# Special lagrangian geometry for Fano varieties

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**Special lagrangian geometry** for Calabi - Yau manifold — extremly popular in view of **Mirror Symmetry conjecture**. Informally, **MS** in the broadest context is a duality

$$\begin{array}{ccc} & | \quad M \quad | & \\ \textit{Complex geometry} & | \quad i \quad | & \textit{Symplectic geometry} \\ & | \quad r \quad | & \end{array}$$

(Yu. Manin)
What is this **duality**:
f.e. **Homological Mirror Symmetry** (M. Kontsevich): $M$ and $W$ are mirror partners if

$$\begin{aligned} \mathcal{DB}^{coh}(M) &\cong \mathcal{FF}(W) \\ \mathcal{DB}^{coh}(W) &\cong \mathcal{FF}(M) \end{aligned}$$

where the categories $\mathcal{DB}^{coh}$ and $\mathcal{FF}$ are defined in terms of Complex geometry and Symplectic geometry respectively.

$(M, I, \omega)$ — Kahler manifold = **complex structure** + **symplectic structure**

Thus $M$ carries derived objects, defined in terms of

(complex structure) $\Longrightarrow$ **vector bundles, coherent sheaves**

(symplectic structure) $\Longrightarrow$ **lagrangian submanifolds**

One needs an appropriate organizations on the unbounded sets

$$\{\text{coherent sheaves}\} \mapsto \mathcal{DB}^{coh},$$

*derived category of bounded coherent sheaves*

$$\{\text{lagrangian submanifolds}\} \mapsto \mathcal{FF},$$

*Fukaya - Floer category*.

But this way of thinking is not unique: recall the original story.

**Mirror symmetry for 3 - dimensional Calabi - Yau**:
$(M, I)$ is complex 3 - dimensional manifold s.t. there exists nonvanishing everywhere top holomorphic form $\theta$.
In local complex coordinates $(z_1, z_2, z_3)$:

$$\Theta = dz_1 \wedge dz_2 \wedge dz_3,$$

and globally $\Theta$ is unique up to $\mathbb{C}^*$.
**Example:** quintic hypersurface $Q_5 \in \mathbb{CP}^4$.
According to **Yau**'s famous result this $M$ admits solutions to the Einstein equation:
**the family of Kahler - Einstein metrics** $\mapsto$ distinguished Kahler structure.
Thus $(M, I, \omega)$ can be regarded as a *symplectic manifold*, but the space of lagrangian submanifolds is too huge —
**Problem:** *impose any natural conditions to derive finite dimensional moduli space*

**J. McLeane, N. Hitchin**: special lagrangian geometry for $\mathrm{CY}_3$.
Recall, $S \subset M$ is **lagrangian** iff $\omega|_S \equiv 0$ and

$$\mathrm{dim}_{\mathbb{R}}\mathrm{S} = \mathrm{dim}_{\mathbb{C}}\mathrm{M} = 3.$$

Thus $\Theta|_S$ is a top complex form on $S$, and one can compare it with $\mathrm{Vol_g}$, where $g$ is the restriction of the Kahler - Einstein metric on $S$.
**Special condition:** $\Theta|_S = re^{ic}\mathrm{Vol_g}$ where $r \in \mathbb{R}$ and $c$ is constant.
**Local deformation theory for SpLag:** McLean's doctor thesis, the main results

- deformation is finite dimensional, $b_1(S)$;
- deformation is unobstructed.

Thus one gets *the moduli space of special lagrangian submanifolds* — a finite dimensional manifold.
**Example:** Mirror symmetry for elliptic curves

$$\textbf{VBAC} \longleftrightarrow \textbf{SpLag}$$

**Special lagrangian geometry for** $\mathrm{CY}_3$ — very popular subject 15 years ago.
The main abbreviation: **SYZ** — A. Strominger, S.-T. Yau, E. Zaslow.
**Main problem:** any $\mathrm{CY}_3$ admits **special lagrangian fibrations**.

$$\begin{array}{cccc} \text{fibration} & M & \supset & \pi^{-1}(b) \\ & \downarrow & & \upharpoonleft \\ \text{the base} & B & & \text{SpLag} \end{array}$$

**is not solved yet**, present status — **conjecture**.

**D. Auroux: Special lagrangian geometry for Fano varieties.**

Formal definition: *$X$ is Fano, if the anticanonical class is ample.*

!For Fano's: the Kahler - Einstein equation admits solutions!

Unformal definition: $(X, I, \omega)$ — Kahler manifold,

$\exists$ $\Theta_D$ holomorphic top form:

with pole on complex subsurface $D \subset X$;

and without zeros on $X \backslash D$;

s.t. the Kahler form turns to be exact $\omega = d\alpha$ on $X \backslash D$.

(of course, such a form is not unique).

The space of such $D \subset X$ is isomorphic to projective space $\mathbb{CP}^k$ where $k$ is sufficiently big. Thus the specialty condition depends on the choice of $D$.

**Auroux's special condition:** *The complement $X \backslash D$ is fibered on special lagrangian submanifolds with respect to $\Theta_D$.*

*- in other words, $X \backslash D$ is open $\mathrm{CY}_3$ with top form $\Theta_D$.*

**Auroux's example:** For $\mathbb{CP}^2$ and $D = Q \cup l$ where $Q$ is a conic and $l$ is a line, such special lagrangian fibration exists.
**Auroux's conjecture:** Special lagrangian fibration exists for $D = \Sigma_3$ — smooth cubic curve.

*- is not proved yet.*

The Auroux's example was reconstructed using **pseudotoric geometry** (**N.T.**), but it doesn't help for the study of Auroux's conjecture.
(**N.T.:** Special lagrangian fibration for flag variety $F^3$.)

Alternative approach: **Special Bohr - Sommerfeld geometry.**
$(X, I, \omega)$ — Fano variety, $\omega$ — canonical Kahler form,
$[\omega] \in H^2(X, \mathbb{Z})$, so $L \to X$ line bundle with $c_1(L) = [\omega]$.
$\Longrightarrow$ the Bohr - Sommerfeld condition can be imposed on lagrangian submanifolds:

$$\begin{array}{ccccc} \text{prequantization data} & (L, a) & \mapsto & (L_S, a_S) \\ F_a = 2\pi i \omega & \downarrow & & \downarrow \\ \text{flat} \quad a_S & X & \supset & S \end{array}$$

and $S \subset X$ is **Bohr - Sommerfeld** iff $a_S$ admits covariantly constant section $\sigma_S \in \Gamma(S, L_S)$.
**General definition:** $S \subset X$ is $s$ - *Special Bohr - Sommerfeld* w.r.t $s \in \Gamma(X, L)$ iff

$$s|_S = f \cdot e^{ic} \sigma_S, \quad \text{where} \quad f \in C^\infty(S, \mathbb{R}_+), \quad c \in \mathbb{R}.$$

*In general* it gives a relation between two approaches in **Geometric Quantization:**

$$
\begin{array}{ccc}
\text{Souriau - Kostant} & \longleftrightarrow & \text{Lag(N.T.)} \\
\downarrow & & \downarrow \\
\mathbb{P}(\Gamma(X,L)) & & \mathcal{B}_S \\
& \searrow \quad \swarrow & \\
& \mathbb{P}(\Gamma(X,L)) \times \mathcal{B}_S & \\
& \cup & \\
& \mathcal{U}_{SBS} &
\end{array}
$$

where $\mathcal{U}_{SBS}$ is the *incidence cycle* consisting of pairs $(s, S)$ where $S$ is $s$ - SBS.

*In particular* for Fano varieties:

- finite dimensional subspace $\mathbb{P}(H^0(X_I, L)) \subset \mathbb{P}(\Gamma(X, L))$;
- for any holomorphic $s \in H^0(X_I, L))$ one has **descrete** set of $s$ - SBS cycles,

**Finite object:** $\mathcal{M}_{SBS}(X)$ — *the moduli space of Special Bohr - Sommerfeld cycles* of fixed topological type for our given Fano variety $X$.

**Toy example:** $X = \mathbb{CP}^1$ — projective line, the simplest Fano variety

$$\mathcal{M}_{SBS}(X) = \mathbb{CP}^2 \backslash Q$$

where $Q$ is a quadric $\Longrightarrow$ **quite nice answer**.

*work in progress...*