^{-\mu}) \frac{1}{2} \left( \frac{1}{q^+ + i\Delta} + \frac{1}{q^+ - i\Delta} \right)$$

$$d_{\text{Adv/Ret}}^{\mu\nu}(q) = -(q^\mu n^{-\nu} + q^\nu n^{-\mu}) \frac{1}{q^+ \mp i\Delta}$$

In what follows, we keep  $\Delta$  small, but finite. Dimensional regularization is used to control UV singularities.

The UV divergent part read:

$$\begin{aligned} \Sigma_{\text{left}}^{UV}(p, \alpha_s; \epsilon) &= \\ &= -\frac{\alpha_s}{\pi} C_F \frac{1}{\epsilon} \left[ -\frac{3}{4} - \ln \frac{\Delta}{p^+} + \frac{i\pi}{2} + i\pi C_\infty \right] + \alpha_s C_F \frac{1}{\epsilon} [iC_\infty] \end{aligned}$$

**prescription dependence is cancelled**

$$C_\infty = \begin{cases} 0, & \text{Advanced: } \frac{1}{[q^+]} = \frac{1}{q^+ - i\delta} \\ -1, & \text{Retarded: } \frac{1}{[q^+]} = \frac{1}{q^+ + i\delta} \\ -\frac{1}{2}, & \text{Principal Value: } \frac{1}{[q^+]} = \frac{1}{2} \left( \frac{1}{q^+ - i\delta} + \frac{1}{q^+ + i\delta} \right) \end{cases}$$

Taking into account (*h.c.*) contributions, one gets total real UV divergent part:

$$\begin{aligned} \Sigma_{\text{tot}}(p, \alpha_s(\mu); \epsilon) &= \Sigma_{\text{left}} + \Sigma_{\text{right}} = \\ &= -\frac{\alpha_s}{4\pi} C_F \frac{2}{\epsilon} \left( -3 - 4 \ln \frac{\Delta}{p^+} \right) \end{aligned}$$

**Anomalous dimension** is defined as

$$\gamma = \frac{1}{2} \frac{1}{Z^{(1)}} \mu \frac{\partial \alpha_s(\mu)}{\partial \mu} \frac{\partial Z^{(1)}(\mu, \alpha_s(\mu); \epsilon)}{\partial \alpha_s}$$

and reads

$$\gamma_{\text{LC}} = \gamma_{\text{smooth}} - \delta\gamma, \quad \gamma_{\text{smooth}} = \frac{3\alpha_s}{4\pi} c_F + O(\alpha_s^2)$$

defect of anomalous dimension

$$\delta\gamma = -\frac{\alpha_s}{\pi} c_F \ln \frac{\Delta}{p^+}$$

contains undesirable  $p^+$ -dependent term which should be removed by a consistent procedure.

Note, that  $\delta\gamma$  is nothing else, but the **cuspidal dimension**:

$$p^+ = (p \cdot n^-)$$

Renormalization of the Wilson operators with obstructions (cusps, self-intersections) requires additional renormalization factor depending on the cusp angle

$$Z_{p^+} = \left[ \langle 0 | \mathcal{P} \exp \left[ ig \int_{\Gamma^\pm} d\zeta^\mu \hat{A}_\mu^a(\zeta) \right] | 0 \rangle \right]^{-1}$$

Choose the integration path as follows

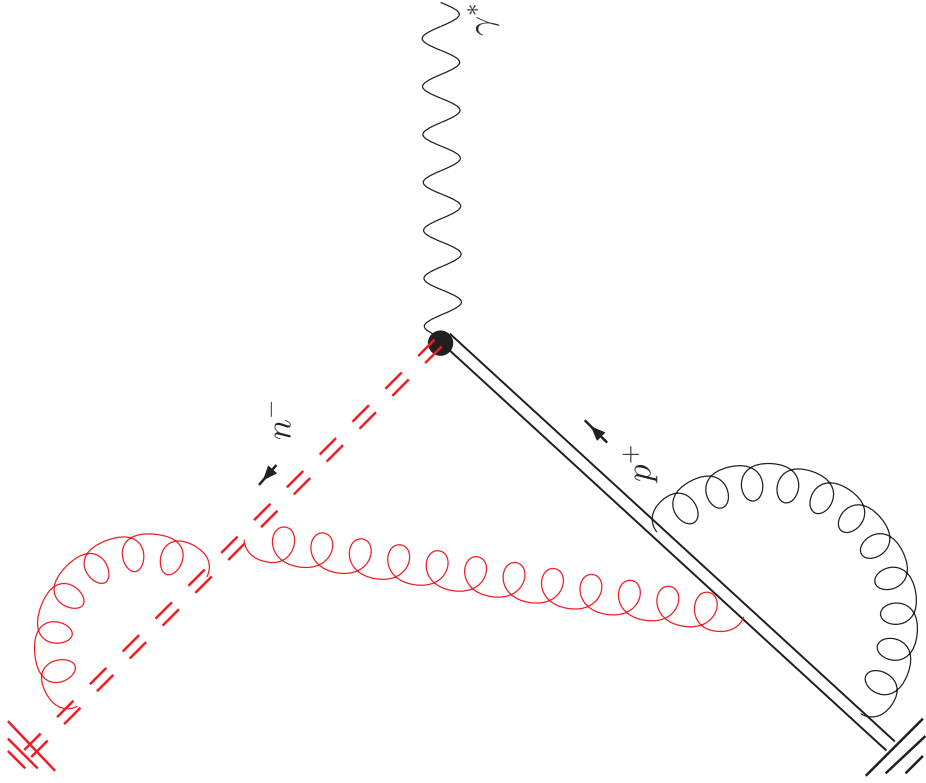


Figure 2: Space-time picture: integration trajectory for the additional cusp-dependent renormalization factor

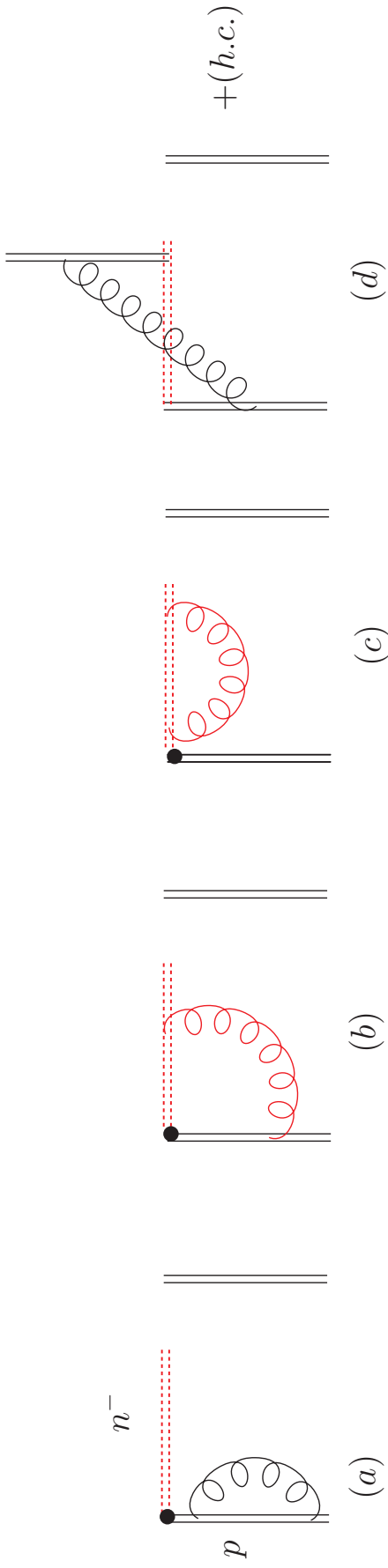


Figure 3: One-gluon exchanges for the generalized multiplicative renormalization factor

The generalized **renormalization constant** reads

$$\hat{Z}_{\text{GEN}} = 1 + \frac{\alpha_s}{4\pi} c_F \frac{2}{\epsilon} \left( -3 - 4 \ln \frac{\Delta}{p^+} + 4 \ln \frac{\Delta}{p^+} \right) = 1 - \frac{3\alpha_s}{4\pi} c_F \frac{2}{\epsilon}$$

so that

$$\frac{1}{2} \mu \frac{d}{d\mu} \ln \hat{Z}_f(\mu, \alpha_s, p^+) = \frac{3\alpha_s}{4\pi} c_F + O(\alpha_s^2)$$

i.e., equal to the anomalous dimension of the corresponding operator with the **smooth gauge connector**, according to the anomalous dimensions sum rule.

ADSR can be formulated in the following form

$$\begin{aligned}
\mathbf{AD} & \frac{1}{2} \int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2} \mathbf{e}^{-ik^+\xi^- + ik_\perp \cdot \xi_\perp} \langle \mathbf{p} | \bar{\psi}(\xi) \gamma^+ [\xi, \mathbf{0}]_{\text{direct link}} \psi(0) | \mathbf{p} \rangle = \\
& = \mathbf{AD} \frac{1}{2} \int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2} \mathbf{e}^{-ik^+\xi^- + ik_\perp \cdot \xi_\perp} \cdot \langle \mathbf{p} | \bar{\Psi}(\xi | \infty) \gamma^+ \Psi(0 | \infty) | \mathbf{p} \rangle \cdot \Phi(\mathbf{p}^+, n^- | 0^-, \mathbf{0}_\perp) \Phi^\dagger(\mathbf{p}^+, n^- | \xi^-, \xi_\perp)
\end{aligned}$$

The definition of the renormalized TMD PDF's reads

$$\begin{aligned}
\hat{f}(x, \mathbf{k}_\perp)_R & = \\
= \hat{Z}_{\text{GEN}} & \frac{1}{2} \int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2} \mathbf{e}^{-ik^+\xi^- + ik_\perp \cdot \xi_\perp} \langle \mathbf{p} | \bar{\Psi}(\xi | \infty) \gamma^+ \Psi(0 | \infty) | \mathbf{p} \rangle
\end{aligned}$$



## Conclusions

- The anomalous dimension of the TMD PDF in the **light-cone gauge** is calculated in 1-loop order. It is shown explicitly, how the transverse semi-infinite gauge link eliminates dependence from the different pole prescriptions in the gluon LC-propagator.
- Anomalous dimension sum rule (ADSR) is introduced, which allows to study the **possible structure of gauge links in the TMD PDF on the basis of their UV renormalization properties**, starting from the smooth connector which provides the simplest way of gauge-invariance restoration and obeys simple and well-known RG properties.
- The **generalized renormalization procedure** is proposed, in analogue to the renormalization of the Wilson exponentials with cusps.