Four types of (Super)Conformal Mechanics

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Based on:

N. L. Holanda and F.T., **JMP** 55, 061703 (2014), arXiv:1402.7290.

Previous works:

Z. Kuznetsova and F. T., **JMP** 53, 043513 (2012), arXiv:1112.0995.

S. Khodaee and F.T., **JMP** 53, 103518 (2012), arXiv:1208.3612.

G. Papadopoulos, **CQG** 30 (2013) 075018; arXiv:1210.1719.

Conformal symmetry in Minkowski D-dimensional spacetime $(D-1,1)$:

$ds^2 = 0$ is preserved.

For $D \neq 2$ the conformal algebra is $so(D,2)$.

Special cases:

$0+1$ conformal algebra: $so(1,2) \approx sl(2)$.

$1+1$ conformal algebra: $\mathfrak{witt} \oplus \mathfrak{witt}$ (holo+antiholo)

$\mathfrak{w}itt$ a.k.a. centerless Virasoro algebra.

$\mathfrak{witt}$ algebra:

$$[L_n, L_m] = (n-m)L_{n+m}, \quad n, m \in \mathbb{Z}$$

$sl(2)$ algebra:

$$\begin{aligned} [D, H] &= H, \\ [D, K] &= -K, \\ [H, K] &= 2D. \end{aligned}$$

$sl(2) \subset \mathfrak{witt}$ for $n, m = 0, \pm 1$: $\quad sl(2) \equiv \{L_{\pm 1}, L_0\}$.

$0 + 1$ Conformal Mechanics:
de Alfaro-Fubini-Furlan, Nuovo Cimento 1976.

Some selected applications:
Test particles near BH horizon (Britto-Pacumio et al. hep-th/9911066),

CFT_1/AdS_2 correspondence(Jackiw, A. Sen ,...),

Computation of BH entropy via l.w.r.'s and ladder operators,

Higher-spin theories in the world-line framework,

Cosmology: supersymmetric mini-superspace.

parabolic versus hyperbolic/trigonometric D-module reps and homogeneous versus inhomogeneous D-module reps.

$$\begin{aligned} par: &\qquad L_n = -t^{n+1}\partial_t - \lambda_n t^n, \\ hyp: &\qquad L_n = -\frac{1}{\mu}e^{n\mu t}\left(\partial_t + \lambda_n\right). \end{aligned}$$

μ dimensional parameter.
$\mathfrak{witt}$ closure for $\lambda_n = n\lambda + \gamma$.

$$\begin{aligned} hom: &\qquad \delta_n(\varphi) = L_n\varphi = a_n\dot{\varphi} + b_n\varphi, \\ inhom: &\qquad \delta_n(\varphi) = L_n\varphi = a_n\dot{\varphi} + b_n. \end{aligned}$$

Let $x_n = a_n\partial_t$, $y_n = b_n$. In matrix form

$$\begin{aligned} L_n^{hom} &= \left(x_n + y_n\right), \\ L_n^{inhom} &= \begin{pmatrix} x_n & y_n \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

$\mathfrak{witt}$ closure for $\lambda_n = n\lambda + \gamma$.

Combining *par*, *hyp* and *hom*, *inhom* we have four distinct actions of the $\mathfrak{witt}$ generators on the bosonic field $\varphi(t)$.

Special choices for γ

$$\begin{aligned} par: &\quad \gamma = \lambda, \\ hyp: &\quad \gamma = 0. \end{aligned}$$

In both cases $b_n = \dot{a}_n$.
One of the L_n operators, proportional to ∂_t, is the "Hamiltonian": L_{-1} for *par* and L_0 for *hyp*.

Dimensional analysis:

hom, *par*: no dimensional parameter,
inhom, *par*: one dimensional parameter, λ,
hom, *hyp*: one dimensional parameter, μ,
inhom, *hyp*: two dimensional parameters, μ and λ.

$D = 1$ conformal actions:

Let $\mathcal{L} = g \cdot \varphi_t^2 + h$.

Up to total derivatives, for *hom*:

$$\begin{array}{rcl}\delta\mathcal{L} & \equiv & [gA_t + 2gB + g_\varphi B\varphi]\varphi_t^2 + [2gB_t\varphi + h_\varphi A + N_\varphi]\varphi_t + \\ & & [h_\varphi B\varphi + N_t].\end{array}$$

For *par*, $B = \lambda A_t$. System to be solved

$$\begin{array}{rcl}A_t[(1+2\lambda)g + \lambda g_\varphi \phi] & = & 0, \\ 2\lambda g A_{tt}\varphi + h_\varphi A + N_\varphi & = & 0, \\ \lambda h_\varphi \varphi A_t + N_t & = & 0.\end{array}$$

Similar system for *hyp*. On the other hand, the relation $A_{tt} = n^2\mu^2 A$, only exists for *hyp* case.

The *Inhom* case variation is

$$\delta\mathcal{L} \equiv [gA_t + Bg_\varphi]\varphi_t^2 + [2gB_t + h_\varphi A + N_\varphi]\varphi_t + [h_\varphi B + N_t].$$

The system to be solved is

$$\begin{array}{rcl} A_t[g+\lambda g_\varphi] & = & 0, \\ 2\lambda g A_{tt} + h_\varphi A + N_\varphi & = & 0, \\ \lambda h_\varphi A_t + N_t & = & 0. \end{array}$$

The $D = 1$ $sl(2)$-invariant actions ($n = 0, \pm 1$):

Homogeneous – parabolic case: power-law Lagrangian

$$\mathcal{L} = C_1 \varphi^{-\frac{(1+2\lambda)}{\lambda}} \dot{\varphi}^2 + C_2 \varphi^{\frac{1}{\lambda}}.$$

for $\lambda \neq 0$. No dimensional parameter.

Homogeneous – hyperbolic case:

$$\mathcal{L} = C_1 [\varphi^{-\frac{(1+2\lambda)}{\lambda}} \dot{\varphi}^2 + \mu^2 \lambda^2 \varphi^{-\frac{1}{\lambda}}] + C_2 \varphi^{\frac{1}{\lambda}}.$$

Inhomogeneous – parabolic case:

$$\mathcal{L} = C_1 e^{-\frac{1}{\rho}\varphi}\dot{\varphi}^2 + C_2 e^{\frac{1}{\rho}\varphi}.$$

Inhomogeneous – hyperbolic case:

$$\mathcal{L} = C_1 e^{-\frac{1}{\rho}\varphi}[\dot{\varphi}^2 + \mu^2\rho^2] + C_2 e^{\frac{1}{\rho}\varphi}.$$

In order to have a dimensionless action $\mathcal{S}$ ($[\mathcal{S}] = 0$), the scaling dimension of the Lagrangian is $[\mathcal{L}] = 1$, if we assign the time coordinate to have scaling dimension -1. Taking into account $[\mu] = 1$ we end up with:

- in both homogeneous cases (*I* and *III*), $[C_1] = [C_2] = 0$, provided that $[\varphi] = \lambda$;
- in both inhomogeneous cases (*II* and *IV*), $[C_1] = -1 - 2s$, $[C_2] = 1$, $[\varphi] = [\rho] = s$, with s arbitrary.

The $D=2$ $\mathfrak{witt}\oplus\mathfrak{witt}$-invariant actions:

The *homogeneous* case
(no dimensional parameter, power-law Lagrangian):

$$\mathcal{L} \;=\; \frac{C_1}{\varphi^2}\varphi_+\varphi_- + C_2\varphi^{\frac{1}{\lambda}}.$$

($\mathcal{S}=\int\int dz_+dz_-\mathcal{L}$, $[C_{1,2}]=0$, $[\varphi]=2\lambda$, $[\lambda]=0$).

The *inhomogeneous* case
(the Liouville equation):

$$\mathcal{L} \;=\; \varphi_+\varphi_- + C_2 e^{\frac{1}{\lambda}\varphi}.$$

($[\varphi]=[\lambda]=\lambda$, $[C_1]=-2\lambda$, $[C_2]=2$).

In the parabolic case, the Hamiltonian is identified with the (positive or negative) $sl(2)$ root generator while, in the hyperbolic case, the Hamiltonian is identified with the $sl(2)$ Cartan generator. This difference proves to be crucial in the construction of conformally invariant actions.

From an algebraic point of view the hyperbolic D-module rep can be recovered from the parabolic D-module rep via a singular transformation. Let us call, for simplicity, $\overline{L}_n = L_n^{hyp.}$ when we fix the values $\mu = 1$ and $\overline{\gamma} = 0$. Therefore $\overline{L}_n = -e^{n\tau}(\partial_\tau + n\overline{\lambda})$. For $t > 0$ the change of variable

$$t \mapsto \tau(t) = ln(t)$$

allows to recover the parabolic rep $\overline{L}_n = -t^{n+1}\partial_t - n\overline{\lambda}t^n$ at the specific values, for its constants, $\tilde{\lambda} = \overline{\lambda}$ and $\tilde{\gamma} = 0$.
Therefore

$$\tilde{\lambda} \quad = \quad \overline{\lambda}$$

Superconformal extensions (main results):

Existence of a critical scaling dimension for $\mathcal{N} = 4, 7, 8$ finite SCA's.

Constraints on superconformal mechanics in the Lagrangian setting.

New type of (target-target) dualities.

Properties of finite SCA's:

Even sector $\mathcal{G}_{even}$: $sl(2) \oplus R$ (R is the R-symmetry).

Odd sector $\mathcal{G}_{odd}$: $2\mathcal{N}$ generators.

The dilatation operator D induces the grading

$$\mathcal{G} \quad = \quad \mathcal{G}_{-1} \oplus \mathcal{G}_{-\frac{1}{2}} \oplus \mathcal{G}_0 \oplus \mathcal{G}_{\frac{1}{2}} \oplus \mathcal{G}_1 .$$

The sector $\mathcal{G}_1$ ($\mathcal{G}_{-1}$) containes a unique generator given by H (K).
The $\mathcal{G}_0$ sector is given by the union of D and the R-symmetry subalgebra ($\mathcal{G}_0 = \{D\} \bigcup \{R\}$).
The odd sectors $\mathcal{G}_{\frac{1}{2}}$ and $\mathcal{G}_{-\frac{1}{2}}$ are spanned by the supercharges Q_i's and their superconformal partners $\widetilde{Q}_i$'s, respectively.
The invariance under the global supercharges Q_i's and the generator K implies the invariance under the full superconformal algebra $\mathcal{G}$.

Most relevant cases: $\mathcal{N} = 4, 7, 8$ finite SCA's

$\mathcal{N} = 4$: simple SCA's are $A(1,1)$ and the exceptional superalgebras $D(2,1;\alpha)$, for $\alpha \in \mathbb{C}\backslash\{0,-1\}$.

Superalgebra isomorphism for α's connected via an S_3 group transformation:

$$\begin{array}{lclclclclcl} \alpha^{(1)} &=& \alpha, & \alpha^{(3)} &=& -(1+\alpha), & \alpha^{(5)} &=& -\frac{1+\alpha}{\alpha}, \\ \alpha^{(2)} &=& \frac{1}{\alpha}, & \alpha^{(4)} &=& -\frac{1}{(1+\alpha)}, & \alpha^{(6)} &=& -\frac{\alpha}{(1+\alpha)}. \end{array}$$

$A(1,1)$ can be regarded as a degenerate superalgebra recovered from $D(2,1;\alpha)$ at the special values $\alpha = 0, -1$.
For α real ($\alpha \in \mathbb{R}$) a fundamental domain under the action of the S_3 group can be chosen to be the closed interval

$$\alpha \quad \in \quad [0,1].$$

Over $\mathbb{C}$, there are four finite $\mathcal{N}=8$ SCA's and one finite $\mathcal{N}=7$ SCA.

The finite $\mathcal{N}=8$ superconformal algebras are:
i) the $A(3,1)=sl(4|2)$ superalgebra, possessing 19 even generators and bosonic sector given by $sl(2)\oplus sl(4)\oplus u(1)$,
ii) the $D(4,1)=osp(8,2)$ superalgebra, possessing 31 even generators and bosonic sector given by $sl(2)\oplus so(8)$,
iii) the $D(2,2)=osp(4|4)$ superalgebra, possessing 16 even generators and bosonic sector given by $sl(2)\oplus so(3)\oplus sp(4)$,
iv) the $F(4)$ exceptional superalgebra, possessing 24 even generators and bosonic sector given by $sl(2)\oplus so(7)$.

The finite $\mathcal{N}=7$ superconformal algebra is the exceptional superalgebra $G(3)$, possessing 17 even generators and bosonic sector given by $sl(2)\oplus g_2$.

Existence of critical scaling dimension λ's:

Results: D-module reps are induced, with the identifications:

$\mathcal{N}=4$: $D(2,1;\alpha)$ reps are recovered from the $(k,4,4-k)$ supermultiplets, with a relation between α and the scaling dimension given by $\alpha=(2-k)\lambda$.

$\mathcal{N}=8$: for $k\neq 4$, all four $\mathcal{N}=8$ finite superconformal algebras are recovered, at the critical values $\lambda_k=\frac{1}{k-4}$, with the identifications:
$D(4,1)$ for $k=0,8$, $F(4)$ for $k=1,7$,
$A(3,1)$ for $k=2,6$ and $D(2,2)$ for $k=3,5$.

$\mathcal{N}=7$: the global supermultiplet $(1,7,7,1)$ induces, at $\lambda=-\frac{1}{4}$, a D-module representation of the exceptional superalgebra $G(3)$.

inhomogeneous transformations exist at $\lambda \neq 0$:

In the parabolic case the general Witt algebra transformations applied on the field φ are

$$L_m^{par}(\varphi) \;=\; -t^{m+1}\dot{\varphi} - \lambda(m+\gamma)t^m\varphi - \rho(m+\beta)t^m.$$

L_{-1}^{par} is proportional to the Hamiltonian if we set $\gamma = 1$ and $\beta = 1$.
For $\lambda \neq 0$ we can write

$$L_m^{par}(\varphi) \;=\; -t^{m+1}\dot{\varphi} - \lambda(m+1)t^m(\varphi + \frac{\rho}{\lambda}),$$

so that the action of the homogeneous transformation with scaling dimension $\lambda \neq 0$ is recovered for the shifted field $\overline{\varphi} = \varphi + \frac{\rho}{\lambda}$. Therefore the (λ, ρ) transformations with $\lambda \neq 0$ are equivalent to the pure homogeneous transformations with scaling parameter λ and $\rho = 0$. The same is true in the hyperbolic case.

The list of inhomogeneous D-module reps for the finite $d = 1$ superconformal algebras is given by

$$
\begin{array}{lcll}
\mathcal{N} = 1 & : & osp(1|2) & - (1,1)_{0,\rho}, \\
\mathcal{N} = 2 & : & sl(2|1) & - \quad (1,2,1)_{0,\rho}, \quad (2,2)_{0,\rho}, \\
\mathcal{N} = 3 & : & B(1,1) & - (1,3,3,1)_{0,\rho}, \\
\mathcal{N} = 4 & : & A(1,1) & - (1,4,3)_{0,\rho}, \quad (2,4,2)_{0,\rho}, \quad (3,4,1)_{0,\rho}, \quad (4,4,0)_{0,\rho}, \\
\mathcal{N} = 8 & : & \text{none} &
\end{array}
$$

Concerning the centerless superVirasoro algebras, the homogeneous supermultiplets are encountered for

$$\mathcal{N}=1 \quad \mathrm{SVir}: \quad (k,1,1-k)_\lambda, \quad k=0,1 \quad \text{with} \quad \lambda \quad \text{arbitrary},$$
$$\mathcal{N}=2 \quad \mathrm{SVir}: \quad (k,2,2-k)_\lambda, \quad k=0,1,2 \quad \text{with} \quad \lambda \quad \text{arbitrary},$$
$$\mathcal{N}=3 \quad \mathrm{SVir}: \quad (1,3,3,1)_\lambda, \quad \text{with} \quad \lambda \quad \text{arbitrary},$$
$$\mathcal{N}=4 \quad \mathrm{SVir}: \quad (k,4,4-k)_\lambda, \quad k=0,1,2,3,4; \lambda=0 \text{ or } \lambda=\frac{1}{k-2}(k\neq 2$$

The inhomogeneous *D*-module reps of the centerless superVirasoro algebras are only encountered for $\mathcal{N}=1,2,3$ but not for $\mathcal{N}=4$:

$$\mathcal{N}=1 \quad \mathrm{SVir}: \quad (1,1)_{0,\rho},$$
$$\mathcal{N}=2 \quad \mathrm{SVir}: \quad (2,2,0)_{0,\rho} \quad \text{and} \quad (1,2,1)_{0,\rho},$$
$$\mathcal{N}=3 \quad \mathrm{SVir}: \quad (1,3,3,1)_{0,\rho},$$
$$\mathcal{N}=4 \quad \mathrm{SVir}: \quad \text{none}.$$

The D-module reps with $\mathcal{N}=1,2,4$ act on the $(\mathcal{N}+1|\mathcal{N})$ supermultiplets m,
$m^T=(\varphi_1,\ldots,\varphi_k,g_1,\ldots,g_{\mathcal{N}-k},1|\psi_1,\ldots,\psi_{\mathcal{N}})$, with component fields $\varphi_a,g_i,\psi_\alpha$ and constant entry 1 in the $(\mathcal{N}+1)$-th position.
The $\mathcal{N}=3$ D-module rep acts on a $(5|4)$ supermultiplet with 1 in the 5-th position. The homogeneous D-module reps are recovered by deleting the row and the column associated with the constant entry 1 in the supermultiplet.
For $\mathcal{N}=1$, in matrix form and in the hyperbolic presentation:

$$
\begin{aligned}
Q_r^0 &= e^{rt}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -\partial_t-2r\lambda & -2r\rho & 0 \end{pmatrix}, \\
L_n &= e^{nt}\begin{pmatrix} -\partial_t-n\lambda & -n\rho & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\partial_t-\frac{1}{2}n(1+2\lambda) \end{pmatrix}.
\end{aligned}
$$

The inhomogeneous D-module rep of $osp(1|2)$ is recovered for $n=0,\pm1$, $r=\pm\frac{1}{2}$ and by setting $\lambda=0$.

For $\mathcal{N}=2$:

the $(2,2,0)$ rep is constructed from

$$\begin{aligned}
Q_r^0 &= e^{rt}[E_{14}+E_{25}-2(E_{43}+E_{53})r\rho-(E_{41}+E_{52})(\partial_t+2r\lambda)],\\
Q_r^1 &= e^{rt}[E_{15}-E_{24}+2(E_{43}-E_{53})r\rho+(E_{42}-E_{51})(\partial_t+2r\lambda)];
\end{aligned}$$

the $(1,2,1)$ rep is constructed from

$$\begin{aligned}
Q_r^0 &= e^{rt}[E_{14}+E_{52}-2E_{43}r\rho-E_{25}r-(E_{25}+E_{41})(\partial_t+2r\lambda)],\\
Q_r^1 &= e^{rt}[E_{15}-E_{42}-2E_{53}r\rho+E_{24}r+(E_{24}-E_{51})(\partial_t+2r\lambda)];
\end{aligned}$$

the $(0,2,2)$ rep is constructed from

$$\begin{aligned}
Q_r^0 &= e^{rt}[E_{41}+E_{52}-(E_{14}+E_{25})(\partial_t+r+2r\lambda)],\\
Q_r^1 &= e^{rt}[E_{51}-E_{42}+(E_{24}-E_{15})(\partial_t+r+2r\lambda)].
\end{aligned}$$

For homogeneous transformations the Lagrangians of the $D(2,1;\alpha)$-invariant actions are:

in the *homogeneous parabolic* case:

$$\mathcal{L} \quad = \quad A(\dot{\varphi}^2 + \psi_I\dot{\psi}_I + g_i^2) + A_\varphi(\psi_0\psi_i g_i + \frac{1}{2}\epsilon^{ijk}\psi_i\psi_j g_k) + \frac{1}{6}A_{\varphi\varphi}\epsilon^{ijk}\psi_0\psi_i\psi_j$$

$$\text{with} \quad A = C\varphi^{-\frac{1+2\alpha}{\alpha}}$$

in the *homogeneous hyperbolic* case:

$$\mathcal{L} \quad = \quad A(\dot{\varphi}^2 + \mu\psi_I\dot{\psi}_I + \mu^2 g_i^2) + \mu^2 A_\varphi(\psi_0\psi_i g_i + \frac{1}{2}\epsilon^{ijk}\psi_i\psi_j g_k) + \frac{1}{6}\mu^2 A_{\varphi\varphi}\epsilon^{ijk}\psi_0\psi_i\psi_j\psi_k + \mu^2\alpha^2 A\varphi^2,$$

$$\text{with} \quad A = C\varphi^{-\frac{1+2\alpha}{\alpha}}.$$

For inhomogeneous transformations, the requirement that $\rho \neq 0$ with $\lambda = 0$ implies that the superconformal actions are only invariant under the $A(1,1)$ superalgebra.

In the *inhomogeneous parabolic* case:

$$\mathcal{L} \quad = \quad A(\dot{\varphi}^2 + \psi_I \dot{\psi}_I + g_i^2) + A_\varphi(\psi_0 \psi_i g_i + \frac{1}{2}\epsilon^{ijk}\psi_i\psi_j g_k) + \frac{1}{6}A_{\varphi\varphi}\epsilon^{ijk}\psi_0\psi_i\psi_j$$

$$\text{with} \quad A = Ce^{-\frac{\varphi}{\rho}}$$

in the *inhomogeneous hyperbolic* case:

$$\begin{aligned} \mathcal{L} \quad = \quad & A(\dot{\varphi}^2 + \mu\psi_I \dot{\psi}_I + \mu^2 g_i^2) + \mu^2 A_\varphi(\psi_0 \psi_i g_i + \frac{1}{2}\epsilon^{ijk}\psi_i\psi_j g_k) + \\ & \frac{1}{6}\mu^2 A_{\varphi\varphi}\epsilon^{ijk}\psi_0\psi_i\psi_j\psi_k + \mu^2\rho^2 A, \end{aligned}$$

$$\text{with} \quad A = Ce^{-\frac{\varphi}{\rho}}.$$

Fields redefinitions:

homogeneous

$$
\begin{aligned}
\overline{\phi} &= -2\alpha\phi^{-\frac{1}{2\alpha}},\\
\overline{\psi}_I &= \phi^{-\frac{1+2\alpha}{2\alpha}}\psi_I,\\
\overline{g}_i &= \phi^{-\frac{1+2\alpha}{2\alpha}}g_i.
\end{aligned}
$$

inhomogeneous

$$
\begin{aligned}
\overline{\phi} &= -2\rho e^{-\frac{\phi}{2\rho}},\\
\overline{\psi}_I &= e^{-\frac{\phi}{2\rho}}\psi_I,\\
\overline{g}_i &= e^{-\frac{\phi}{2\rho}}g_i.
\end{aligned}
$$

The actions in their respective “constant kinetic basis”, are

Homogeneous parabolic case:

$$\mathcal{L} = C(\dot{\bar{\phi}}^2 + \bar{\psi}_I\dot{\bar{\psi}}_I + \bar{g}_i{}^2) + \frac{2(1+2\alpha)C}{\bar{\phi}}\left(\bar{\psi}_0\bar{\psi}_i\bar{g}_i + \frac{1}{2}\epsilon^{ijk}\bar{\psi}_i\bar{\psi}_j\bar{g}_k\right) + \frac{2(1+2\alpha)(1+3\alpha)C}{3\bar{\phi}^2}\epsilon^{ijk}\bar{\psi}_0\bar{\psi}_i\bar{\psi}_j\bar{\psi}_k,$$

Inhomogeneous parabolic case:

$$\mathcal{L} = C(\dot{\bar{\phi}}^2 + \bar{\psi}_I\dot{\bar{\psi}}_I + \bar{g}_i{}^2) + \frac{2C}{\bar{\phi}}\left(\bar{\psi}_0\bar{\psi}_i\bar{g}_i + \frac{1}{2}\epsilon^{ijk}\bar{\psi}_i\bar{\psi}_j\bar{g}_k\right) + \frac{2C}{3\bar{\phi}^2}\epsilon^{ijk}\bar{\psi}_0\bar{\psi}_i\bar{\psi}_j\bar{\psi}_k,$$

Homogeneous hyperbolic case:

$$\mathcal{L} = C(\dot{\bar{\phi}}^2 + \mu\bar{\psi}_I\dot{\bar{\psi}}_I + \mu^2\bar{g}_i{}^2) + \frac{2(1+2\alpha)\mu^2 C}{\bar{\phi}}\Big(\bar{\psi}_0\bar{\psi}_i\bar{g}_i + \frac{1}{2}\epsilon^{ijk}\bar{\psi}_i\bar{\psi}_j\bar{g}_k\Big) + \frac{2(1+2\alpha)(1+3\alpha)\mu^2 C}{3\bar{\phi}^2}\epsilon^{ijk}\bar{\psi}_0\bar{\psi}_i\bar{\psi}_j\bar{\psi}_k + \frac{\mu^2 C}{4}\bar{\phi}^2,$$

Inhomogeneous hyperbolic case:

$$\mathcal{L} = C(\dot{\bar{\phi}}^2 + \mu\bar{\psi}_I\dot{\bar{\psi}}_I + \mu^2\bar{g}_i{}^2) + \frac{2\mu^2 C}{\bar{\phi}}\Big(\bar{\psi}_0\bar{\psi}_i\bar{g}_i + \frac{1}{2}\epsilon^{ijk}\bar{\psi}_i\bar{\psi}_j\bar{g}_k\Big) + \frac{2\mu^2 C}{3\bar{\phi}^2}\epsilon^{ijk}\bar{\psi}_0\bar{\psi}_i\bar{\psi}_j\bar{\psi}_k + \frac{\mu^2 C}{4}\bar{\phi}^2.$$

1D superconformal invariance does not imply supersymmetry:

Two types of grading (leading to parabolic or hyperbolic/trigonometric D-module reps).

The ordinary supersymmetry requires, for a given $\mathcal{N}$, that a set of $\mathcal{N}$ fermionic symmetry generators Q_i closes the supersymmetry algebra $\{Q_i, Q_j\} = 2\delta_{ij}H$, $[H, Q_i] = 0$ $(i, j = 1, \ldots, \mathcal{N})$, where H is the time-derivative operator (the "Hamiltonian").

In the hyperbolic/trigonometric cases, $\mathcal{N}$ fermionic symmetry generators can be found. They are the square roots of a symmetry generator (let's call it Z), which does not coincide with the Hamiltonian H. In the hyperbolic/trigonometric cases, two independent symmetry subalgebras $\{Q_i^\pm, Q_j^\pm\} = 2\delta_{ij}Z^\pm$, $[Z^\pm, Q_i^\pm] = 0$ (with $Z^+ \neq H$ and $Z^- \neq H$) are encountered. In the parabolic cases two independent symmetry subalgebras are also encountered and one of them can be identified with the ordinary supersymmetry ($Z^- = H$, $Z^+ \neq H$).

$osp(1|2)$-invariant example.

The hyperbolic action is

$$\mathcal{S} \;=\; \int dt(\dot{\varphi}^2 - \psi\dot{\psi} + \varphi^2).$$

The five invariant operators (closing the $osp(1|2)$ algebra) are given by

$$\begin{aligned} Q^{\pm}\varphi &= e^{\pm t}\psi, & Q^{\pm}\psi &= e^{\pm t}(\dot{\varphi} \mp \varphi),\\ Z^{\pm}\varphi &= e^{\pm 2t}(\dot{\varphi} \mp \varphi), & Z^{\pm}\psi &= e^{\pm 2t}\dot{\psi},\\ H\varphi &= \dot{\varphi}, & H\psi &= \dot{\psi}. \end{aligned}$$

One shoulde note that $Z^{\pm} = (Q^{\pm})^2$.
No change of time variable $t \mapsto \tau(t)$ allows to represent either Z^{+} or Z^{-} as a time-derivative operator with respect to the new time τ.

Further developments:

Test particles near BH horizon,

CFT_1/AdS_2 correspondence,

Computation of BH entropy via l.w.r.'s and ladder operators,

Higher-spin theories in the world-line framework,

Cosmology: supersymmetric mini-superspace.

Conformal topological theories (via twisted supersymmetry).

Extension to affine superalgebras.

Extension to Galileian superconformal theories.

Thanks for the attention!