On the Localization of Energy-Momentum and Spin in Classical Field Theory

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In this talk, I will try to partly respond to issues related to the quark/gluon spin discussed in papers of Becattini & Tinti 2011, Teryaev SPIN-2012, Lorcé 2013, Leader & Lorcé 2013. On gravitational gauge theory, I collaborated, amongst others, with Yuri Obukhov (Moscow) and Milutin Blagojević (Belgrade).

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1. Action principle, translational invariance

- SR, Minkowski spacetime $M_4$, Lorentz metric $g_{ij} = \delta_{ij}$ := diag($+\ldots$), coordinates $x^i$, $i,j,k,\ldots = 0,1,2,3$; here Cartesian coo., matter field $\Psi$, could be a scalar, Weyl, Dirac, Maxwell, Proca, Rarita-Schwinger, Fierz-Pauli field etc.). Isolated material system with 1st order action (see Landau-Lifshitz, Corson): $W_{\text{mat}} := \frac{1}{c} \int d\Omega L(\Psi, \partial\Psi)$.

- Invariance under 4 transl.: $x'{}^i = x^i + a^i$. Noether theorem and $\frac{\delta L}{\delta \Psi} = 0$, $\partial_j \Sigma^{i}{}_{j} = 0$, $\Sigma^{i}{}_{j} := L \delta^{i}{}_{j} - \frac{\partial L}{\partial \partial_i \Psi} \partial_j \Psi$

  canonical energy-momentum tensor of type $(\frac{1}{1})$, Noether energy-momentum (or momentum current density), 16 indep. comps., Whittaker: Minkowski’s most important discovery; is asymmetric a priori

- Physical components of components of $\Sigma^{i}{}_{j}$ (a,b=1,2,3):

$$\Sigma = \begin{pmatrix} \Sigma^0{}^0 &=& -\text{energy d.} \\ \Sigma^a{}^0 &=& -\text{(momentum d.)/}c \end{pmatrix} \quad \Sigma^0{}^b = (\text{energy flux d.}) \times c \quad \Sigma^a{}^b = -\text{mom. flux d.}$$

$$\begin{pmatrix} \Sigma^{i}{}_{j} \end{pmatrix} = h \ell^{-3} t^{-1} \times \begin{pmatrix} 1 \\ (\ell/t)^{-1} \\ \ell/t \\ 1 \end{pmatrix}$$

  here $h := \text{[action]}$, $\ell := \text{[length]}$, $t := \text{[time]}$, method of Dorlego-Schouten. Lagrangian: $[L] = \frac{h}{\ell^3 t}$, $\frac{h}{t} = \text{[energy]}$. 

Note, the spatial components \([T^a_b] = (mv)u \frac{1}{\ell^3} = \frac{E}{\ell^3} = \frac{f}{\ell^2} = \text{stress}\), see Lorentz’s interpretation of the Maxwell stress.

Semiclassical Weyssenhoff ansatz for a fluid:
\[
[T^i_j] = [p_i] [v^j], \quad \text{observe natural index positions!}
\]

If \(p_i = \rho \ g_{ik} v^k\), then \(T^i_j = T^j_i\) symm.. Is not the case for spin fluids

Superfluid \(^3\)He in the A-phase, is as spin fluid (Lee, Osheroff, Richardson 1972). Take the angular momentum law, see D. Vollhardt and P. Wölfle, *The Superfluid Phases of Helium 3*, London 1990, p.427: The antisym. piece of stress reads:

\[
\epsilon_{ijk} \Pi^i_{jk} = -\left(\frac{\partial}{\partial t} + v_n \cdot \nabla\right) (t_0 l_i) + \nabla_j B_{ji} - \nabla_j \left\{\frac{\hbar}{2m} g_{s,j} l_i + \hat{l} \times T \frac{\partial s}{\partial (\nabla_j \hat{l})}_i\right\}
\]

\(v_n = \text{velocity of normal fluid, } t_0 = \text{modulus of intrinsic angular momentum } t = t_0 \hat{l}, \ l_i = \text{preferred direction of A-phase order parameter,}\)
\(s = \text{entropy density, } T = \text{temperature, } g_s = \text{momentum density of superfluid component; this is an irrefutable proof that asymmetric stress tensors exist in nature (see Pascal-Euler-Cauchy-Boltzmann-Voigt-E. & F.Cosserat-E.Cartan...)}\)
In a spacetime with metric, as in the $M_4$, we can decompose $\mathcal{T}_{ij}$ irreducible wrt the Lorentz group:

$$\mathcal{T}_{ij} = \mathcal{T}_{ij} + \mathcal{T}_{[ij]} + \frac{1}{4}g_{ij}\mathcal{T}^k_k,$$

$$16 = 9 \text{(sym.tracefree)} \oplus 6 \text{(antisym.)} \oplus 1 \text{(trace)},$$

$$\mathcal{T}_{ij} := \mathcal{T}_{(ij)} - \frac{1}{4}g_{ij}\mathcal{T}^k_k,$$

Bach parentheses

$$(ij) := \frac{1}{2}\{i + j\}, \quad [ij] := \frac{1}{2}\{i - j\}.$$ 

In electromagnetism, only $\mathcal{T}_{ij}$ survives (9 components), since it is massless, that is, $\mathcal{T}^k_k = 0$, and carries helicity, but no (Lorentz) spin, i.e., $\mathcal{T}_{[ij]} = 0$, see below.

Classical ideal (perfect, Euler) fluid of GR ($\rho = \text{mass/energy density}, \ p = \text{pressure}, \ u_i = \text{velocity of fluid}$):

$$\mathcal{T}_{ij} = (\rho + p)u_iu_j - pg_{ij}, \quad \mathcal{T}_{[ij]} = 0, \quad \mathcal{T}^k_k = \rho - 3p.$$ 

Where took Einstein the symmetry of the energy-momentum tensor from? Einstein (The Meaning of Relativity, 1922, p.50) discussed the symmetry of the energy-momentum tensor of Maxwell’s theory. Subsequently, he argued: “We can hardly avoid making the assumption that in all other cases, also, the space distribution of energy is given by a symmetrical tensor, $T_{\mu\nu}, ...$” This is hardly a convincing argument if one recalls that the Maxwell field is massless.
2. Lorentz invariance

- Invariance under 3+3 Lorentz transf.: \( x'{}^i = x^i + \omega^{ij} x_j \), with \( \omega^{(ij)} = 0 \)

Noether theorem and \( \frac{\delta L}{\delta \Psi} = 0 \),

\[
\partial_k \left( \mathcal{S}_{ij}{}^k + \frac{1}{2} x_i \mathcal{T}_{jk} - \frac{1}{2} x_j \mathcal{T}_{ik} \right) = 0,
\]

\( \mathcal{S}_{ij}{}^k := - \frac{\partial L}{\partial \partial_k \Psi} f_{ij} \Psi \)

Noether spin \( \mathcal{S}_{ij}{}^k = - \mathcal{S}_{ji}{}^k \), the spin current density, is a tensor of type \( (\frac{1}{2}) \), see also Einstein-de Haas effect (1915).

- Physical components of \( \mathcal{S}_{ij}{}^k \) (a,b=1,2,3; here mom.=moment!):

\[
\begin{pmatrix}
\mathcal{S}_{ij}{}^k \\
\mathcal{S}_{ij}{}^k
\end{pmatrix} = \begin{pmatrix}
\mathcal{S}_{00}{}^0 = \text{en.-dipole mom. d.} & \mathcal{S}_{0b}{}^c = (\text{en.-dipole mom. flux d.})/c \\
\mathcal{S}_{ab}{}^0 = (\text{spin density}) \times c & \mathcal{S}_{ab}{}^c = \text{spin flux density}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\mathcal{S}_{i}{}^{jk} \\
\mathcal{S}_{i}{}^{jk}
\end{pmatrix} = h \ell^{-2} t^{-1} \times \begin{pmatrix}
1 & \ell/t \\
(\ell/t)^{-1} & 1
\end{pmatrix}
\]

- \( [\mathcal{S}_{a}{}^{bc}] = (m v \ell) v \frac{1}{\ell^3} = \frac{f \ell}{\ell^2} = \text{moment stress} \), known from Voigt (1887) and from the Cosserat brothers (1909), from micropolar media,...
Convective Weyssenhoff ansatz (distinguish spin current from spin):

\[ \mathcal{S}_{ij}^k = \mathfrak{s}_{ij} \mathfrak{v}^k = -\mathcal{S}_{ji}^k \]

spin curr. d. spin velocity

Irreducible decomposition:

\[ \mathcal{S}_{ij}^k = \text{TEN} \mathcal{S}_{ij}^k + \text{VEC} \mathcal{S}_{ij}^k + \text{AX} \mathcal{S}_{ij}^k \]

24 = 16 \oplus 4 \oplus 4

with \( \text{AX} \mathcal{S}_{ijk} := \mathcal{S}_{[ijk]} \) and \( \text{VEC} \mathcal{S}_{ij}^k := \frac{2}{3} \mathcal{S}_{[i|\ell]j} \delta^k_{[j]} \)

For the Dirac field: \( D \mathcal{S}_{ijk} = D \mathcal{S}_{[ijk]} \). Thus, only \( \text{AX} D \mathcal{S}_{ijk} \neq 0 \) and we can introduce the spin flux vector (Dirac field is highly symmetric):

\[ S^i := \frac{1}{3!} \epsilon^{ijkl} \mathcal{S}_{jkl} \sim \text{(spin flux density 1, spin density 3)} \]

Back to the angular momentum law. Differentiate and apply \( \partial_k \mathcal{Z}_i^k = 0 \):

\[ \partial_k \left( \mathcal{S}^{ijk} + x^{[i} \mathcal{Z}^{j]k} \right) = 0 \quad \Rightarrow \quad \partial_k \mathcal{S}^{ijk} - \mathcal{Z}^{[ij]} = 0 \]

The boxed version can be generalized to Riemann(-Cartan) spacetimes directly, see below. If \( \mathcal{S}^{ijk} = 0 \), then \( \mathcal{Z}^{[ij]} = 0 \) (symmetric energy-momentum tensor), but not necessarily vice versa.
3. Poincaré invariance

Thus, Poincaré invariance yields the $4 + 6$ conservation laws

$$\partial_k \mathcal{T}_i{}^k = 0 \quad \text{(energy-momentum)}$$
$$\partial_k \mathcal{S}_{ij}{}^k - \mathcal{T}_{[ij]} = 0 \quad \text{(angular momentum)}$$

Semi-direct product structure! Lie algebra of the Poincaré group:

$$[P_i, P_j] = 0,$$
$$[L_{ij}, P_k] = g_k[iP_j], \quad \text{(transl. and Lorentz transf. mix in } \mathcal{S}_{ijk} + x[i\mathcal{T}_j]_k \text{)}$$
$$[L_{ij}, L_{k\ell}] = g_k[i L_j]_\ell - g_\ell[i L_j]_k.$$ 

The Casimirs $P^2$ (mass square) and $W^2$ (spin square), with

$$W_i := -\frac{1}{2} \epsilon_{ijkl} P^j P^l$$

correspond to $\mathcal{T}_i{}^k$ and $\mathcal{S}_{ij}{}^k$.

The rigid Poincaré group of SR can be gauged $\Rightarrow$ Poincaré gauge theory of gravity (PG), yielding, in particular, a Riemann-Cartan spacetime. The conservation laws generalize to

$$\nabla_k \mathcal{T}_i{}^k = \text{torsion } C_{ik\ell} \mathcal{T}_\ell{}^k + R_{iklm} \mathcal{S}_{lm}{}^k,$$

$$\nabla_k \mathcal{S}_{ij}{}^k - \mathcal{T}_{[ij]} = 0.$$

Here $\nabla_k := \nabla_k + C_{k\ell} \mathcal{T}_\ell{}^k$. General relativity (GR) is the subcase for $\mathcal{S}_{ij}{}^k = 0$. Otherwise, the viable Einstein-Cartan(-Sciama-Kibble) theory (EC) with $C_{ij}{}^k \neq 0$. In GR and in EC the Noether theorems for transl. + Lorentz can be mapped to the 1st and 2nd Bianchi identities.
During the last few decades, gravity, one of the fundamental forces of nature, has been formulated as a gauge field theory of the Weyl–Corman–Yang–Mills type. The resulting theory, the Poncaré gauge theory of gravity, encompasses Einstein's gravitational theory as its degenerate limit of gravity in vacuo. In general, the spacetime structure is enriched by Cartan's theory and the new theory can accommodate fermionic matter and its spin in a perfectly natural way.

The present reprint volume contains articles from the most prominent proponents of the theory, and is supplemented by detailed commentaries from Milutin Blagojević and Friedrich W. Hehl. This guide first starts from special relativity and leads, in the first part, to general relativity and its gauge type extensions. In Weyl and Cartan, subsequent steps are the theories of Yang-Mills and Einstein and, as a particular milestone, the theory of Einstein and Riemann. Later, the Poncaré gauge theory and its generalizations are explored and specific topics, such as the Hamiltonian form and exact solutions, are studied.

This guide to the literature on gauge theories of gravity is intended to be a stimulating and unique introduction to the field of classical gauge theories of gravity for graduate and advanced undergraduate students of theoretical and mathematical physics, in particular for those studying gravity and/or elementary particles, and for other interested readers.

**Gauge Theories of Gravitation**

* A Reader with Commentaries

Milutin Blagojević • Friedrich W. Hehl editors

foreword by

T. W. B. Kibble, FRS

Imperial College Press
4. On exterior calculus, the electromagnetic/gluon energy-momentum, and on the Dirac field

- Introduce the calculus of exterior differential forms in order to streamline the Lagrange-Noether formalism, see Hehl et al. Phys.Rep. 1995. In such a formalism one works with an orthonormal coframe (tetrad) \( \vartheta^\alpha = e_i^\alpha \, dx^i \), a Lorentz connection \( \Gamma^{\alpha\beta} = \Gamma^i_{\alpha\beta} \, dx^i = -\Gamma^\beta_{\alpha} \), and the fields are exterior forms (0-forms, 1-forms,..., 4-forms) with values in the algebra of some Lie group. The electromagnetic potential is a 1-form \( A = A_i \, dx^i \), the field strength a 2-form \( F := dA = \frac{1}{2} F_{ij} \, dx^i \wedge dx^j \), for details see H. & Yuri Obukhov, *Foundations of Electrodynamics*, Birkhäuser, Boston (2003).

- Translation from tensor to exterior calculus: Energy-momentum 3-form \( T_\alpha = T_\alpha^\gamma \, \vartheta^\gamma = \delta L_{\text{mat}} / \delta \vartheta^\alpha \), spin 3-form \( S_{\alpha\beta} = S_{\alpha\beta}^\gamma \, \vartheta^\gamma = \delta L_{\text{mat}} / \delta \Gamma_{\alpha\beta} \)

- Maxwell’s vacuum field \( A(x) \) is a 1-form (a geometrical object independent of coordinates and frames). As such, it has Lorentz-spin 0, but helicity \( \pm 1 \) \( \implies \) the analogous is true for the gluon field. As a consequence, the canonical (i.e. Noether) energy-momentum 3-form is symmetric and gauge invariant directly.

- **Thesis 1:** The energy-momentum current 3-form of the Maxwell/gluon field \( F = DA \) is given by the Minkowski expression

\[
T_\alpha = \frac{1}{2} [F \wedge (e_\alpha] * F') - * F \wedge (e_\alpha] F')] \quad \text{or} \quad T_i^j = \frac{1}{4} \delta_i^j F_{kl} F^{kl} - F_{ik} F^{jk}.
\]
The second example, Dirac field in exterior calculus for illustration:

\[ L_D = \frac{i}{2} (\overline{\Psi}^* \gamma \wedge D\Psi + \overline{D\Psi} \wedge *\gamma\Psi) + *m\overline{\Psi}\Psi \]

with \( \gamma := \gamma_\alpha \theta^\alpha \) and \( \gamma_{(\alpha \gamma_\beta)} = \delta_{\alpha \beta} 1_4 \). The 3-forms of the canonical momentum and spin current densities (\( D_\alpha := e_\alpha \lrcorner D \)):

\[
\begin{align*}
\mathcal{I}_\alpha &= \frac{i}{2} (\overline{\Psi}^* \gamma \wedge D_\alpha \Psi + \overline{D_\alpha \Psi} \wedge *\gamma\Psi), \\
\mathcal{S}_{\alpha\beta} &= \frac{1}{4} \theta_\alpha \wedge \theta_\beta \wedge \overline{\Psi} \gamma_5 \Psi
\end{align*}
\]

In Ricci calculus \( \mathcal{S}_{\alpha\beta\gamma} = \mathcal{S}_{[\alpha\beta\gamma]} = \frac{1}{4} \epsilon_{\alpha\beta\gamma\delta} \overline{\Psi} \gamma_5 \gamma^\delta \Psi \) and \( t_{\alpha\beta} = \mathcal{I}_{(\alpha\beta)} \) (Tetrode), see subsequent slide. These are the inertial currents (and thus the gravitational currents) of the classical Dirac field. A decomposition of \( (\mathcal{I}_\alpha, \mathcal{S}_{\alpha\beta}) \) à la Gordon, yields the gravitational moment densities of the Dirac field (arXiv:gr-qc/9706009); is a special case of relocalization, see below.

**Thesis 2:** The canonical (Noether) energy-momentum and the canonical (Noether) spin current 3-forms of a Dirac/quark field are given by the expressions (\( \star \)) and (\( \star \star \)), respectively.
5. Relocalization of energy-momentum and spin

- Canonical currents are not uniquely defined. Relocalization

\[
\hat{T}_{ij} := \mathcal{T}_{ij} - \partial_t X_{ij}^l, \\
\hat{\mathcal{S}}_{klj} := \mathcal{S}_{klj} - 2X_{[kl]}^j + \partial_l Y_{kl}^{ji}.
\]

Still,

\[
\partial_j \hat{T}_{ij} = 0, \\
\partial_j \hat{\mathcal{S}}_{klj} - \hat{\mathcal{T}}_{[kl]} = 0.
\]

Arbitrary \(X_{ij}^l = -X_{lj}^i, Y_{kl}^{ji} = -Y_{kl}^{ij} = -Y_{lk}^{ji}\) (H. 1976). Integrated total energy-mom. and ang. mom. remain the same. However, “relocalization invariance” is not a generally valid physical principle.

- Belinfante relocalization (1939) is a specialization: Require \(\hat{\mathcal{S}}_{klj} = 0\). Resolve with respect to \(X_{ij}^l\). Then,

\[
X_{ij}^l = -\frac{1}{2} \left( \mathcal{S}_{ij}^l + \mathcal{S}_{li}^j - \mathcal{S}_{lj}^i \right) - \frac{1}{2} \partial_n \left( Y_{ij}^l n + Y_{lj}^i n - Y_{ij}^l n \right),
\]

and the relocalized e.-m., \(\text{Bel} t_{ij} := \hat{T}_{ij}\), with \(\hat{\mathcal{S}}_{klj} = 0, Y_{ij}^{kl} = 0\), reads \(\text{Bel} t_{ij} = \mathcal{T}_{ij} + \frac{1}{2} \partial_k \left( \mathcal{S}_{jk}^i + \mathcal{S}_{ki}^j - \mathcal{S}_{ij}^k \right)\) with \(\partial_j \text{Bel} t_{ij} = 0, \text{Bel} t_{[kl]} = 0\).

- The Gordon relocalization, mentioned above, differs from the Belinfante relocalization.
6. Dynamic Hilbert energy-momentum in general relativity

- How can we choose amongst the multitude of relocalized energy-momentum tensors, and how can we find the correct physical one? The Belinfante recipe was to kill $\mathcal{S}_{[kl]}$. This does not yield a unique relocalized tensor (we had to require additionally $Y_{ij}^{kl} = 0$).

- Hilbert defined already in 1915 the dynamic energy-momentum as the response of the matter Lagrangian to the variation of the metric:

$$\mathcal{H}_{ij} := 2\frac{\delta \mathcal{L}_{\text{mat}}(g, \Psi, \nabla \Psi)}{\delta g_{ij}}.$$  

$g^{ij}$ (or its reciprocal $g_{kl}$) is the gravitational potential in GR. The matter Lagrangian is supposed to be minimally coupled to $g^{ij}$, in accordance with the equivalence principle. Only in gravitational theory, in which spacetime can be deformed, we find a real local definition of the material energy-momentum tensor (see Weyl).

- The Hilbert definition is analogous to the relation from elasticity theory

$$\text{stress} \sim \frac{\delta(\text{elastic energy})}{\delta(\text{strain})}.$$  

Recall that strain $\varepsilon^{ab} := \frac{1}{2} (\text{def} g^{ab} - \text{undef} g^{ab})$, see LL. Even the factor 2 is reflected in the Hilbert formula.
Rosenfeld (1940) has shown, via Noether type theorems, that the Belinfante tensor $^{\text{Bel}}t_{ij}$, derived within SR, coincides with the Hilbert tensor $^{\text{Hi}}t_{ij}$ of GR. Thus, the Belinfante-Rosenfeld recipe yields

**Thesis 3:** In the framework of GR, the Hilbert energy-momentum tensor

\[
^{\text{Hi}}t_{ij} = ^{\text{Bel}}t_{ij} = \Xi_{ij} + \frac{1}{2} \partial_k \left( S^{jk}i + S^{i}kj - S^{ik}j \right)
\]

(localizes the energy-momentum distribution correctly; here $(\Xi_{ij}, S^{i}j_k)$ are the canonical Noether currents.

The Rosenfeld formula ($\ast$) identifies the Belinfante with the Hilbert tensor. In other words, the Belinfante tensor provides the correct source for Einstein’s field equation.

As long as we accept GR as the correct theory of gravity, the localization of energy-momentum and spin of matter is solved. This state of mind is conventionally kept till today by most theoretical physicists. One should note that the spin of matter has a rather auxiliary function in this approach. After all, the spin of the Hilbert-Belinfante-Rosenfeld tensor vanishes.

However, the Poincaré gauge theory of gravity (PG; Sciama, Kibble 1961), in particular the viable Einstein-Cartan theory (EC) with the curvature scalar as gravitational Lagrangian, has turned the Rosenfeld formula ($\ast$) upside down...
7. Dynamic Sciama-Kibble spin in Poincaré gauge theory

- Gauging of the Poincaré group, gauge potentials orthonormal coframe and Lorentz connection \( \partial^{\alpha} = e_{i}^{\alpha} \, dx^{i}, \Gamma^{\alpha\beta} = \Gamma_{i}^{\alpha\beta} \, dx^{i} = -\Gamma^{\beta\alpha} \) → Poincaré gauge theory of gravity (PG) with a Riemann-Cartan space with Cartan’s torsion and with Riemann-Cartan curvature, respectively:

\[
C_{ij}^{\alpha} := D_{[i} e_{j]}^{\alpha}, \quad R_{ij}^{\alpha\beta} := \text{"D"}_{[i} \Gamma_{j]}^{\alpha\beta} \quad \text{(or} \quad C^{\alpha} = D \partial^{\alpha}, \quad R^{\alpha\beta} = \text{"D"} \Gamma^{\alpha\beta} \text{)}.
\]

- The currents are defined by variations with respect to the potentials:

\[
SK \zeta_{\alpha}^{i} = \frac{\delta \mathcal{L}_{\text{mat}}(e, \Gamma, \Psi, D \Psi)}{\delta e_{i}^{\alpha}}, \quad SK \mathcal{G}_{\alpha\beta}^{i} = \frac{\delta \mathcal{L}_{\text{mat}}(e, \Gamma, \Psi, D \Psi)}{\delta \Gamma_{i}^{\alpha\beta}}.
\]

- This Sciama-Kibble definition of the spin (1961) is only possible in the Riemann-Cartan spacetime of PG. It is analogous to the relation

\[
\text{moment stress} \sim \delta(\text{elastic energy}) / \delta(\text{contortion})
\]

in a Cosserat type medium, contortion is a “rotational strain”, see H & Obukhov, Elie Cartan’s torsion in geometry and in field theory, an essay, arXiv:0711.1535.

- Applic. of Noether identities yields, after a lot of algebra, the final result

\[
SK \zeta_{\alpha}^{i} = \zeta_{\alpha}^{i}, \quad SK \mathcal{G}_{\alpha\beta}^{i} = \mathcal{G}_{\alpha\beta}^{i}.
\]
Thesis 4: The dynamically defined currents à la Sciama-Kibble coincide with the canonical Noether currents of classical field theory. Within PG, the quark spin is determined by the canonical energy-momentum current.—

This is in marked contrast to the doctrine in the context of GR.

We express the canonical energy-momentum tensor in the Hilbert one:

\[
\text{SK} \mathcal{T}_{\alpha}^i = \text{Hi} t_{\alpha}^i - \frac{1}{2} * D_k \left( \mathcal{S}_{\alpha}^{ik} - \mathcal{S}_{ik}^\alpha + \mathcal{S}_{k}^{\alpha i} \right),
\]

\[
\text{SK} \mathcal{S}_{\alpha\beta}^i = \mathcal{S}_{\alpha\beta}^i.
\]

The new Rosenfeld type formula (**) reverses its original meaning in (*). Within PG, the canonical tensor represents the energy-momentum distribution of matter and the (sym)metric Hilbert tensor now plays an auxiliary role. Moreover, we are now provided with a dynamic definition of the canonical spin tensor. In GR, the spin was only a kinematic quantity floating around freely.

These results on the correct distribution of material energy-momentum and spin in the framework of PG are independent of a specific choice of the gravitational Lagrangian.

However, if we choose the RC curvature scalar as a gravitational Lagrangian, we arrive at the Einstein-Cartan(-Sciama-Kibble) theory of gravitation, which is a viable theory of gravity competing with GR.
8. An algebra of the momentum and the spin currents?

- I discussed exclusively classical field theory. Can we learn something for a corresponding quantization of gravity? Our classical analysis has led us to the gravitational currents $\mathcal{F}_\alpha$ and $\mathcal{G}_{\alpha\beta}$. They represent the sources of gravity.

- In strong and in electroweak interaction, before the standard model had been worked out, one started with the current algebra of the phenomenologically known strong and the electroweak currents (Gell-Mann 1961, see also T.Y. Cao, *From Current Algebra to Quantum Chromodynamics*, Cambridge 2010).

- Schwinger (1963) studied, e.g., the equal time commutators of the components of the Hilbert e.-m. tensor. Should one try to include also the spin tensor components and turn to the canonical tensors?

- In the Sugawara model (1968), *A field theory of currents*, 8 vector and 8 axial vector currents for strong interaction are introduced and a symmetric e.-m. current expressed bilinearly in terms of these currents. Now that we have good arguments that the gravitational currents are $\mathcal{F}_\alpha$ and $\mathcal{G}_{\alpha\beta}$, one may want to develop a corresponding current algebra by determining the equal time commutator of these currents.....
Appendix. Extra dilation invariance and improved energy-momentum current

- Matter Lagrangian is assumed to be, in addition to Poincaré invariance, scale invariant, then we have the canon. Noether dilation current (3-form)

\[ \Delta = \Delta^\alpha \eta_\alpha, \quad \Delta := w \Psi \wedge \frac{\partial L}{\partial D \Psi} \]

(here \( \eta_\alpha := e_\alpha \lceil \eta \), with frame \( e_\alpha \) and volume 4-form \( \eta \)). The dilation current \( \Delta^\alpha \) is somewhat analogous to the electric current \( J^\alpha \) (1-parameter gauge transformation).

\( w \) is weight of scale transformation:

\[ \Psi(x) \to \Psi'(x') = (e^\omega)^w \Psi(e^\omega x). \]

Noether law: \( D \Delta + \vartheta^\alpha \wedge \mathcal{I}_\alpha = D(\Delta + x^\alpha \wedge \mathcal{I}_\alpha) = 0 \).

- Three types of Noether theorems (Poincaré \( \otimes \) dilation):

\[ D \mathcal{I}_\alpha = 0 \quad (4 \text{ cons. momentum currents}), \]

\[ D \mathcal{G}_{\alpha\beta} + \vartheta_{[\alpha} \wedge \mathcal{I}_{\beta]} = 0 \quad (6 \text{ cons. angular momentum currents}), \]

\[ D \Delta + \vartheta^\alpha \wedge \mathcal{I}_\alpha = 0 \quad (1 \text{ cons. dilation current}) \]

(in the literature, intrinsic and orbital dil. current are not cleanly defined).
Take the superpotential 2-forms $M_\alpha, Y_{\alpha\beta}, Z$ such that

$$\widehat{\mathcal{I}}_\alpha(M) = \mathcal{I}_\alpha - DM_\alpha,$$

$$\widehat{\mathcal{G}}_{\alpha\beta}(M, Y) = \mathcal{G}_{\alpha\beta} - \vartheta_{[\alpha \wedge M_\beta]} - DY_{\alpha\beta},$$

$$\widehat{\Delta}(M, Z) = \Delta - \vartheta^\alpha \wedge M_\alpha - DZ.$$

The hatted quantities fulfill again the $4 + 6 + 1$ conservation laws. The total charges remain the same.

For the improved energy-momentum tensor $\mathcal{T}_\alpha$ of Chernikov-Tagirov (1968) and Callan-Coleman-Jackiw (1970),

$$\mathcal{T}_\alpha := \widehat{\mathcal{I}}_\alpha(M) \quad \text{for} \quad \widehat{\mathcal{G}}_{\alpha\beta}(M, Y) \overset{!}{=} 0 \quad \text{and} \quad \widehat{\Delta}(M, Z) \overset{!}{=} 0,$$

we require additionally that its trace vanishes ($\rightarrow$ soft pions):

$$\vartheta^\alpha \wedge \mathcal{T}_\alpha = \vartheta^\alpha \wedge \widehat{\mathcal{I}}_\alpha + D\Delta - DDZ \overset{!}{=} 0.$$

This can be achieved and, accordingly, the improved energy-momentum tensor is symmetric, traceless, and divergencefree:

$$\begin{align*}
\vartheta_{[\alpha \wedge \mathcal{T}_\beta]} &= 0, \\
\vartheta^{\alpha} \wedge \mathcal{T}_\alpha &= 0 \\
D \mathcal{T}_\alpha &= 0.
\end{align*}$$

(In Ricci calculus: $\mathcal{T}_{[\alpha\beta]} = 0$, $\mathcal{T}_{\gamma\gamma} = 0$, $\nabla_\beta \mathcal{T}_{\alpha\beta} = 0$.)

$\mathcal{I}_\alpha, \mathcal{G}_{\alpha\beta}$, and, for massless fields, additionally $\Delta$ are the inertial (and thus the gravitational) currents.