Higher-Spin Theory,

Multiparticle Symmetry

and

Operator Algebra of Free Currents

arXiv:1212.6071 MV, arXiv:1301.3123 O.Gelfond, MV

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Plan

HS algebra

Current operator algebra as a multiparticle extension of the HS algebra

$3d$ conformal equations and HS symmetry

Conformal invariant massless equations in $d=3$

$$(\frac{\partial}{\partial x^{\alpha\beta}} \pm i\frac{\partial^2}{\partial y^\alpha \partial y^\beta})C_j^\pm(y|x) = 0\,, \qquad \alpha,\beta = 1,2\,, \quad j = 1,\dots\mathcal{N}$$

$$C_j^\pm(y|x) = \sum_{n=0}^{\infty} C_j^{\pm\alpha_1\dots\alpha_{2n}}(x)y_{\alpha_1}\dots y_{\alpha_{2n}}$$

Shaynkman, MV (2001)

IRREPS of Lorentz algebra: totally symmetric multispinors $A_{\alpha_1\dots\alpha_n}$

Bosons (fermions) are even (odd) functions of y: $C_i(-y|x) = (-1)^{p_i}C_i(y|x)$

Dynamical fields: $C(0|x)$ **and** $C_\alpha(x) = \frac{\partial}{\partial y^\alpha}C(y|x)|_{y=0}$

$3d$ conformal HS algebra is the algebra of various differential operators

$\epsilon(y,\frac{\partial}{\partial y})$ **obeying** $\epsilon(-y,-\frac{\partial}{\partial y}) = \epsilon(y,\frac{\partial}{\partial y})$

$$\delta C(y|x) = \epsilon(y,\frac{\partial}{\partial y}|x)C(y|x)$$

$$\epsilon(y,\frac{\partial}{\partial y}|x) = \exp\left[\mp i x^{\alpha\beta}\frac{\partial^2}{\partial y^\alpha\partial y^\beta}\right]\epsilon_{gl}(y,\frac{\partial}{\partial y})\exp\left[\pm i x^{\alpha\beta}\frac{\partial^2}{\partial y^\alpha\partial y^\beta}\right]$$

$\epsilon_{gl}(y,\frac{\partial}{\partial y})$ **describes global HS transformations**

Weyl algebra and star product

Weyl algebra A_n: associative algebra of polynomials of oscillators $\hat{Y}_A$

$$[\hat{Y}_A\,,\hat{Y}_B] = 2iC_{AB}\,, \qquad A, B, \ldots = 1, \ldots 2n\,, \qquad C_{AB} = -C_{BA}$$

$3d$ **CHS algebra = AdS_4 HS algebra is (even part of) Weyl algebra** A_2

$$\hat{Y}_A = \begin{pmatrix} y^\alpha \\ \frac{\partial}{\partial y^\beta} \end{pmatrix}$$

Weyl star–product language

$$[Y_A, Y_B]_* = 2iC_{AB}\,, \qquad [a\,,b]_* = a*b - b*a$$

Weyl-Moyal formula

$$(f_1 * f_2)(Y) = f_1(Y)\ \mathrm{exp}\,[i\overleftarrow{\partial^A}\overrightarrow{\partial^B}C_{AB}]\,f_2(Y)\ , \qquad \partial^A \equiv \frac{\partial}{\partial Y_A}$$

Currents

Rank-two equations: conserved currents

$$\left\{ \frac{\partial}{\partial x^{\alpha\beta}} - \frac{\partial^2}{\partial y^{(\alpha}\partial u^{\beta)}} \right\} J(u,\, y|x) = 0$$

Gelfond, MV (2003)

$J(u,\, y|x)$: **generalized stress tensor. Rank-two equation is obeyed by**

$$J(u,\, y\,|x) = \sum_{i=1}^{\mathcal{N}} \overline{\Phi}_i(u+y|x)\, \Phi_i(y-u|x)$$

Full fields: $\Phi_j(y|x) = C_j^+(y|x) + i^{p_j} C_j^-(iy|x)\,, \qquad \overline{\Phi}_j(y|x) = C_j^-(y|x) + i^{p_j} C_j^+(iy|$

$$\left(\frac{\partial}{\partial x^{\alpha\beta}} + i \frac{\partial^2}{\partial y^\alpha \partial y^\beta} \right) \Phi_j(y|x) = 0\,, \qquad \left(\frac{\partial}{\partial x^{\alpha\beta}} - i \frac{\partial^2}{\partial y^\alpha \partial y^\beta} \right) \overline{\Phi}_j(y|x) = 0$$

Rank-two fields: bilocal fields in the twistor space.

Primaries: $3d$ **currents of all integer and half-integer spins**

$$J(u,0|x) = \sum_{2s=0}^{\infty} u^{\alpha_1} \dots u^{\alpha_{2s}} J_{\alpha_1 \dots \alpha_{2s}}(x)\,, \qquad \tilde{J}(0,y|x) = \sum_{2s=0}^{\infty} y^{\alpha_1} \dots y^{\alpha_{2s}} \tilde{J}_{\alpha_1 \dots \alpha_{2s}}(x)$$

$$J^{asym}(u,y|x) = u_\alpha y^\alpha J^{asym}(x)$$

$$\Delta J_{\alpha_1 \dots \alpha_{2s}}(x) = \Delta \tilde{J}_{\alpha_1 \dots \alpha_{2s}}(x) = s+1 \qquad \Delta J^{asym}(x) = 2\,.$$

Quantization

Operator fields obey

$$[\widehat{C}_j^-(y|x)\,,\widehat{C}_k^+(y'|x')] = \frac{1}{2i}\delta_{jk}\left(\mathcal{D}^-(y-y'|x-x') + (-1)^{p_j p_k}\mathcal{D}^-(y+y'|x-x')\right)$$

$$\mathcal{D}^\pm(y|x) = \pm\frac{i}{4\pi}\,\mathrm{exp}\pm\frac{i\pi I_x}{4}|det|x||^{-1/2}\,\mathrm{exp}[-\frac{i}{4}x^{-1}_{\alpha\beta}y^\alpha y^\beta]$$

Since

$$\mathcal{D}^\pm(y|0) = \mp i\delta^M(y)$$

commutation relations make sense at $x = x'$

$$[\widehat{C}_j^-(y|x)\,,\widehat{C}_k^+(y'|x)] = \frac{1}{2}\delta_{jk}\left(\delta(y-y') + (-1)^{p_j p_k}\delta(y+y')\right)$$

Singularity at $(y,x) = (y',x')$ does not imply singularity at $x = x'$.

Rank-one twistor to boundary evolution

$$C^\pm(y|x) = \mp i\int d^2y'\mathcal{D}^\mp(y'-y|x'-x)C^\pm(y'|x')\,.$$

Bulk extension via twistor-to-bulk $\mathcal{D}$-function

$$\mathcal{D}(y|X)\,,\qquad X = (x,z)\,,\qquad D_0\mathcal{D}(y|X) = 0\,,\qquad \mathcal{D}^\pm(y|0) = \mp i\delta^2(y)$$

Quantum currents

$J_{jk}(y_1, y_2|x) =: \hat{\Phi}_j(y_1|x)\overline{\hat{\Phi}}_k(y_2|x) :$

Generating function J_g^2 with test function g

$$J_g^2 = \int dw_1 dw_2\, g^{mn}(w_1, w_2) J_{mn}(w_1, w_2|0)\,,$$

$$J_{j\,k}(w_1, w_2|x) = \sum_{a,b=+,-} (\kappa_1^a)^{p_j}(\kappa_2^b)^{p_k} J_{j\,k}^{ab}(\kappa_1^a y_1, \kappa_2^b p_2|x)\,,$$

$$J_{j\,k}^{a\,b}(w_1, w_2|x) =: \hat{C}_j^a(w_1|x)\hat{C}_k^b(w_2|x) :$$

$$\kappa_1^+ = \kappa_2^- = 1\,, \qquad \kappa_2^+ = \kappa_1^- = i$$

where

$$J_g^2(x) = \int dw_1 dw_2\, g_{ab}^{mn}(w_1, w_2) J_{mn}^{ab}(w_1, w_2|x) = J_{g(x)}^2$$

x**-dependence of** $g_{ab}^{mn}(x)$ $(a, b = \pm)$ **is reconstructed by** $\mathcal{D}$**-functions**

Twistor current algebra

Elementary computation gives

$$J^2_g J^2_{g'} =: J^2_g J^2_{g'} : + J^2{}_{[g\,,g']_\star} + \mathcal{N} tr_\star(g \star g') J^0$$

Convolution product $\star$ is related to HS star-product via half-Fourier transform

$$\tilde{g}(w,v) = (2\pi)^{-1} \int d^2u \exp[iw_\alpha u^\alpha] g(v+u, v-u)$$

Star product of AdS_4 HS theory results from OPE of boundary currents

Full set of operators

$$J^{2m}_g =: \underbrace{J^2_g \dots J^2_g}_{m} : \qquad J^0_g = Id$$

What is the associative twistor operator algebra?!

Since

$$J^2_{g_1} J^2_{g_2} - J^2_{g_2} J^2_{g_1} = 2 J^2_{[g_1\,,g_2]_\star}$$

universal enveloping algebra $U(h)$ of the HS algebra h Gelfond, MV 2013

Explicit construction of multiparticle algebra

Being maximal symmetry $gl(V)$ HS algebra is associated with associative algebra $End(V)$

Universal enveloping algebra $U(l(A))$ of a Lie algebra $l(A)$ associated with an associative algebra A has remarkable properties allowing explicit description of the operator product algebra

Let $\{t_i\}$ be some basis of A

$$a \in A: \qquad a = a^i t_i\,, \qquad\quad t_i \star t_j = f_{ij}^k t_k$$

$$t_i \sim J^2\,, \qquad\quad a^i \sim g(w_1, w_2)$$

$U(l(A))$ is algebra of functions of α_i (commutative analogue of t_i)

Explicit composition law of $M(A)$ 2012

$$F(\alpha) \circ G(\alpha) = F(\alpha) \exp\left(\frac{\overleftarrow{\partial}}{\partial \alpha_i} f_{ij}^n \alpha_n \frac{\overrightarrow{\partial}}{\partial \alpha_j}\right) G(\alpha)$$

where derivatives $\frac{\overleftarrow{\partial}}{\partial \alpha_i}$ and $\frac{\overrightarrow{\partial}}{\partial \alpha_j}$ act on F and G, respectively.

Associativity of $\star$ of A implies associativity of $\circ$ of $M(A)$

Different bases

Composition law for linear functions

$$F(\alpha)\circ G(\alpha) = F(\alpha)G(\alpha) + F(\alpha)\star G(\alpha)$$

differs from current operator algebra

$$F(\alpha)\diamond G(\alpha) = F(\alpha)G(\alpha) + \frac{1}{2}[F(\alpha)\,,G(\alpha)]_\star + \mathcal{N} tr_\star(F(\alpha)G(\alpha))$$

Uniqueness of the Universal enveloping algebra implies that the two composition laws are related by a basis change

Class of basis changes:

Generating function $G_\nu = \exp\nu \quad \nu = \nu^i\alpha_i \in A$ **is replaced by**

$$U_u(G_\nu) := \tilde{G}_\nu = G_{u(\nu)}\,, \qquad \tilde{T}^u_{i_1\ldots i_n} = \frac{\partial^n}{\partial\nu^{i_1}\ldots\partial\nu^{i_n}}\tilde{G}_{(\nu)}\Big|_{\nu=0}$$

$$u(a) = ({u_1}^1 a + {u_1}^2 e_\star)\star({u_2}^1 a + {u_2}^2 e_\star)^{-1}_\star\,, \qquad (e_\star + \beta a)^{-1}_\star := \sum_{n=0}^{\infty}(-\beta)^n a_n$$

Composition: $\mathbf{U}_u\mathbf{U}_v = \mathbf{U}_{uv}\,, \qquad {(uv)_i}^j = {u_i}^k {v_k}^j$

Affine subgroup: $u(a) = b(e_\star + \beta a)^{-1}_\star$

Operator product

For affine maps, the composition law is

$$\tilde{G}_\nu \diamond \tilde{G}_\mu = \tilde{G}_{\sigma_{b,\beta}(\nu,\mu)}$$

$$\sigma_{b,\beta}(\nu,\mu) = -\beta^{-1}(e_\star - (e_\star + \beta\mu) \star (e_\star - \beta(b+\beta)\nu \star \mu)^{-1} \star (e_\star + \beta\nu)) \,.$$

The distinguished case of $\sigma_{1,-\frac{1}{2}}$

$$\sigma_{1,-\frac{1}{2}}(\nu,\mu) = 2(e_\star - (2e_\star - \mu) \star (4e_\star + \nu \star \mu)_\star^{-1} \star (2e_\star - \nu))$$

reproduces OPE of the currents after an appropriate further rescaling

$$\tilde{G}_\nu = \exp[-\frac{\mathcal{N}}{4} tr_\star ln_\star (e_\star - \frac{1}{4}\nu \star \nu)] \exp[\nu \star (e_\star - \frac{1}{2}v)_\star^{-1}]$$

$$\tilde{G}_\nu \diamond \tilde{G}_\mu = \left(\frac{det_\star|e_\star - \frac{1}{4}\nu \star \nu| \, det_\star|e_\star - \frac{1}{4}\mu \star \mu|}{det_\star|e_\star - \frac{1}{4}\sigma_{1,-\frac{1}{2}}(\nu,\mu) \star \sigma_{1,-\frac{1}{2}}(\nu,\mu)|} \right)^{\frac{\mathcal{N}}{4}} \tilde{G}_{\sigma_{1,-\frac{1}{2}}(\nu,\mu)}$$

Correlators

Generating function for correlators $\langle J^{2n} J^{2m} \rangle$ of all currents

$$\langle \tilde{G}_\nu \tilde{G}_\mu \rangle = \left(\frac{det_\star |e_\star - \frac{1}{4}\nu \star \nu|\, det_\star |e_\star - \frac{1}{4}\mu \star \mu|}{det_\star |e_\star - \frac{1}{4}\sigma_{1,-\frac{1}{2}}(\nu,\mu) \star \sigma_{1,-\frac{1}{2}}(\nu,\mu)|} \right)^{\frac{\mathcal{N}}{4}}$$

$$J^{2n}_{g_1 \ldots g_n} = g^{i_1} \ldots g^{i_n} \frac{\partial^n}{\partial \nu^{i_1} \ldots \partial \nu^{i_n}} \tilde{G}_\nu \Big|_{\nu=0}$$

Theories with different $\mathcal{N}$: different frames of the same algebra!

$U(h)$ **possesses different invariants generating different n-point functions**

What are models associated with different frame choices?!

Multiparticle algebra as a symmetry of a multiparticle theory

$l(U(h))$

- contains h as a subalgebra
- admits quotients containing up to k^{th} tensor products of h:

  k Regge trajectories?!
- Acts on all multiparticle states of HS theory
- Resolves the problem with lower energies

  Oscillator realization: $[Y_i^A, Y_j^B] = \delta_{ij} C^{AB} \mathbf{E_i}$

Promising candidate for a HS symmetry algebra of HS theory with mixed symmetry fields like String Theory

String Theory as a theory of bound states of HS theory

Chang, Minwalla, Sharma and Yin (2012)

Conclusion

A multiparticle theory: quantum HS theory and String theory

Multiparticle algebra is a Hopf algebra.

Relation with integrable structures underlying both String theory and analysis of amplitudes?!

HS Theory and String Theory

HS theories: $\Lambda \neq 0$, $m = 0$

symmetric fields $s = 0, 1, 2, \dots \infty$

String Theory: $\Lambda = 0$, $m \neq 0$ **except for a few zero modes**

mixed symmetry fields $\vec{s} = 0, 1, 2, \dots \infty$

String theory has much larger spectrum:

HS Theory: first Regge trajectory

Pattern of HS gauge theory is determined by HS symmetry

What is a string-like extension of a global HS symmetry underlying a string-like extension of HS theory?

- **String Theory as spontaneously broken HS theory?!** $(s > 2, m > 0)$

  Singleton String Engquist, Sundell (2005, 2007)

  Recent conjecture (Chang, Minwalla, Sharma and Yin (2012)**):**

  String Theory = Quantum HS theory?!

Failure of naive string-like extension of HS algebra

$$P^\nu = P^\nu_{AB}\{Y^A, Y^B\}, \qquad M^{\nu\mu} = M^{\nu\mu}_{AB}\{Y^A, Y^B\}, \qquad [Y_A, Y_B] = C_{AB}$$

Tensoring modules: $Y^A \to Y^A_i$, $[Y^A_i, Y^B_j] = \delta_{ij} C^{AB}$, $i, j = 1, \dots N$

$$P^\nu = P^\nu_{AB} \sum_i \{Y^A_i, Y^B_i\}, \qquad M^{\nu\mu} = M^{\nu\mu}_{AB} \sum_i \{Y^A_i, Y^B_i\}$$

If $|E_0(2)\rangle$ vacuum was a Fock vacuum for Y^A E_0 increases as NE_0.

If there was gravity at $N = 1$: no gravity at $N > 1$.

Incompatibility of AdS extension of Minkowski first quantized string

$$M^{\nu\mu} = \sum_{n\neq 0} \frac{1}{n} x^{[\nu}_{-n} x^{\mu]}_{n} + p^{[\nu} x^{\mu]}, \qquad P^\nu = p^\nu$$

since $[P^\nu, P^\mu] = -\lambda^2 M^{\nu\mu}$ implies that P^ν should involve all modes and hence lead to the infinite vacuum energy: no graviton

What symmetry can unify HS gauge theory with String?

Current operator algebra