N=4 HARMONIC SUPERSPACE

B.M. ZUPNIK

Harmonic superspaces for the group $SU(4)$:

E. Ivanov, S. Kalitzin, Nguen Ai Viet, V. Ogievetsky, *Harmonic superspaces of extended supersymmetry. The calculus of harmonic variables,* **J. Phys. A 18 (1985) 3433.**

G.G. Hartwell, P.S. Howe, *(N, p, q) harmonic superspace*, **Int. J. Mod. Phys. A 10 (1995) 3901, hep-th/9412147.**

B.M. Zupnik, *$SU(4)$ harmonic superspace and supersymmetric gauge theory*, **arxiv: 1406.7602 (hep-th)**

$SU(4)/SU(2)\times SU(2)\times U(1)$ harmonic coset space

Harmonic variables

We use the harmonics parameterizing the 8 dimensional coset space $\mathcal{U}_8 = G/H$

$$u_k^{+a} \ , \qquad u_k^{-\hat{a}}$$

where $G = SU(4)$, $H = SU_L(2) \times SU_R(2) \times U(1)$, $k = 1,2,3,4$ is the spinor index of $SU(4)$, $a = 1,2$ describe the $SU_L(2)$ doublet and $\hat{a} = \hat{1},\hat{2}$ corresponds to the $SU_R(2)$ doublet, and $\pm$ are charges of $U(1)$. They form an $SU(4)$ matrix and are covariant under the independent G and H transformations.

The Hermitian conjugation of the harmonic matrix $SU(4)$ gives us the conjugated harmonics

$$\overline{u_k^{+a}} = \bar{u}_a^{-k} \ , \qquad \overline{u_k^{-\hat{a}}} = \bar{u}_{\hat{a}}^{+k}$$

The harmonics satisfy the following basic relations :

$$u_i^{+a}\bar{u}_b^{-i} = \delta_b^a \ , \qquad u_i^{+a}\bar{u}_{\hat{b}}^{+i} = 0$$

$$u_i^{-\hat{a}}\bar{u}_{\hat{b}}^{+i} = \delta_{\hat{b}}^{\hat{a}}, \qquad u_i^{-\hat{a}}\bar{u}_b^{-i} = 0$$

$$u_i^{+a}\bar{u}_a^{-k} + u_i^{-\hat{a}}\bar{u}_{\hat{a}}^{+k} = \delta_i^k$$

$$\varepsilon^{ijkl}u_i^{+a}u_j^{+b}u_k^{-\hat{a}}u_l^{-\hat{b}} = \varepsilon^{ab}\varepsilon^{\hat{a}\hat{b}} \ .$$

The special $\sim$-conjugation for these harmonics and other quantities preserves $U(1)$ charge and changes indices of two $SU(2)$ subgroups

$$(u_i^{+a})^\sim = -\bar{u}_{\hat{a}}^{+i} \ , \quad (u_i^{-\hat{a}})^\sim = -\bar{u}_a^{-i} \ , \quad (\varepsilon^{ab}) = \varepsilon_{\hat{b}\hat{a}}$$

$$(\bar{u}_a^{-i})^\sim = -u_i^{-\hat{a}} \ , \quad (\bar{u}_{\hat{a}}^{+i})^\sim = -u_i^{+a}$$

it is consistent with the basic harmonic relations.

The $U(1)$ neutral $SU(4)$-invariant harmonic derivatives

$$\partial_b^a = u_i^{+a}\partial_b^{-i} - \bar{u}_b^{-i}\bar{\partial}_i^{+a} - \frac{1}{2}\delta_b^a(u_i^{+f}\partial_f^{-i} - \bar{u}_f^{-i}\bar{\partial}_i^{+f})$$

$$\hat{\partial}_{\hat{b}}^{\hat{a}} = u_i^{-\hat{a}}\partial_{\hat{b}}^{+i} - \bar{u}_{\hat{b}}^{+i}\bar{\partial}_i^{-\hat{a}} - \frac{1}{2}\delta_{\hat{b}}^{\hat{a}}(u_i^{-\hat{f}}\partial_{\hat{f}}^{+i} - \bar{u}_{\hat{f}}^{+i}\bar{\partial}_i^{-\hat{f}})$$

$$\partial^0 = u_i^{+f}\partial_f^{-i} - \bar{u}_f^{-i}\bar{\partial}_i^{+f} + \bar{u}_{\hat{f}}^{+i}\bar{\partial}_i^{-\hat{f}} - u_i^{-\hat{f}}\partial_{\hat{f}}^{+i}$$

contain partial harmonic derivatives

$$\partial_b^{-i} = \frac{\partial}{\partial u_i^{+b}}, \quad \partial_{\hat{b}}^{+i} = \frac{\partial}{\partial u_i^{-\hat{b}}}, \quad \bar{\partial}_i^{-\hat{b}} = \frac{\partial}{\partial \bar{u}_{\hat{b}}^{+i}}, \quad \bar{\partial}_i^{+b} = \frac{\partial}{\partial \bar{u}_b^{-i}}$$

These neutral harmonic derivatives satisfy the $SU_L(2) \times SU_R(2) \times U(1)$ Lie algebra.

The eight charged coset harmonic derivatives

$$\partial^{++a}_{\hat b} = u^{+a}_i \partial^{+i}_{\hat b} - \bar u^{+i}_{\hat b} \bar\partial^{+a}_i$$

$$\partial^{--\hat a}_{b} = u^{-\hat a}_i \partial^{-i}_{b} - \bar u^{-i}_{b} \bar\partial^{-\hat a}_i$$

satisfy the $SU(4)$ Lie algebra

The special Hermitian conjugation is defined for the harmonic derivatives

$$(\partial^{+i}_{\hat b})^\dagger = \bar\partial^{+b}_i, \quad (\partial^{-i}_{b})^\dagger = \bar\partial^{-\hat b}_i$$

$$(\partial^{a}_{b})^\dagger = \hat\partial^{\hat b}_{\hat a}, \quad (\partial^0)^\dagger = -\partial^0, \quad (\partial^{++a}_{\hat b})^\dagger = \partial^{++b}_{\hat a},$$

$$(\partial^{--\hat a}_{b})^\dagger = \partial^{--\hat b}_{a},$$

$$(\partial^{++a}_{\hat b} f)^\sim = [\partial^{++a}_{\hat b}, f]^\dagger = -\partial^{++b}_{\hat a} \tilde f,$$

$$(\partial^{a}_{b} f)^\sim = [\partial^{a}_{b}, f]^\dagger = -\hat\partial^{\hat b}_{\hat a} \tilde f$$

where f and $\tilde f$ are the $\sim$-conjugated harmonic functions.

Irreducible harmonic combinations

We study irreducible in $SU(4)$ and $SU_L(2) \times SU_R(2)$ group indices combinations of the harmonic coordinates with diferent $U(1)$ charge q

$q = 0$ harmonics

We consider the bilinear traceless neutral combinations of harmonics

$$U^k_l = u^{+b}_l \bar{u}^{-k}_b - \bar{u}^{+k}_{\hat{b}} u^{-\hat{b}}_l = -(U^l_k)^{\sim}, \quad U^k_l U^l_j = \delta^k_j$$

The simplest bilinear combinations with $SU_L(2)\times SU_R(2)$ indices have the form

$$U^{a\hat{a}}_{[ij]} = \frac{1}{2}u^{+a}_i u^{-\hat{a}}_j - \frac{1}{2}u^{+a}_j u^{-\hat{a}}_i, \quad U^{a\hat{a}}_{(ij)} = \frac{1}{2}u^{+a}_i u^{-\hat{a}}_j + \frac{1}{2}u^{+a}_j u^{-\hat{a}}_i$$

$$\tilde{U}^{[ij]}_{a\hat{a}} = (U^{a\hat{a}}_{[ij]})^{\sim}, \quad \tilde{U}^{(ij)}_{a\hat{a}} = (U^{a\hat{a}}_{(ij)})^{\sim}$$

We consider the important $U(1)$ neutral self-duality relation connecting conjugated harmonics

$$U^{a\hat{a}}_{[ij]} = \frac{1}{2}\varepsilon_{ijkl}\varepsilon^{ab}\varepsilon^{\hat{a}\hat{b}}\tilde{U}^{[kl]}_{b\hat{b}}$$

$q = 2$ harmonics

The $q = 2$ charged self-duality condition

$$\frac{1}{2}\varepsilon^{klij}U^{++}_{[ij]} = \tilde{U}^{++[kl]}$$

connects the corresponding combinations of harmonics

$$U^{++}_{[ij]} = \varepsilon_{ab}u^{+a}_i u^{+b}_j, \quad \tilde{U}^{++[ij]} = \varepsilon^{\hat{c}\hat{e}}\bar{u}^{+i}_{\hat{c}}\bar{u}^{+j}_{\hat{e}} = -(U^{++}_{[ij]})^{\sim}$$

The $q = -2$ charged self-duality condition

$$\frac{1}{2}\varepsilon^{klij} U^{--}_{[ij]} = \tilde{U}^{--[kl]}$$

connects the corresponding combinations of harmonics

$$U^{--}_{[ij]} = \varepsilon_{\hat{a}\hat{b}} u_i^{-\hat{a}} u_j^{-\hat{b}}, \quad \tilde{U}^{--[ij]} = \varepsilon^{ce} \bar{u}_c^{-i} \bar{u}_e^{-j} = -(U^{--}_{[ij]})^{\sim}$$

$$\varepsilon^{ijkl} U^{--}_{[ij]} U^{++}_{[kl]} = 4$$

Analytic basis of the $SU(4)/SU(2) \times SU(2) \times U(1)$ harmonic superspace

We use the $N = 4$ superspace $R(4|16)$ with 4 space-time and 8+8 spinor coordinates

$$z = (x^m, \ \theta^\alpha_k, \ \bar{\theta}^{k\dot{\alpha}})$$

where $k = 1, 2, 3, 4$ is the $SU(4)$ index, $m = 0, 1, 2, 3$ is the vector index and α, $\dot{\alpha}$ are the $SL(2, C)$ spinor indices. The supersymmetry transformations have the form

$$\delta x^m = -i(\epsilon_k \sigma^m \bar{\theta}^k) + i(\theta_k \sigma^m \bar{\epsilon}^k), \quad \delta\theta^\alpha_k = \epsilon^\alpha_k, \quad \delta\bar{\theta}^{k\dot{\alpha}} = \bar{\epsilon}^{k\dot{\alpha}}$$

We construct the analytic coordinates in the $\mathcal{U}_8$ harmonic superspace $\mathcal{H}(4+8|8)$

$$x^m_A = x^m - i(\theta^-_a \sigma^m \bar\theta^{+a}) + i(\theta^+_{\hat a}\sigma^m \bar\theta^{-\hat a})$$

$$\delta_\epsilon x^m_A = 2i(\theta^+_{\hat a}\sigma^m \bar\epsilon^{-\hat a}) - 2i(\epsilon^-_a \sigma^m \bar\theta^{+a})$$

$$\theta^{+\alpha}_{\hat a} = \theta^\alpha_i \bar u^{+i}_{\hat a}, \qquad \bar\theta^{+a\dot\alpha} = \bar\theta^{i\dot\alpha} u^{+a}_i = -(\theta^{+\alpha}_{\hat a})^\sim$$

We add also the nonanalytic spinor coordinates in the analytic basis (AB)

$$\theta^{-\alpha}_a = \theta^\alpha_i \bar u^{-i}_a, \qquad \bar\theta^{-\hat a\dot\alpha} = \bar\theta^{i\dot\alpha} u^{-\hat a}_i = -(\theta^{-\alpha}_a)^\sim, \qquad (\theta^{-a\alpha})^\sim = \bar\theta^{-\dot\alpha}_{\hat a}$$

We consider bilinear products of spinor coordinates

$$\Theta^{++\alpha\beta} = \varepsilon^{\hat b\hat a}\theta^{+\alpha}_{\hat a}\theta^{+\beta}_{\hat b}, \qquad \bar\Theta^{++\dot\alpha\dot\beta} = \varepsilon_{ba}\bar\theta^{+a\dot\alpha}\bar\theta^{+b\dot\beta}$$

$$\Theta^{++\alpha\beta}(\theta^+_{\hat a}\theta^+_{\hat b}) = 0$$

We use the spinor derivatives in the analytic basis

$$D^{+b}_\alpha = \partial^{+b}_\alpha, \qquad \bar D^+_{\hat b\dot\alpha} = -\bar\partial^+_{\hat b\dot\alpha}$$

$$D^{-\hat a}_\alpha = \partial^{-\hat a}_\alpha + 2i\bar\theta^{-\hat a\dot\alpha}\partial^A_{\alpha\dot\alpha}, \qquad \bar D^-_{b\dot\alpha} = -\bar\partial^-_{b\dot\alpha} - 2i\theta^{-\alpha}_b \partial^A_{\alpha\dot\alpha}$$

$$\partial^{+b}_\alpha = \frac{\partial}{\partial\theta^{-\alpha}_b}, \qquad \partial^{-\hat a}_\alpha = \frac{\partial}{\partial\theta^{+\alpha}_{\hat a}}, \qquad \bar\partial^+_{\hat b\dot\alpha} = \frac{\partial}{\partial\bar\theta^{-\hat b\dot\alpha}}, \qquad \bar\partial^-_{b\dot\alpha} = \frac{\partial}{\partial\bar\theta^{+b\dot\alpha}}$$

$$\partial^A_{\alpha\dot\alpha} = (\sigma^m)_{\alpha\dot\alpha}\frac{\partial}{\partial x^m_A}$$

The CR harmonic derivative in AB

$$D^{++b}_{\hat{b}} = \partial^{++b}_{\hat{b}} + 2i\theta^{+\beta}_{\hat{b}}\bar{\theta}^{+b\dot{\beta}}\partial^{A}_{\beta\dot{\beta}} - \theta^{+\beta}_{\hat{b}}\partial^{+b}_{\beta} + \bar{\theta}^{+b\dot{\beta}}\bar{\partial}^{+}_{\hat{b}\dot{\beta}} = (D^{++b}_{\hat{b}})^{\dagger}$$

preserves analyticity

$$[D^{++b}_{\hat{b}}, D^{+a}_{\alpha}] = 0, \qquad [D^{++b}_{\hat{b}}, \bar{D}^{+}_{\hat{a}\dot{\alpha}}] = 0$$

We also define the nonanalytic and neutral harmonic derivatives in AB

$$D^{--\hat{a}}_{b} = \partial^{--\hat{a}}_{b} - 2i(\theta^{-}_{b}\sigma^{m}\bar{\theta}^{-\hat{a}})\partial^{A}_{m} - \theta^{-\alpha}_{b}\partial^{-\hat{a}}_{\alpha} + \bar{\theta}^{-\hat{a}\dot{\alpha}}\bar{\partial}^{-}_{b\dot{\alpha}}$$

$$D^{0} = \partial^{0} + \theta^{+\alpha}_{\hat{a}}\partial^{-\hat{a}}_{\alpha} + \bar{\theta}^{+a\dot{\alpha}}\bar{\partial}^{-}_{a\dot{\alpha}} - \theta^{-\alpha}_{a}\partial^{+a}_{\alpha} - \bar{\theta}^{-\hat{a}\dot{\alpha}}\bar{\partial}^{+}_{\hat{a}\dot{\alpha}}$$

$$D^{a}_{b} = \partial^{a}_{b} + \bar{\theta}^{+a\dot{\alpha}}\bar{\partial}^{-}_{b\dot{\alpha}} - \frac{1}{2}\delta^{a}_{b}\bar{\theta}^{+c\dot{\alpha}}\bar{\partial}^{-}_{c\dot{\alpha}} - \theta^{-\alpha}_{b}\partial^{+a}_{\alpha} + \frac{1}{2}\delta^{a}_{b}\theta^{-\alpha}_{c}\partial^{+c}_{\alpha}$$

$$\hat{D}^{\hat{a}}_{\hat{b}} = \partial^{\hat{a}}_{\hat{b}} - \theta^{+\alpha}_{\hat{b}}\partial^{-\hat{a}}_{\alpha} + \frac{1}{2}\delta^{\hat{a}}_{\hat{b}}\theta^{+\alpha}_{\hat{c}}\partial^{-\hat{c}}_{\alpha} + \bar{\theta}^{-\hat{a}\dot{\alpha}}\bar{\partial}^{+}_{\hat{b}\dot{\alpha}} - \frac{1}{2}\delta^{\hat{a}}_{\hat{b}}\bar{\theta}^{-\hat{c}\dot{\alpha}}\bar{\partial}^{+}_{\hat{c}\dot{\alpha}}$$

The integral measure in the analytic superspace has the form

$$d\zeta^{(-8)} = d^{4}x_{A}D^{-4}\bar{D}^{-4}$$

$$D^{-4} = \frac{1}{24}(D^{-\hat{a}}D^{-\hat{b}})(D^{-}_{\hat{a}}D^{-}_{\hat{b}}), \qquad \bar{D}^{-4} = \frac{1}{24}(\bar{D}^{-a}\bar{D}^{-b})(\bar{D}^{-}_{a}\bar{D}^{-}_{b})$$

$$D^{-4}\Theta^{+4} = 1, \qquad \bar{D}^{-4}\bar{\Theta}^{+4} = 1$$

$N = 4$ superfield constraints in the harmonic superspace

The on-shell superfield constraints of the $N = 4$ Yang–Mills theory in the ordinary superspace are described by the following equations (M.F. Sohnius, Nucl. Phys. B 136 (1978) 461)

$$\{\nabla^k_\alpha, \nabla^j_\beta\} = \varepsilon_{\alpha\beta} W^{kj} \ , \quad \{\bar\nabla_{k\dot\alpha}, \bar\nabla_{j\dot\beta}\} = \varepsilon_{\dot\alpha\dot\beta} \bar W_{kj} \ ,$$

$$\{\nabla^k_\alpha, \bar\nabla_{j\dot\beta}\} = -2i\delta^k_j \nabla_{\alpha\dot\beta}$$

where ∇ are the spinor and vector covariant derivatives in the the central basis (CB).

We consider harmonic projections of the $N = 4$ constraints

$$\{\mathbf{D}^{+a}_\alpha, \mathbf{D}^{+b}_\beta\} = \frac{1}{2}\varepsilon_{\alpha\beta}\varepsilon^{ba}\mathbf{W}^{++}$$

$$\{\bar{\mathbf{D}}^+_{\hat a\dot\alpha}, \bar{\mathbf{D}}^+_{\hat b\dot\beta}\} = \frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\varepsilon_{\hat b\hat a}\bar{\mathbf{W}}^{++} = \frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\varepsilon_{\hat b\hat a}\mathbf{W}^{++}$$

$$\{\mathbf{D}^{+a}_\alpha, \bar{\mathbf{D}}^+_{\hat a\dot\beta}\} = 0$$

where

$$\mathbf{D}^{+a}_\alpha = u^{+a}_k \nabla^k_\alpha = D^{+a}_\alpha + A^{+a}_\alpha$$

$$\bar{\mathbf{D}}^+_{\hat a\dot\alpha} = \bar u^{+k}_{\hat a} \bar\nabla_{k\dot\alpha} = \bar D^+_{\hat a\dot\alpha} + \bar A^+_{\hat a\dot\alpha}$$

$$\mathbf{W}^{++} = \varepsilon_{ab} u^{+a}_k u^{+b}_l W^{kl}, \quad \bar{\mathbf{W}}^{++} = \varepsilon^{\hat a\hat b} \bar u^{+k}_{\hat a} \bar u^{+l}_{\hat b} \bar W_{kl}$$

We connect the self-duality condition for W^{kl} with the harmonic condition of anti-Hermicity

$$[(\mathbf{W}^{++})^A_B]^\sim = -(\mathbf{W}^{++})^B_A, \quad A, B = 1, 2, \ldots n$$

$$\frac{1}{2}\varepsilon^{klij}\varepsilon_{ab}u^{+a}_i u^{+b}_j = \varepsilon^{\hat{a}\hat{b}}\bar{u}^{+k}_{\hat{a}}\bar{u}^{+l}_{\hat{b}} = -[\varepsilon_{ab}u^{+a}_k u^{+b}_l]^\sim$$

We analyze the superfield equations

$$[\partial^{++a}_{\hat{a}}, \mathbf{D}^{+b}_\alpha] = 0, \quad [\partial^{++a}_{\hat{a}}, \bar{\mathbf{D}}^+_{\hat{b}\dot{\alpha}}] = 0, \quad [\partial^{++a}_{\hat{a}}, \partial^{++b}_{\hat{b}}] = 0$$

$$\partial^{++a}_{\hat{a}}\mathbf{W}^{++} = 0, \quad \partial^{++a}_{\hat{a}}\bar{\mathbf{W}}^{++} = 0$$

$$\mathbf{D}^{+a}_\alpha \mathbf{W}^{++} = \bar{\mathbf{D}}^+_{\hat{a}\dot{\alpha}}\mathbf{W}^{++}$$

which are evident in the central basis.

In the analytic basis of the non-abelian theory, we define the Hermitian analytic gauge 4-prepotential

$$(V^{++a}_{\hat{b}})^B_A = [(V^{++b}_{\hat{a}})^A_B]^\sim, \quad D^{+c}_\alpha V^{++a}_{\hat{b}} = \bar{D}^+_{\hat{c}\dot{\alpha}}V^{++a}_{\hat{b}} = 0$$

in the adjoint representation of the gauge group $SU(n)$

$$\delta_\lambda V^{++a}_{\hat{b}} = -D^{++a}_{\hat{b}}\lambda + [\lambda, V^{++a}_{\hat{b}}]$$

where we use the analytical superfield gauge parameters $\lambda^B_A = -(\lambda^A_B)^\sim$ **The pure analytic harmonic covariant derivative is defined off mass shell** $\mathcal{D}^{++a}_{\hat{b}} = D^{++a}_{\hat{b}} + V^{++a}_{\hat{b}}$ **The commutator of**

pure analytic covariant harmonic derivatives

$$[\mathcal{D}^{++a}_{\hat{a}}, \mathcal{D}^{++b}_{\hat{b}}] = \frac{1}{2}F^{(+4)ab}\varepsilon_{\hat{a}\hat{b}} + \frac{1}{2}\hat{F}^{(+4)}_{\hat{a}\hat{b}}\varepsilon^{ba}$$

is expressed via two dimensionless analytic gauge covariant superfields

$$\hat{F}^{(+4)}_{\hat{a}\hat{b}} = \varepsilon_{ab}(D^{++a}_{\hat{a}}V^{++b}_{\hat{b}} - D^{++b}_{\hat{b}}V^{++a}_{\hat{a}} + [V^{++a}_{\hat{a}}, V^{++b}_{\hat{b}}]) = \hat{F}^{(+4)}_{\hat{b}\hat{a}}$$

$$F^{(+4)ab} = -(\hat{F}^{(+4)}_{\hat{a}\hat{b}})^{\dagger}$$

We consider the HS transform from the central basis to the analytic basis

$$\mathbf{D}^{+a}_{\alpha} = e^{-v}\nabla^{+a}_{\alpha}e^{v}, \quad \bar{\mathbf{D}}^{+}_{\hat{a}\dot{\alpha}} = e^{-v}\bar{\nabla}^{+}_{\hat{a}\dot{\alpha}}e^{v}, \quad \mathbf{W}^{++} = e^{-v}W^{++}e^{v}$$

where $v(z,u)$ is an anti-Hermitian superfield matrix on the mass shell, and W^{++} is a harmonic superfield in AB. We use the $N=4$ non-covariant on-shell representation of the spinor covariant derivatives in the analytic basis

$$\nabla^{+a}_{\alpha} = D^{+a}_{\alpha} - \frac{1}{4}\theta^{-a}_{\alpha}W^{++}, \quad \bar{\nabla}^{+}_{\hat{a}\dot{\alpha}} = \bar{D}^{+}_{\hat{a}\dot{\alpha}} + \frac{1}{4}\bar{\theta}^{-}_{\hat{a}\dot{\alpha}}W^{++} = (\nabla^{+a}_{\alpha})^{\dagger}$$

where W^{++} is an independent analytic anti-Hermitian covariant superfield

$$\delta_{\lambda}W^{++} = [\lambda, W^{++}], \quad (W^{++})^{\dagger} = -W^{++}$$

These spinor AB covariant derivatives satisfy the constraints

$$\{\nabla^{+a}_{\alpha}, \nabla^{+b}_{\beta}\} = \frac{1}{2}\varepsilon_{\alpha\beta}\varepsilon^{ba}W^{++}, \quad \{\bar{\nabla}^{+}_{\hat{a}\dot{\alpha}}, \bar{\nabla}^{+}_{\hat{b}\dot{\beta}}\} = \frac{1}{2}\varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon_{\hat{b}\hat{a}}W^{++}$$

$$\{\nabla^{+a}_{\alpha}, \bar{\nabla}^{+}_{\hat{b}\beta}\} = 0$$

where we use the linear relations

$$D^{+b}_{\beta} W^{++} = \bar{D}^{+}_{\hat{b}\dot{\alpha}} W^{++} = 0, \quad D^{+a}_{\alpha}\theta^{-b}_{\beta} = -\varepsilon_{\alpha\beta}\varepsilon^{ba}$$

We define the on-shell harmonic covariant AB derivative

$$e^{v}\partial^{++a}_{\hat{a}} e^{-v} = \nabla^{++a}_{\hat{a}} = D^{++a}_{\hat{a}} + V^{++a}_{\hat{a}}$$

$$-\frac{1}{4}[(\theta^{-a}\theta^{+}_{\hat{a}}) + (\bar{\theta}^{-}_{\hat{a}}\bar{\theta}^{+a})]W^{++}$$

The nonlinear consistency equations for these AB constraints have the form

$$I) \quad \mathbf{dim.}\, 1: \quad \mathcal{D}^{++a}_{\hat{a}} W^{++} = D^{++a}_{\hat{a}} W^{++} + [V^{++a}_{\hat{a}}, W^{++}] = 0$$

$$\mathbf{dim.}\, 0: \quad E^{(+4)ab} = F^{(+4)ab} + (\bar{\theta}^{+a}\bar{\theta}^{+b})W^{++} = 0,$$

$$\hat{E}^{(+4)}_{\hat{a}\hat{b}} = \hat{F}^{(+4)}_{\hat{a}\hat{b}} + (\theta^{+}_{\hat{a}}\theta^{+}_{\hat{b}})W^{++} = 0$$

The constraint equations are covariant under the nonstandard $N = 4$ supersymmetry transformations

$$\delta_{\epsilon} W^{++} = [(\epsilon_k Q^k) + (\bar{\epsilon}^k \bar{Q}_k)]W^{++}$$

$$\delta_{\epsilon} V^{++a}_{\hat{a}} = [(\epsilon_k Q^k) + (\bar{\epsilon}^k \bar{Q}_k)]V^{++a}_{\hat{a}} + \frac{1}{2}[(\epsilon^{-a}\theta^{+}_{\hat{a}}) + (\bar{\epsilon}^{-}_{\hat{a}}\bar{\theta}^{+a})]W^{++}$$

$$\delta_{\epsilon} \hat{E}^{(+4)}_{\hat{a}\hat{b}} = [(\epsilon_k Q^k) + (\bar{\epsilon}^k \bar{Q}_k)]\hat{E}^{(+4)}_{\hat{a}\hat{b}}$$

$$+\frac{1}{2}\varepsilon_{ab}\{[(\epsilon^{-b}\theta^{+}_{\hat{b}}) + (\bar{\epsilon}^{-}_{\hat{b}}\bar{\theta}^{+b})]\mathcal{D}^{++a}_{\hat{a}} - [(\epsilon^{-a}\theta^{+}_{\hat{a}}) + (\bar{\epsilon}^{-}_{\hat{a}}\bar{\theta}^{+a})]\mathcal{D}^{++b}_{\hat{b}}\}W^{++}$$

Action of supergauge model in the harmonic $N=4$ superspace and component representation of equations of motion

We consider an alternative harmonic formalism of the $N=4$ gauge theory off mass shell and construct a gauge invariant and $SU(4)$ invariant action (A-model) including the independent analytic superfield W^{++} and the prepotential $V_{\hat{a}}^{++a}$

$$A \sim \frac{1}{g^2}\int d^{(-8)}\zeta du \mathbf{Tr}\{W^{++}[(\theta^{+\hat{a}}\theta^{+\hat{b}})\hat{F}^{(+4)}_{\hat{a}\hat{b}} + (\bar{\theta}^{+a}\bar{\theta}^{+b})F^{(+4)}_{ab}$$

$$+\frac{1}{2}W^{++}(\Theta^{+4}+\bar{\Theta}^{+4})]\}$$

Varying the action in the superfield $V_{\hat{a}}^{++a}$ gives us the equation

$$II) \quad \mathbf{dim.}0: \quad [\varepsilon_{ab}(\theta^{+\hat{a}}\theta^{+\hat{b}}) + \varepsilon^{\hat{a}\hat{b}}(\bar{\theta}^+_a\bar{\theta}^+_b)]\mathcal{D}^{++b}_{\hat{b}}W^{++} = 0$$

Varying A in W^{++} we obtain the equation

$$II) \quad (\theta^{+\hat{a}}\theta^{+\hat{b}})\hat{F}^{(+4)}_{\hat{a}\hat{b}} + (\bar{\theta}^{+a}\bar{\theta}^{+b})F^{(+4)}_{ab} + W^{++}(\Theta^{+4}+\bar{\Theta}^{+4})$$

$$= (\theta^{+\hat{a}}\theta^{+\hat{b}})\hat{E}^{(+4)}_{\hat{a}\hat{b}} + (\bar{\theta}^{+a}\bar{\theta}^{+b})E^{(+4)}_{ab} = 0$$

Component analysis of the A-model

WZ**-type gauge condition for the prepotential** $(V^{++a}_{\hat{a}})_{WZ} = v^{++a}_{\hat{a}} + \mathcal{V}^{++a}_{\hat{a}}$ **where** $v^{++a}_{\hat{a}}$ **contains the standard** $N = 4$ **supermultiplet** $\phi^{[kl]}, \; A_m, \; \lambda_{k\alpha}, \; \bar{\lambda}^k_{\dot{\alpha}}$

$$v^{++a}_{\hat{a}} = -2\theta^{+\alpha}_{\hat{a}}\bar{\theta}^{+a\dot{\alpha}}A_{\alpha\dot{\alpha}} + \phi^{[kl]}[(\theta^{+}_{\hat{a}}\theta^{+}_{\hat{c}})U^{a\hat{c}}_{[kl]} + (\bar{\theta}^{+a}\bar{\theta}^{+c})U_{c\hat{a}[kl]}]$$
$$+(\theta^{+}_{\hat{a}}\theta^{+}_{\hat{c}})\bar{\theta}^{+a\dot{\alpha}}u^{-\hat{c}}_{k}\bar{\lambda}^k_{\dot{\alpha}} - (\bar{\theta}^{+a}\bar{\theta}^{+c})\theta^{+\alpha}_{\hat{a}}\bar{u}^{-k}_{c}\lambda_{k\alpha}$$

and $\mathcal{V}^{++a}_{\hat{a}}$ **includes an infinite number of additional bosonic and fermionic component fields. The off-shell decomposition of the independent** $\sim$**-imaginary abelian superfield strength contains independent component fields**

$$W^{++} = U^{++}_{[kl]}F^{[kl]} + \varepsilon^{\hat{b}\hat{c}}\theta^{+\beta}_{\hat{b}}\bar{u}^{+k}_{\hat{c}}\Lambda_{k\beta} - \varepsilon_{bc}\bar{\theta}^{+b\dot{\alpha}}u^{+c}_{k}\bar{\Lambda}^k_{\dot{\alpha}}$$
$$+i\Theta^{++}_{\alpha\beta}F^{\alpha\beta} + i\bar{\Theta}^{++}_{\dot{\alpha}\dot{\beta}}\bar{F}^{\dot{\alpha}\dot{\beta}}$$
$$+\theta^{+\alpha}_{\hat{b}}\bar{\theta}^{+\dot{\alpha}}_{b}[U^{b\hat{b}}_{[kl]}W^{[kl]}_{\alpha\dot{\alpha}} + U^{b\hat{b}}_{(kl)}W^{(kl)}_{\alpha\dot{\alpha}} - \tilde{U}^{b\hat{b}(kl)}\bar{W}_{(kl)\alpha\dot{\alpha}}]$$
$$+(\theta^{+}_{\hat{b}}\theta^{+\hat{c}})\tilde{U}^{k\hat{b}}_{\hat{c}l}V^l_k - (\bar{\theta}^{+b}\bar{\theta}^{+}_{c})U^{ck}_{bl}\bar{V}^l_k + \dots$$

The abelian version of equation gives us algebraic restrictions for the infinite supermultiplet

$$V^k_l = 0, \quad T^{[kl]} = 0, \quad P^k_\alpha = 0, \dots$$

and differential constraints

$$\partial^{\dot{\alpha}\beta}F_{\alpha\beta} = 0, \quad \partial^{\dot{\alpha}\beta}\Lambda_{k\beta} = 0, \quad W^{[kl]}_{\alpha\dot{\alpha}} = -4i\partial_{\alpha\dot{\beta}}F^{[kl]}$$
$$\partial^{\dot{\alpha}\alpha}W^{[kl]}_{\alpha\dot{\alpha}} = 0, \quad R_{k(\alpha\beta)\dot{\alpha}} = i\partial_{(\alpha\dot{\alpha}}\Lambda_{k\beta)}$$

Nonlinear interactions in the abelian $N = 4$ gauge theory

We consider the bilinear component electromagnetic terms from the abelian version A_0

$$\frac{1}{64}[F^{\alpha\beta}F_{\alpha\beta} + 4F^{\alpha\beta}(\partial_{\alpha\dot{\beta}}A^{\dot{\beta}}_{\beta} + \partial_{\beta\dot{\beta}}A^{\dot{\beta}}_{\alpha}) + \mathbf{c.c}]$$

$$F^2 = F^{\alpha\beta}F_{\alpha\beta}, \quad \bar{F}^2 = \bar{F}^{\dot{\alpha}\dot{\beta}}\bar{F}_{\dot{\alpha}\dot{\beta}}$$

The $N = 4$ **supersymmetric fourth-order interaction of the abelian harmonic superfield**

$$S_4 = N_1 f^2 \int d^{(-8)}\zeta du (W^{++})^4$$

In particular, this superfield term yields the component interaction

$$L_4 \sim f^2 F^2 \bar{F}^2$$

Conclusion

1. We solve the $N = 4$ **Yang-Mills superfield constraints in the** $SU(4)/SU(2) \times SU(2) \times U(1)$ **harmonic superspace. The constraint equations connect independent analytic harmonic superfields** W^{++} **and** $V_{\hat{a}}^{++a}$**. In the component fields these equations are equivalent to the known** $N = 4$ **SYM equations.**

2. We construct the alternative off-shell formalism using the superfield action for W^{++} **and** $V_{\hat{a}}^{++a}$ **which contain manifestly the Grassmann coordinates.**

3. We construct the nonlinear effective supersymmetric interaction of the abelian superfield W^{++}**.**