Inflation and Integrable One-Field Cosmologies Embedded in Gauged Supergravity

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Supersymmetry in Integrable Systems
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One scalar flat cosmologies

$$\mathcal{L}(g,\phi) = \left[R[g] + \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\right]\sqrt{-g}$$

Generalized ansatz for spatially flat metric

$$ds^2 = -\,e^{2\mathcal{B}(t)}\,\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}\,,$$

Friedman equations

when B(t) = 0

$$\left\{\begin{aligned} H^2 &= \frac{1}{3}\,\dot{\phi}^2 + \frac{2}{3}\,V(\phi)\,, \\ \dot{H} &= -\,\dot{\phi}^2\,, \\ \ddot{\phi} + 3\,H\,\dot{\phi} + V' &= 0\,, \end{aligned}\right.$$

$$ds^2 = - e^{2\mathcal{B}(t)} \mathrm{d}t^2 + a^2(t) \, \mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x} ,$$

$$a(t) = e^{A(t)} , \qquad H(t) \equiv \frac{\dot{a}(t)}{a(t)} = \dot{A}(t) ,$$

$$dt_c = e^{\mathcal{B}(t)} dt ,$$

$$\mathcal{H} = \dot{\mathcal{A}} \equiv H\,(d-1)$$

$$\mathcal{L} = e^{\mathcal{A} - \mathcal{B}} \left[- \frac{1}{2} \dot{\mathcal{A}}^2 + \frac{1}{2} \dot{\varphi}^2 - e^{2\mathcal{B}} \mathcal{V}(\varphi) \right] ,$$

$$\ddot{\varphi} + \left(\dot{\mathcal{A}} - \dot{\mathcal{B}} \right) \dot{\varphi} + e^{2\mathcal{B}} \mathcal{V}'(\varphi) = 0 ,$$

$$\ddot{\mathcal{A}} = \dot{\mathcal{A}} \dot{\mathcal{B}} - \dot{\varphi}^2 ,$$

$$\dot{\mathcal{A}}^2 - \dot{\varphi}^2 = 2 e^{2\mathcal{B}} \mathcal{V}(\varphi) ,$$

Subsystem I

$$\dot{\varphi} = v \,,$$
$$\dot{v} = -\sigma\, v \sqrt{v^2 + 2\mathcal{V}(\varphi)} - \mathcal{V}'(\varphi)\,,$$
$$\sigma = \pm 1$$

where one should *exclude* possible branches satisfying the conditions

$$v^2 + 2\mathcal{V}(\varphi) = 0 \quad \text{if} \quad \mathcal{V}(\varphi) \neq 0\,,$$
$$v^2 + 2\mathcal{V}(\varphi) < 0\,.$$

$$\mathcal{H} = \sigma \sqrt{v^2 + 2\mathcal{V}(\varphi)}\,.$$

Subsystem II

$$\dot{\varphi} = \sigma \left(\mathcal{H}^2 - 2\mathcal{V}(\varphi) \right)^{\frac{1}{2}} ,$$

$$\dot{\mathcal{H}} = - \left(\mathcal{H}^2 - 2\mathcal{V}(\varphi) \right) ,$$

where one should *exclude* possible branches satisfying the conditions

$$\mathcal{H}^2 - 2\mathcal{V}(\varphi) = 0 \qquad \text{if} \qquad \mathcal{V}'(\varphi) \neq 0 ,$$
$$\mathcal{H}^2 - 2\mathcal{V}(\varphi) < 0 .$$

A well–developed theory of planar dynamical systems makes it possible to analyze qualitatively local and global properties of their phase portraits. Generic planar systems are indeed very regular and can have only a few different types of trajectories and limit sets. There are thus:

- **fixed points** (critical or stationary);
- **periodic orbits** (cycles);
- **homoclinic orbits**: connecting a given fixed point with itself;
- **heteroclinic orbits**: connecting pairs of different fixed points.

An implication of these results is that generic planar systems cannot be chaotic [1], and are therefore very special if compared with dynamical systems in more than two dimensions, where chaotic regimes are frequently present.

System I: Admissible fixed points

$$v_c = 0 \,, \qquad \mathcal{V}'(\varphi_c) = 0 \,,$$

if $$\mathcal{V}(\varphi_c) \geq 0$$

If Subsystem I is linearized around a fixed point, the resulting equations read

$$\begin{aligned} \dot{\phi} &= v \,, \\ \dot{v} &= -\sigma \sqrt{2\,\mathcal{V}(\varphi_c)}\, v \;-\; \mathcal{V}''(\varphi_c)\, \phi \,, \end{aligned}$$

where ϕ denotes the displacement of φ from its critical value,

$$\phi = \varphi - \varphi_c \,.$$

The corresponding eigenvalues

$$\lambda_\pm = -\,\sigma \sqrt{\frac{\mathcal{V}(\varphi_c)}{2}} \pm \sqrt{\frac{\mathcal{V}(\varphi_c)}{2} - \mathcal{V}''(\varphi_c)}$$

- **Hyperbolic fixed point:** $Re\,(\lambda_{\pm}) \neq 0$
  - **saddle:** if the two eigenvalues are real and have opposite signs;
  - **node** *(attracting or repelling)*: if the eigenvalues are real and have the same sign;
  - **improper node** *(attracting or repelling)*: if the two eigenvalues coincide;
  - **focus** *(attracting or repelling)*: if the two eigenvalues have the same real part;
- **Elliptic fixed point:** if the eigenvalues are purely imaginary.

An important result is that the phase portraits of a nonlinear system and of its linearization are qualitatively equivalent in a neighborhood of a *hyperbolic fixed point*, where $Re\,(\lambda_{\pm}) \neq 0$. Let us add that Subsystem I does not possess periodic trajectories on account of Dulac's criterion, since the expression

$$\frac{\partial \dot{\varphi}}{\partial \varphi} + \frac{\partial \dot{v}}{\partial v} = -2\sigma\,\frac{v^2 + \mathcal{V}(\varphi)}{\sqrt{v^2 + 2\mathcal{V}(\varphi)}} \tag{4.14}$$

does not change sign on the whole two–dimensional plane. We are thus led to conclude that Subsystem I can only have *fixed points*, *heteroclinic orbits* or *homoclinic orbits*.

In order to understand qualitatively the phase portrait in a neighborhood of an admissible fixed point one can analyze its structural stability. If the fixed point φ_c is a local *minimum* of the potential $\mathcal{V}(\varphi)$, one should define the weak Lyapunov function with the required properties,

$$\begin{aligned} f(\varphi, v) \; &\equiv \; \sqrt{v^2 \, + \, 2\,\mathcal{V}(\varphi)} \; - \; \sqrt{2\,\mathcal{V}(\varphi_c)} \; > \; 0 \; , \quad f(\varphi_c, v_c) \; = \; 0 \; , \\ \dot{f}(\varphi, \, v) \; &= \; - \, \sigma \, v^2 \; . \end{aligned} \tag{4.15}$$

By construction this function is positive definite in the domain of phase space delimited by the corresponding inequality (4.15) and vanish only at the fixed point, while its time derivative is negative or positive semi–definite depending on the sign of $\sigma = \pm 1$ and do not vanish identically on any trajectory other than the fixed point itself. From the constructed Lyapunov function one can conclude that this fixed point is unstable for $\sigma = -1$ and asymptotically stable for $\sigma = +1$. The inequality (4.15) defines explicitly the basin of attraction, i.e. the phase-space domain of asymptotic stability, and all trajectories crossing it approach asymptotically the fixed point as $t \to +\infty$.

The asymptotic behavior as $t \to +\infty$ of the Hubble function and of the scale factor that apply if the fixed point is asymptotically stable have the form

$$\mathcal{H} \; = \; \sqrt{2\,\mathcal{V}(\varphi_c)} \; , \quad \mathcal{A} \; = \; \sqrt{2\,\mathcal{V}(\varphi_c)}\,(t - t_0) \quad \text{if} \quad \mathcal{V}(\varphi_c) \; > \; 0 \; , \tag{4.16}$$

as pertains to an expanding de Sitter patch, while the exponential behavior leaves way to a power–like behavior if $\mathcal{V}(\varphi_c) = 0$. Let us also recall that in four dimensions $H = \frac{\mathcal{H}}{3}$ and $a = e^{\mathcal{A}/3}$.

Qualitative Analysis of Subsystem II and the "separatrix"

$$\mathcal{V}(\varphi) \;=\; \frac{1}{2}\left(\mathcal{P}(\varphi)^2 \;-\; \mathcal{P}'(\varphi)^2\right) \,,$$

$$\mathcal{P}(\varphi) \;\equiv\; \sqrt{\frac{2\,\mathcal{V}(\varphi)}{1 \;-\; (\mathfrak{P}(\varphi))^{-2}}} \,,$$

Abel equation of the first kind:

$$\mathfrak{P}'(\varphi) \;=\; \left((\mathfrak{P}(\varphi))^2 \;-\; 1\right)\left(\frac{(\log|\mathcal{V}(\varphi)|)'}{2}\,\mathfrak{P}(\varphi) \;\mp 1\right)$$

$$\begin{aligned}\dot{\varphi} \;&=\; \sigma\,\mathcal{P}'(\varphi) \,,\\ \mathcal{H} \;&=\; -\sigma\,\mathcal{P}(\varphi) \,,\end{aligned}$$

$$\epsilon = \frac{1}{2}\left(\frac{V'}{V}\right)^2 , \quad \eta = \frac{V''}{V} \quad N = \int_{\varphi_{end}}^{\varphi} \frac{V}{V'}\, d\varphi \qquad (N = 50 - 60)$$

$$n_s - 1 = -6\,\epsilon + 2\,\eta \approx -\frac{2}{N} \quad (n_s \approx 0.96) ,$$

$$r = 16\epsilon \approx \frac{8}{b^2 N^2} \quad (< 0.08)$$

Integrable natural two-dimensional systems

$$\mathcal{L} = e^{\mathcal{A} - \mathcal{B}} \left[- \frac{1}{2} \dot{\mathcal{A}}^2 + \frac{1}{2} \dot{\varphi}^2 - e^{2\mathcal{B}} \mathcal{V}(\varphi) \right] ,$$

h= $\dot{\mathcal{A}}^2 - \dot{\varphi}^2 - 2e^{2\mathcal{B}} \mathcal{V}(\varphi)$

	Infinite families of Integrable Potentials
I_1	$C_{11}\, e^{\sqrt{\frac{3}{2}}\,\hat{\phi}} + 2\, C_{12} + C_{22}\, e^{-\sqrt{\frac{3}{2}}\,\hat{\phi}}$
I_2	$C_1\, e^{,\gamma\,\sqrt{6}\,\hat{\phi}} + C_2 e^{(\gamma+1)\,\sqrt{\frac{3}{2}}\,\hat{\phi}}$
I_3	$C_1\, e^{\sqrt{6}\,\hat{\phi}} + C_2$
I_4	$C_1\, \hat{\phi}\, e^{\sqrt{6}\,\hat{\phi}}$

I_5	$C_1 \log\left(\coth[\sqrt{\frac{3}{2}}\,\hat{\phi}]\right) + C_2$
I_6	$C_1 \arctan\left(e^{-\sqrt{6}\,\hat{\phi}}\right) + C_2$
I_7	$C_1\left(\cosh\gamma\sqrt{\frac{3}{2}}\,\hat{\phi}\right)^{\frac{2}{\gamma}-2} + C_2\left(\sinh\gamma\sqrt{\frac{3}{2}}\,\hat{\phi}\right)^{\frac{2}{\gamma}-2}$
I_8	$C_1\left(\cosh[\gamma\sqrt{6}\,\hat{\phi}]\right)^{\frac{1}{\gamma}-1}\cos\left[\left(\frac{1}{\gamma}-1\right)\arccos\left(\tanh[\gamma\sqrt{6}\,\hat{\phi}] + C_2\right)\right]$

I_6	$C_1 \arctan\left(e^{-\sqrt{6}\,\hat{\phi}}\right) + C_2$
I_7	$C_1 \left(\cosh \gamma \sqrt{\frac{3}{2}}\,\hat{\phi}\right)^{\frac{2}{\gamma}-2} + C_2\left(\sinh \gamma \sqrt{\frac{3}{2}}\,\hat{\phi}\right)^{\frac{2}{\gamma}-2}$
I_8	$C_1 \left(\cosh[\gamma\sqrt{6}\,\hat{\phi}]\right)^{\frac{1}{\gamma}-1} \cos\left[\left(\frac{1}{\gamma}-1\right) \arccos\left(\tanh[\gamma\sqrt{6}\,\hat{\phi}] + C_2\right)\right]$
I_9	$C_1\, e^{\gamma\sqrt{6}\,\hat{\phi}} + C_2\, e^{\frac{1}{\gamma}\sqrt{6}\,\hat{\phi}}$
I_{10}	$C_1\, e^{\gamma\sqrt{6}\,\hat{\phi}} \cos\left(\sqrt{6}\,\hat{\phi}\sqrt{1-\gamma^2} + C_2\right)$

Table 1: Families of integrable potential functions for the Lagrangians of eq. (2.11)

Potential function $\mathcal{V}$	$\mathcal{A}, \varphi, \mathcal{B}$	$\mathcal{L}$, Hamilt. Constr., dt_c
$C_{11}\, e^{\varphi} + 2\, C_{12} + C_{22}\, e^{-\varphi}$	$\mathcal{A} = \log(x\, y)$ $\varphi = \log\left(\frac{x}{y}\right)$ $\mathcal{B} = 0$	$\mathcal{L} = -\,2\dot{x}\,\dot{y} - C_{11}\, x^2 - 2\, C_{12}\, x\, y - C_{22}\, y^2$ $2\,\dot{x}\,\dot{y} = C_{11}\, x^2 + 2\, C_{12}\, x\, y + C_{22}\, y^2$ $dt_c = dt$
$C_1\, e^{2\,\gamma\,\varphi} + C_2 e^{(\gamma+1)\,\varphi} \quad (\gamma^2 \neq 1)$	$\mathcal{A} = \log\left(x^{\frac{1}{1+\gamma}}\, y^{\frac{1}{1-\gamma}}\right)$ $\varphi = \log\left(x^{\frac{1}{1+\gamma}}\, y^{\frac{-1}{1-\gamma}}\right)$ $\mathcal{B} = \log\left(x^{\frac{-\gamma}{1+\gamma}}\, y^{\frac{\gamma}{1-\gamma}}\right)$	$\mathcal{L} = -\,4\dot{x}\,\dot{y} - 2(1-\gamma^2)\left[C_1\, x\, y + C_2\, x^{\frac{2}{1+\gamma}}\right]$ $2\,\dot{x}\,\dot{y} = (1-\gamma^2)\left[C_1\, x\, y + C_2\, x^{\frac{2}{1+\gamma}}\right]$ $dt_c = x^{-\frac{\gamma}{1+\gamma}}\, y^{\frac{\gamma}{1-\gamma}}\, dt$
$C_1\, e^{2\,\varphi} + C_2$	$\mathcal{A} = \frac{1}{2}\log x + v$ $\varphi = \frac{1}{2}\log x - v$ $\mathcal{B} = -\,\frac{1}{2}\log x + v$	$\mathcal{L} = -\,2\dot{x}\dot{v} - 2\, C_1 x - 2\, C_2\, e^{2v}$ $\dot{x}\,\dot{v} = C_1\, e^{2v} + C_2\, x$ $dt_c = e^{v}\, x^{-\frac{1}{2}}\, dt$
$C\,\varphi\, e^{2\,\varphi}$	$\mathcal{A} = \frac{1}{4}\log x + v$ $\varphi = \frac{1}{4}\log x - v$ $\mathcal{B} = -\,\frac{3}{4}\log x + v$	$\mathcal{L} = -\,\frac{1}{2}\,\dot{x}\,\dot{v} - C\left(\frac{1}{4}\log x - v\right)$ $\dot{x}\,\dot{v} = C\left(\frac{1}{2}\log x - 2\, v\right)$ $dt_c = x^{-\frac{3}{4}}\, e^{v}\, dt$
$C\, \log(\coth\varphi) + D$	$\mathcal{A} = \frac{1}{2}\log\left(\frac{\xi^2 - \eta^2}{2}\right)$ $\varphi = \frac{1}{2}\log\left(\frac{\xi + \eta}{\xi - \eta}\right)$ $\mathcal{B} = -\,\frac{1}{2}\log\left(\frac{\xi^2 - \eta^2}{2}\right)$	$\mathcal{L} = -\,\dot{\xi}^{\,2} + \dot{\eta}^{\,2} - 8\, C\, \log\left(\frac{\xi}{\eta}\right) - 8\, D$ $\dot{\xi}^{\,2} - \dot{\eta}^{\,2} = 8\, C\, \log\left(\frac{\xi}{\eta}\right) + 8\, D$ $dt_c = \frac{2\, dt}{\sqrt{\xi^2 - \eta^2}}$

$$C\,\mathrm{Im}\left[\log\left(\frac{e^{2\varphi}+i}{e^{2\varphi}-i}\right)\;+\;D\right]$$

$$\mathcal{A}=\tfrac{1}{2}\,\log\left(\frac{\xi^2-\eta^2}{2}\right)$$
$$\varphi=\tfrac{1}{2}\,\log\left(\frac{\xi+\eta}{\xi-\eta}\right)$$
$$\mathcal{B}=-\,\tfrac{1}{2}\,\log\left(\frac{\xi^2-\eta^2}{2}\right)$$
$$z=\frac{1}{\sqrt{2}}\,\left(\xi\,e^{\frac{i\pi}{4}}+\eta\,e^{-\frac{i\pi}{4}}\right)$$

$$\mathcal{L}=2\,\mathrm{Im}\left[-\dot{z}^2-8\,C\,\log z-8\,D\right]$$
$$\mathrm{Im}\left[\dot{z}^2-8\,C\,\log z-8\,D\right]=0$$
$$dt_C=\frac{2\,dt}{\sqrt{\xi^2-\eta^2}}$$

$$C_1\left(\cosh\gamma\,\varphi\right)^{\frac{2}{\gamma}-2}+\,C_2\left(\sinh\gamma\,\varphi\right)^{\frac{2}{\gamma}-2}$$

$$\mathcal{A}=\tfrac{1}{2\gamma}\,\log\left(\xi^2-\eta^2\right)$$
$$\varphi=\tfrac{1}{2\gamma}\,\log\left(\frac{\xi+\eta}{\xi-\eta}\right)$$
$$\mathcal{B}=\left(\tfrac{1}{2\gamma}-1\right)\,\log(\xi^2-\eta^2)$$

$$\mathcal{L}=-\,\frac{\dot{\xi}^2-\dot{\eta}^2}{2\,\gamma^2}-C_1\,\xi^{\frac{2}{\gamma}-2}-C_2\,\eta^{\frac{2}{\gamma}-2}$$
$$\dot{\xi}^2-\dot{\eta}^2=2\,\gamma^2\left[C_1\,\xi^{\frac{2}{\gamma}-2}+C_2\,\eta^{\frac{2}{\gamma}-2}\right]$$
$$dt_C=(\xi^2-\eta^2)^{\frac{1}{2\gamma}-1}\,dt$$

$$\mathrm{Im}\left[\,C\,\left(i\,+\,\sinh\,2\,\gamma\,\varphi\right)^{\frac{1}{\gamma}-1}\right]$$

$$\mathcal{A}=\tfrac{1}{2\gamma}\,\log\left(\xi^2-\eta^2\right)$$
$$\varphi=\tfrac{1}{2\gamma}\,\log\left(\frac{\xi+\eta}{\xi-\eta}\right)$$
$$\mathcal{B}=\left(\tfrac{1}{2\gamma}-1\right)\,\log(\xi^2-\eta^2)$$
$$z=\frac{1}{\sqrt{2}}\,\left(\xi\,e^{\frac{i\pi}{4}}+\eta\,e^{-\frac{i\pi}{4}}\right)$$

$$\mathcal{L}=\mathrm{Im}\left[-\,\frac{1}{2\,\gamma^2}\,\dot{z}^2-\frac{C}{2}\,z^{\frac{2}{\gamma}-2}\right]$$
$$\mathrm{Im}\left[\frac{1}{2\,\gamma^2}\,\dot{z}^2-\frac{C}{2}\,z^{\frac{2}{\gamma}-2}\right]=0$$
$$dt_C=(\xi^2-\eta^2)^{\frac{1}{2\gamma}-1}\,dt$$

$$C_1\,e^{2\,\gamma\,\varphi}\;+\;C_2\,e^{\frac{2}{\gamma}\,\varphi}\qquad(\gamma^2\neq1)$$

$$\mathcal{A}=\frac{1}{\sqrt{1-\gamma^2}}\left(\widehat{\mathcal{A}}-\gamma\,\widehat{\varphi}\right)$$
$$\varphi=\frac{1}{\sqrt{1-\gamma^2}}\left(\widehat{\varphi}-\gamma\,\widehat{\mathcal{A}}\right)$$
$$\mathcal{B}=\mathcal{A}$$

$$\mathcal{L}=\frac{\dot{\widehat{\varphi}}^2-\dot{\widehat{\mathcal{A}}}^2}{2}-C_1\,e^{2\widehat{\mathcal{A}}\sqrt{1-\gamma^2}}-C_2\,e^{\frac{2\widehat{\varphi}\sqrt{1-\gamma^2}}{\gamma}}$$
$$\frac{\dot{\widehat{\varphi}}^2-\dot{\widehat{\mathcal{A}}}^2}{2}=-\,C_1\,e^{2\widehat{\mathcal{A}}\sqrt{1-\gamma^2}}-C_2\,e^{\frac{2\widehat{\varphi}\sqrt{1-\gamma^2}}{\gamma}}$$
$$dt_C=\exp\left[\frac{\widehat{\mathcal{A}}-\gamma\,\widehat{\varphi}}{\sqrt{1-\gamma^2}}\right]dt$$

Sporadic integrable potentials

$$\xi = \frac{1}{2}\left(e^{\gamma(\mathcal{A}+\varphi)} + e^{\gamma(\mathcal{A}-\varphi)}\right) , \quad \eta = \frac{1}{2}\left(e^{\gamma(\mathcal{A}+\varphi)} - e^{\gamma(\mathcal{A}-\varphi)}\right) , \quad \mathcal{B} = \mathcal{A}(1-2\gamma)$$

$$\mathcal{L} = -\frac{1}{2}\left(\dot{\xi}^2 - \dot{\eta}^2\right) - \mathcal{V}_c(\xi,\eta) ,$$

$$\mathcal{V}_c(\xi,\eta) = \gamma^2 \left(\xi^2 - \eta^2\right)^{\left(\frac{1}{\gamma}-1\right)} \mathcal{V}(\varphi) ,$$

Group Ia ($\gamma = \frac{2}{5}$)

$$\mathcal{V}_{Ia}(\varphi) \;=\; \lambda\left[a\,\cosh^3\left(\frac{2\varphi}{5}\right) \,+\, b\,\sinh^2\left(\frac{2\varphi}{5}\right)\cosh\left(\frac{2\varphi}{5}\right)\right]$$

$$\mathcal{V}_{c,\,Ia}(\xi,\eta) \,=\, \frac{4}{25}\,\lambda\left[a\,\xi^3 \,+\, b\,\xi\,\eta^2\right].$$

$$\{a,b\} \,=\, \left\{\begin{array}{cc} 1 & -3 \\ 1 & -\frac{1}{2} \\ 1 & -\frac{3}{16} \end{array}\right\},$$

$$\mathcal{Q}^{(1)}_{Ia}(\xi,\eta) \,=\, 25\,\dot{\xi}\,\dot{\eta} \,+\, 4\,\lambda\,\eta\left(\eta^2 \,-\, 3\,\xi^2\right),$$

$$\mathcal{Q}^{(2)}_{Ia}(\xi,\eta) \,=\, 50\,\dot{\eta}\left(\eta\,\dot{\xi} \,-\, \dot{\eta}\,\xi\right) \,+\, \lambda\,\eta^2\left(\eta^2 \,-\, 4\,\xi^2\right),$$

$$\mathcal{Q}^{(3)}_{Ia}(\xi,\eta)) \,=\, \dot{\eta}^4 \,+\, \frac{\lambda}{25}\,\dot{\eta}\,\eta^2\left(\eta\,\dot{\xi} \,-\, 3\,\dot{\eta}\,\xi\right) \,+\, \frac{\lambda^2}{5000}\,\eta^4\left(\eta^2 \,-\, 6\,\xi^2\right).$$

Group Ib $(\gamma = \frac{2}{5})$

$$\mathcal{V}_{Ib}(\varphi) \;\; = \;\; \lambda \left[a \sinh^3 \left(\frac{2\varphi}{5} \right) + b \cosh^2 \left(\frac{2\varphi}{5} \right) \sinh \left(\frac{2\varphi}{5} \right) \right]$$

$$\mathcal{V}_{c,\,Ib}(\xi, \eta) \, = \, - \, \frac{4}{25} \, \lambda \left[a \; \eta^3 \; + \; b \; \xi^2 \; \eta \right] .$$

$$\{a, b\} \, = \left\{ \begin{array}{cc} 1 & -3 \\ 1 & -\frac{1}{2} \\ 1 & -\frac{3}{16} \end{array} \right\} \, ,$$

$$\xi \; \rightarrow \; i \, \eta \; , \qquad \eta \; \rightarrow \; i \, \xi \; , \qquad \lambda \; \rightarrow - \, i \, \lambda \; .$$

Group II $(\gamma = \frac{1}{3})$

$$\mathcal{V}_{II}(\varphi) = \lambda \left[a \cosh^4\left(\frac{\varphi}{3}\right) + b \sinh^4\left(\frac{\varphi}{3}\right) + c \sinh^2\left(\frac{\varphi}{3}\right)\cosh^2\left(\frac{\varphi}{3}\right)\right]$$

$$\mathcal{V}_{c,II}(\xi,\eta) = \frac{1}{9}\lambda\left[a\,\xi^4 + b\,\eta^4 + c\,\xi^2\,\eta^2\right]$$

$$\{a,b,c\} = \left\{\begin{array}{ccc} 1 & 1 & -2 \\ 1 & 1 & -6 \\ 1 & 8 & -6 \\ 1 & 16 & -12 \\ 1 & \frac{1}{8} & -\frac{3}{4} \\ 1 & \frac{1}{16} & -\frac{3}{4} \end{array}\right\},$$

$$\mathcal{Q}^{(1)}_{II}(\xi,\eta) = \eta\,\dot{\xi} - \dot{\eta}\,\xi\,,$$

$$\mathcal{Q}^{(2)}_{II}(\xi,\eta) = \frac{9}{4}\dot{\xi}\,\dot{\eta} + \lambda\,\eta\,\xi\left(\eta^2 - \xi^2\right),$$

$$\mathcal{Q}^{(3)}_{II}(\xi,\eta) = \left[\frac{9}{2}\,\dot{\xi}^2 + \lambda\,\xi^2\,(2\,\eta^2 - \xi^2)\right]^2 + 9\,\lambda\,\xi^2\left(\xi\,\dot{\eta} - 2\,\dot{\xi}\,\eta\right)^2,$$

$$\mathcal{Q}^{(4)}_{II}(\xi,\eta) = (\xi\,\dot{\eta} - \dot{\xi}\,\eta)\,\dot{\xi} - \frac{4}{9}\,\lambda\,\eta\,\xi^2\,(\xi^2 - 2\,\eta^2)\,,$$

$$\mathcal{Q}^{(5)}_{II}(\xi,\eta) = \left[\frac{9}{2}\dot{\eta}^2 + \frac{1}{8}\lambda\left(\eta^2 - 2\xi^2\right)\eta^2\right]^2 - \frac{9}{8}\lambda\eta^2\left(2\xi\dot{\eta} - \eta\dot{\xi}\right)^2,$$

$$\mathcal{Q}^{(6)}_{II}(\xi,\eta) = (\eta\,\dot{\xi} - \dot{\eta}\,\xi)\,\dot{\eta} + \frac{1}{36}\,\lambda\,\xi\,\eta^2\,(\eta^2 - 2\,\xi^2)\,.$$

Group III $(\gamma = \frac{2}{5})$

$$\mathcal{V}_{IIIa}(\varphi) = \lambda \left[a \cosh^3\left(\frac{2\varphi}{5}\right) + b \sinh^2\left(\frac{2\varphi}{5}\right) \cosh\left(\frac{2\varphi}{5}\right) + c \sinh^3\left(\frac{2\varphi}{5}\right) \right]$$

$$\mathcal{V}_{c,\,IIIa}(\xi,\eta) = \frac{4}{25}\,\lambda \left[a\,\xi^3 + b\,\xi\,\eta^2 + c\,\eta^3 \right] ,$$

$$\{a,b,c\} = \left\{ 1\,,\ -\frac{1}{2}\,,\ \frac{1}{6\sqrt{3}} \right\} .$$

$$\begin{aligned} \mathcal{Q}_{IIIa}(\xi,\eta) &= \dot{\eta}^4 + \frac{2}{\sqrt{3}}\,\dot{\xi}\,\dot{\eta}^3 - \frac{4\,\lambda}{25\sqrt{3}}\,\dot{\xi}^2\,\eta^3 + \frac{4\,\lambda}{25}\,(\eta^3 + \sqrt{3}\,\xi\,\eta^2)\,\dot{\xi}\,\dot{\eta} \\ &+ \frac{4\,\lambda}{25}\left(-2\sqrt{3}\,\xi^2\,\eta + \frac{1}{\sqrt{3}}\,\eta^3 - 2\,\xi\,\eta^2 \right)\dot{\eta}^2 \\ &+ \frac{4\,\lambda^2}{25^2}\left(\frac{4}{\sqrt{3}}\,\xi^3\,\eta^3 - \frac{2}{\sqrt{3}}\,\xi\,\eta^5 - \xi^2\,\eta^4 + \frac{5}{9}\,\eta^6 \right) . \end{aligned}$$

Group III $\left(\gamma = \frac{2}{5}\right)$

$$\mathcal{V}_{IIIb}(\varphi) = a \cosh^3\left(\frac{2\varphi}{5}\right) + b \sinh\left(\frac{2\varphi}{5}\right)\cosh^2\left(\frac{2\varphi}{5}\right) + c \sinh^3\left(\frac{2\varphi}{5}\right) ,$$

$$\mathcal{V}_{c,\,IIIb}(\xi,\eta) = -\frac{4}{25}\,\lambda\left[a\,\xi^3 + b\,\xi^2\,\eta + c\,\eta^3\right] ,$$

$$\{a,b,c\} = \left\{1,\ -3\sqrt{3},\ 6\sqrt{3}\right\}$$

$$\xi \to i\,\eta\,, \qquad \eta \to i\,\xi\,, \qquad \lambda \to -\,i\,6\sqrt{3}\,\lambda$$

Group IV $(\gamma = 3)$

$$\mathcal{V}_{IV\,a} \;=\; \frac{\lambda}{\cosh^{\frac{2}{3}}(6\varphi)}\;,$$

$$\mathcal{V}_{IV\,b} \;=\; \frac{\lambda}{\sinh^{\frac{2}{3}}(6\varphi)}\;,$$

$$\mathcal{V}_{IV\,ac} \;=\; 9\,\frac{\lambda}{(\xi^2 + \eta^2)^{\frac{2}{3}}}\;,$$

$$\mathcal{V}_{IV\,bc} \;=\; 9\,\frac{\lambda}{(2\,\xi\,\eta)^{\frac{2}{3}}}\;.$$

$$\mathcal{Q}_{IV\,a}(\xi,\eta) \;=\; (\dot{\eta}^2 + \dot{\xi}^2)\,(\eta\,\dot{\xi} - \dot{\eta}\,\xi) - 4\,\lambda\,(\eta\,\dot{\xi} + \dot{\eta}\,\xi)\,(\eta^2 + \xi^2)^{-\frac{2}{3}}\;,$$

$$\mathcal{Q}_{IV\,b}(\xi,\eta) \;=\; (\eta\,\dot{\xi} - \dot{\eta}\,\xi)\,\dot{\xi}\,\dot{\eta} - 2\,\lambda\,(\eta\,\dot{\eta} + \xi\,\dot{\xi})\,(2\,\eta\,\xi)^{-\frac{2}{3}}\;.$$

Group Va $(\gamma = \frac{3}{5})$

$$\mathcal{V}_{Va}(\varphi) \;=\; \lambda\left[a\,\cosh^{\frac{4}{3}}\left(\frac{3\varphi}{5}\right) \;+\; b\,\frac{\sinh^2\left(\frac{3\varphi}{5}\right)}{\cosh^{\frac{2}{3}}\left(\frac{3\varphi}{5}\right)}\right]$$

$$\mathcal{V}_{c,Va}(\xi,\eta) = \frac{9}{25}\,\lambda\Big[a\,\xi^{\frac{4}{3}} \;+\; b\,\xi^{-\frac{2}{3}}\,\eta^2\Big]\;,$$

$$\{b,a\} = \left\{\begin{array}{cc} 1 & -\frac{3}{4} \\ 1 & -\frac{9}{2} \\ 1 & -12 \end{array}\right\}$$

$$\begin{aligned}
\mathcal{Q}^{(1)}_{Va}(\xi,\eta) \;&=\; \Big[\dot\eta^2 \;-\; \frac{3}{2}\,\dot\xi^2 \;+\; \frac{9}{25}\,\lambda\Big(\frac{9}{2}\,\xi^{\frac{4}{3}} \;+\; 3\,\xi^{-\frac{2}{3}}\,\eta^2\Big)\Big]\,\dot\eta \;-\; \frac{81}{25}\,\lambda\,\eta\,\xi^{\frac{1}{3}}\,\dot\xi\;, \\
\mathcal{Q}^{(2)}_{Va}(\xi,\eta) \;&=\; \Big(\dot\eta^2 \;-\; 2\,\dot\xi^2 \;+\; \frac{36}{25}\,\lambda\,\xi^{-\frac{2}{3}}\,\eta^2\Big)\,\dot\eta^2 \;-\; \frac{216}{25}\,\lambda\,\eta\,\xi^{\frac{1}{3}}\,\dot\eta\,\dot\xi \;-\; \frac{5832}{625}\,\lambda\,\eta^2\,\xi^{\frac{2}{3}}\;, \\
\mathcal{Q}^{(3)}_{Va}(\xi,\eta) \;&=\; \Big[\dot\eta^2 \;-\; 3\,\dot\xi^2 \;-\; \frac{54}{25}\,\lambda\Big(3\,\xi^{\frac{4}{3}} \;-\; \xi^{-\frac{2}{3}}\,\eta^2\Big)\Big]\,\dot\eta^4 \;-\; \frac{648}{25}\,\lambda\,\eta\,\xi^{\frac{1}{3}}\,\dot\eta^3\,\dot\xi \\
&\;-\; \Big(\frac{9}{25}\,\lambda\Big)^2\,648\,\eta^2\,\xi^{\frac{2}{3}}\,\dot\eta^2 \;-\; \Big(\frac{9}{25}\,\lambda\Big)^3\,648\,\eta^4\;.
\end{aligned}$$

Group Vb ($\gamma = \frac{3}{5}$)

$$\mathcal{V}_{Vb}(\varphi) = \lambda \left[a \sinh^{\frac{4}{3}}\left(\frac{3\varphi}{5}\right) + b \, \frac{\cosh^2\left(\frac{3\varphi}{5}\right)}{\sinh^{\frac{2}{3}}\left(\frac{3\varphi}{5}\right)} \right]$$

$$\mathcal{V}_{Va}(\varphi) = -\frac{9}{25}\,\lambda\Big[a\,\eta^{\frac{4}{3}} + b\,\xi^2\,\eta^{-\frac{2}{3}}\Big] ,$$

$$\{b, a\} = \left\{ \begin{array}{cc} 1 & -\frac{3}{4} \\ 1 & -\frac{9}{2} \\ 1 & -12 \end{array} \right\}$$

$$\xi \to i\,\eta\,, \quad \eta \to i\,\xi \quad \lambda \to -\lambda\,.$$

Group VI $(\gamma = \frac{6}{7})$

$$\mathcal{V}_{VIa} = \frac{\lambda \sinh\left(\frac{6\varphi}{7}\right)}{\cosh^{\frac{2}{3}}\left(\frac{6\varphi}{7}\right)} ,$$

$$\mathcal{V}_{VIb} = \frac{\lambda \cosh\left(\frac{6\varphi}{7}\right)}{\sinh^{\frac{2}{3}}\left(\frac{6\varphi}{7}\right)} ,$$

$$\mathcal{V}_{VIac} = \frac{36}{49} \lambda \, \eta \, \xi^{-\frac{2}{3}} ,$$

$$\mathcal{V}_{VIbc} = \frac{36}{49} \lambda \, \xi \, \eta^{-\frac{2}{3}} .$$

$$\mathcal{Q}_{VIa}(\xi,\eta) = \dot{\xi}\,(3\,\dot{\eta}^2 - 2\,\dot{\xi}^2) + \frac{18}{49}\lambda\,\eta\,\xi^{-\frac{5}{3}}\left(\frac{2}{3}\,\eta\,\dot{\eta} - \xi\,\dot{\xi}\right) ,$$

$$\mathcal{Q}_{VIb}(\xi,\eta) = \dot{\eta}\,(3\,\dot{\xi}^2 - 2\,\dot{\eta}^2) - \frac{18}{49}\lambda\,\xi\,\eta^{-\frac{5}{3}}\left(\frac{2}{3}\,\xi\,\dot{\xi} - \eta\,\dot{\eta}\right) .$$

Group VII $(\gamma = -1)$

$$\mathcal{V}_{VIIa}(\varphi) = \lambda\left[-\frac{1}{4}(\cosh\varphi)^{-4} + (\sinh\varphi)^2(\cosh\varphi)^{-6}\right] ,$$
$$\mathcal{V}_{VIIb}(\varphi) = \lambda\left[-\frac{1}{4}(\sinh\varphi)^{-4} + (\cosh\varphi)^2(\sinh\varphi)^{-6}\right] .$$

$$\mathcal{V}_{c,VIIa}(\xi,\eta) = \lambda\left[-\frac{1}{4}\xi^{-4} + \eta^2\xi^{-6}\right] ,$$
$$\mathcal{V}_{c,VIIb}(\xi,\eta) = \lambda\left[-\frac{1}{4}\eta^{-4} + \xi^2\eta^{-6}\right] .$$

$$\mathcal{Q}_{VIIa}(\xi,\eta) = (\dot\eta\,\xi - \eta\,\dot\xi)\,\dot\xi - \lambda\,\eta\,\xi^{-4} + 2\,\lambda\,\eta^3\,\xi^{-6} ,$$
$$\mathcal{Q}_{VIIb}(\xi,\eta) = (\dot\xi\,\eta - \xi\,\dot\eta)\,\dot\eta + \lambda\,\xi\,\eta^{-4} - 2\,\lambda\,\xi^3\,\eta^{-6} .$$

An integrable trigonometric potential

Relations to Toda systems

D_2

$$\mathcal{V}_{trig}(\varphi) \;=\; \lambda \;\cos 2\,\varphi \;,$$

A_2 $SL(3)/SO(3)$ and $SL(3)/SO(1,2)$

$$\mathcal{V}_{A_2}(\varphi) \;=\; \lambda \cos\left(2\,\sqrt{3}\,\varphi\right)$$

N=1 Gauged SUGRA

$$g_{ij^\star} = \partial_i \partial_{j^\star} \mathcal{K} \quad \Rightarrow \quad ds^2_K = \partial_i \partial_{j^\star} \mathcal{K} \, dz^i \otimes dz^{j^\star}$$

$$\mathrm{K} = \mathrm{i} \, g_{ij^\star} \, dz^i \wedge dz^{j^\star}$$

$$G(z,\bar{z}) = \log ||W||^2 = \mathcal{K} + \log W + \log \overline{W}$$

$$z^i \to z^i + \epsilon^\Lambda \, k^i_\Lambda(z)$$

$$k^i_\Lambda(z) = \mathrm{i} \, g^{ij^\star} \, \partial_{j^\star} \mathcal{P}_\Lambda \quad ; \quad k^{j^\star}_\Lambda(\bar{z}) = -\mathrm{i} \, g^{ij^\star} \, \partial_i \mathcal{P}_\Lambda$$

$$\mathcal{P}_\Lambda = -\mathrm{i} \, \tfrac{1}{2} \left(k^i_\Lambda \, \partial_i G - k^{i^\star}_\Lambda \, \partial_{i^\star} G \right)$$

$$\begin{aligned} \mathcal{L}^{(4)} &= \sqrt{|\det g|} \left[R[g] - \frac{1}{2} \nabla_\mu z^i \, \nabla_\nu z^{j^\star} g_{ij^\star} \, g^{\mu\nu} + \mathrm{Im}\mathcal{N}_{\Lambda\Sigma} \, F^\Lambda_{\mu\nu} F^{\Sigma|\mu\nu} - V \right] \\ &\quad + \frac{1}{2} \mathrm{Re}\mathcal{N}_{\Lambda\Sigma} \, F^\Lambda_{\mu\nu} F^\Sigma_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} \, , \end{aligned}$$

$$\begin{aligned} V &= \underbrace{\tfrac{1}{4} \, g_{ij^\star} \, \mathcal{H}^i \, \mathcal{H}^{j^\star}}_{\text{chiralinos contr.}} - \underbrace{3 \, S \, S^\star}_{\text{gravitino contr.}} + \underbrace{\tfrac{1}{3} \left(\mathrm{Im} \, \mathcal{N}^{-1} \right)_{\Lambda\Sigma} D^\Lambda \, D^\Sigma}_{\text{gaugino contr.}} \\ &= e^2 \exp\left[\mathcal{K}\right] \left(g^{ij^\star} \, \mathcal{D}_i \, W \, D_{j^\star} \overline{W} - 3 \, |W|^2 \right) + \tfrac{1}{3} \, g^2 \, \mathrm{Im} \, \mathcal{N}_{\Lambda\Sigma} \, \mathcal{P}^\Lambda \, \mathcal{P}^\Sigma \end{aligned}$$

$$\frac{1}{\sqrt{-g}}\,\mathcal{L} = -\frac{1}{2}\,\mathrm{R} - \frac{1}{4}\,F_{\mu\nu}^2(B) + \frac{1}{2}\,J''(C)(\partial C)^2$$
$$+\frac{g^2}{2}\,J''(C)B_\mu^2 - \frac{g^2}{2}\,J'(C)^2 \qquad \left(B_\mu = A_\mu + \frac{1}{g}\,\partial_\mu B\right)$$

The potential is the square of a D-term contribution:

$$P(C) = J'(C)\,, \qquad V(C) = \frac{g^2}{2}\,P^2(C)$$

Starting from $\quad P'(C) = J''(C) = \left(\frac{d\varphi}{dC}\right)^2$

with P(C(φ)) = P(φ) → $\quad P'(\varphi) = \frac{d\varphi}{dC}\quad$ so that

$$C(\varphi) = \int d\varphi\,\frac{dC}{d\varphi} = \int d\varphi\,\frac{1}{P'(\varphi)}\,,$$
$$J(C) = \int dC\,J'(C) = \int P(\varphi)\frac{dC}{d\varphi}\,d\varphi = \frac{P(\varphi)}{P'(\varphi)}\,d\varphi$$

These relations link the (C, φ) variables, where

$$\frac{1}{2}\,J''(C)\,(\partial C)^2 = \frac{1}{2}\,(\partial\varphi)^2$$

Integration constant in P(φ) → Fayet-Iliopoulos term.

The Σ Kahler surface

$$
\begin{aligned}
ds^2_{\Sigma} &= \frac{1}{2}\frac{d^2 J}{dC^2}\left(dC^2 + dB^2\right) \\
&= d\phi^2 + \left(\partial_\phi \mathcal{P}(\phi)\right)^2 dB^2
\end{aligned}
$$

C = Van Proeyen coordinate

φ= canonical coordinate = the inflaton scalar

2 J(C) = the Kahler potential

$\mathcal{P}(\phi)$ = the momentum map of the isometry

Killing vector $\vec{k} = \partial_B$

THE SCALAR POTENTIAL $V(\phi) = (\mathcal{P}(\phi))^2$

The curvature of the 1D Kahler manifold $\mathrm{R} \;=\; \mathrm{R}_{z\bar{z}}\; g^{z\bar{z}} \;=\; -\,\partial_z \partial_{\bar{z}} \log g_{z\bar{z}} g_{z\bar{z}}^{-1}$

can be expressed in terms of J(C) as

$$\mathrm{R}(C) = \frac{J'''(C)^2 \;-\; J''(C)\, J''''(C)}{2\, J''(C)^2}$$

and in terms of P(φ) as

$$\mathrm{R}(\varphi) \;=\; -\,4\,\frac{P'''(\varphi)}{P'(\varphi)}$$

Flat case: P(φ) **linear** or **quadratic** → V **quadratic** or **quartic**

Curved case: combination of exponentials:

$$P(\varphi) \;=\; a\,\exp(\nu\varphi) \;+\; b\,\exp(-\nu\varphi) \;+\mu$$

Starobinsky corresponds to a=0, $\nu > 0$ and $b\mu < 0$. For $\nu = \sqrt{\frac{2}{3}}$ one obtains the theory dual to R + R^2 supergravity (in the "new minimal" formulation):

ν → Kahler curvature , μ → Fayet-Iliopoulos term

The three different solutions correspond to the gauging of different 1D subgroups of SL(2,R): **elliptic** (U(1)); **hyperbolic** (SO(1,1)); **parabolic** (translations)

Topological properties of the inflaton field:

A topology of the Kahler manifold is defined by the topological properties of the inflaton field (the inflaton potential). It is encoded in the definition range of the coordinates C and B of the Kahler manifold.

Three types of isometries

Definition B.1 *A Hadamard manifold is a simply connected, geodesically complete Riemannian manifold* $\mathcal{H} = (\mathcal{M}, g)$ *whose scalar curvature* $R(x)$ *is* nonpositive definite and finite, *namely* $-\infty < R(x) \le 0, \quad \forall x \in \mathcal{M}$. **(Generalization to the case of CAT(k) manifolds!?)**

Σ is the Kahler surface associated with the D-type potential. We assume that Σ is simply connected ($\pi_1(\Sigma) = 1$) and that its curvature is everywhere finite.

We can classify the isometry groups.

1) **The isometry group is elliptic if there exists a fixed point in the interior of Σ**

$$\forall \Gamma \in \mathcal{G}_\Sigma \qquad \Gamma . p_0 = p_0 \qquad p_0 \in \Sigma$$

Asymptotic behavior of the J(C) function at the fixed point of elliptic isometries

$$C_0 = -\infty \qquad J(C) \overset{C \to C_0}{\approx} \exp\left[\delta\, C\right]$$

2) **The isometry group is parabolic if there exists no fixed point in the interior of Σ and just one fixed point on the boundary $\partial\Sigma$**

The range of C is infinite

$$C \in [-\infty, \infty] \quad \text{or} \quad C \in [0, \pm\infty]$$

$$B \in [-\infty, \infty]$$

3) **The isometry group is hyperbolic if there exists no fixed point in the interior of Σ and just two fixed points on the boundary $\partial\Sigma$**

The range of C is finite

$$C \in [A_1, A_2] \quad |A_1| < \infty \quad |A_2| < \infty$$

$$B \in [-\infty, \infty]$$

Curv.	Gauge Group	$V(\sqrt{\frac{1}{2}}\,\hat{\phi})$	Values of ν	Values of μ	Mother series
$-8\,\hat{\nu}^2$	U(1)	$\left(\cosh\left(\hat{\nu}\,\hat{\phi}\right)+\mu\right)^2$	$\hat{\nu}=\frac{1}{2}\sqrt{\frac{3}{2}}$	$\mu=0$	I_1 or I_7 with $\gamma=\frac{1}{2}$
$-8\,\hat{\nu}^2$	U(1)	$\left(\cosh\left(\hat{\nu}\,\hat{\phi}\right)+\mu\right)^2$	$\hat{\nu}=\frac{1}{\sqrt{6}}$	$\mu=1$	I_7 with $\gamma=\frac{1}{3}$
$-8\,\hat{\nu}^2$	U(1)	$\left(\cosh\left(\hat{\nu}\,\hat{\phi}\right)+\mu\right)^2$	$\hat{\nu}=\frac{1}{\sqrt{6}}$	$\mu=-1$	I_7 with $\gamma=\frac{1}{3}$
$-8\,\hat{\nu}^2$	SO(1,1)	$\left(\sinh\left(\hat{\nu}\,\hat{\phi}\right)+\mu\right)^2$	$\hat{\nu}=\frac{1}{2}\sqrt{\frac{3}{2}}$	$\mu=0$	I_1 or I_7 with $\gamma=\frac{1}{2}$
$-8\,\hat{\nu}^2$	parabolic	$\left(\exp\left(\hat{\nu}\,\hat{\phi}\right)+\mu\right)^2$	$\hat{\nu}=$ any	$\mu=0$	all pure exp are integ.

Table 4: In this table we mention which particular values of the curvature and of the Fayet Iliopoulos constant yield cosmological potentials that are both associated to constant curvature and integrable according to the classification of [24]

Flat potentials → chaotic inflation models

Parabolic gauging → Starobinsky model

Results on α-attractors

Definition by Kallosh, Linde and Roest

$$V(\widehat{\phi}) = \left[\mathcal{P}\left(\widehat{\phi}\right)\right]^2$$

$$\boxed{V(\widehat{\phi}) \overset{\widehat{\phi}\to\infty}{\approx} V_0\left(1 - \exp\left[-\sqrt{\frac{2}{3\alpha}}\,\widehat{\phi}\right] + \mathcal{O}\left(\exp\left[-\sqrt{\frac{2}{3\alpha}}\,\widehat{\phi}\right]\right)^2\right)}$$

Universal predictions as $N \to \infty$ $\boxed{n_s = 1 - \frac{2}{N} \quad ; \quad r = \alpha\,\frac{12}{N^2}}$

THE SIMPLEST ATTRACTORS

$$\mathcal{P}_{(n)}(\widehat{\phi}) = \lambda \tanh^n\left(\frac{\widehat{\phi}}{\sqrt{6}}\right)$$

RESULTS: The corresponding surface Σ is non-singular only if $n \leq 2$

For n= 1 the isometry iss parabolic
For n=2 the isometry is elliptic !

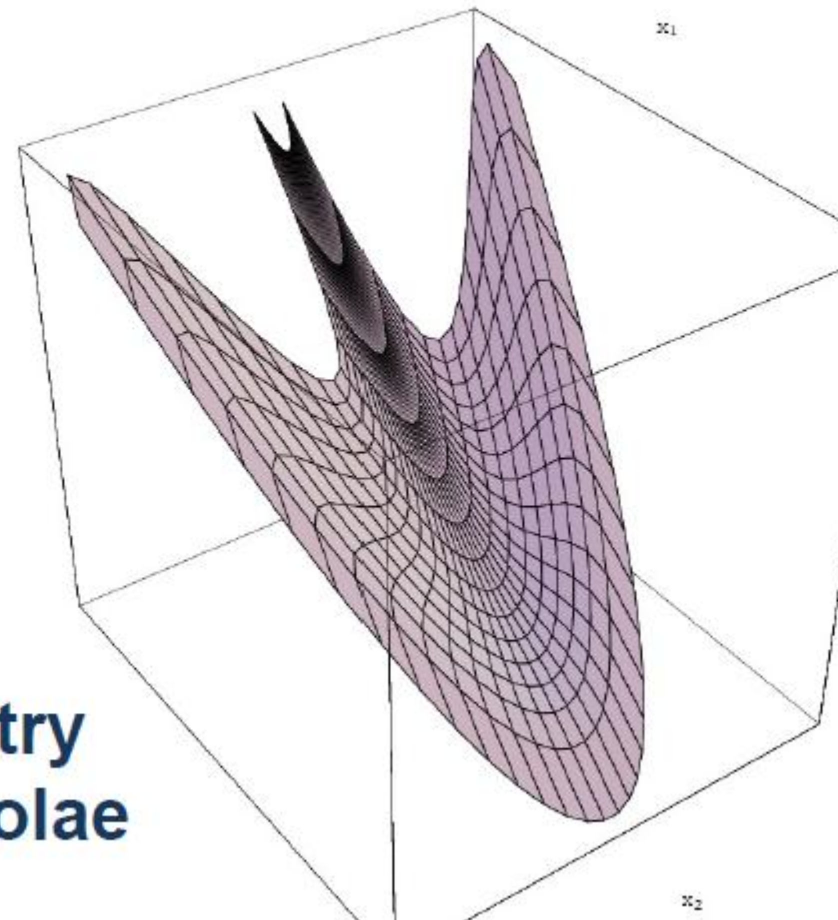
The simplest linear attractor n=1

$$\boxed{ds^2 = d\phi^2 + \mathrm{sech}^4\left(\frac{\phi}{\sqrt{3}}\right)\,\mathrm{dB}^2}$$

is the metric on the following parametric surface in $\mathrm{Mink}_{1,2}$

$$\begin{aligned} X_1 &= \frac{1}{2}\left(-f(\phi)B^2 + f(\phi) + g(\phi)\right) \\ X_2 &= Bf(\phi) \\ X_3 &= \frac{1}{2}\left(f(\phi)B^2 + f(\phi) - g(\phi)\right) \end{aligned}$$

$$\begin{aligned} f(\phi) &= \mathrm{sech}^2\left(\frac{\phi}{\sqrt{3}}\right) \\ g(\phi) &= -\frac{3}{8}\left(\cosh\left(\frac{2\phi}{\sqrt{3}}\right) + 4\log\left(\sinh\left(\frac{\phi}{\sqrt{3}}\right)\right)\right) \end{aligned}$$

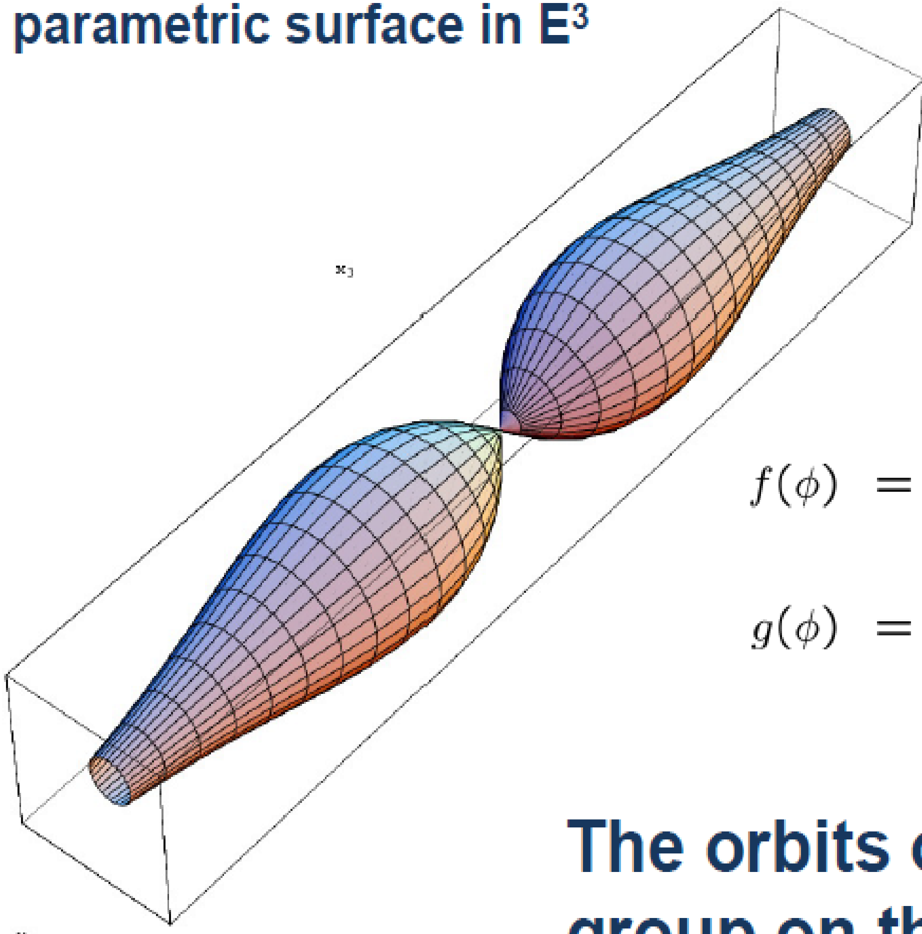


The orbits of the gauged isometry group on the surface are parabolae

The simplest quadratic attractor n=2

$$\boxed{ds^2_\Sigma = \mathrm{d}\phi^2 + \frac{4}{3}\mathrm{sech}^4\left(\frac{\phi}{\sqrt{3}}\right)\tanh^2\left(\frac{\phi}{\sqrt{3}}\right)\,\mathrm{d}B^2}$$

is the metric on the following parametric surface in E³



$$\begin{aligned} X_1 &= f(\phi)\cos(B) \\ X_2 &= f(\phi)\sin(B) \\ X_3 &= g(\phi) \end{aligned}$$

$$\begin{aligned} f(\phi) &= \frac{2\mathrm{sech}^2\left(\frac{\phi}{\sqrt{3}}\right)\tanh\left(\frac{\phi}{\sqrt{3}}\right)}{\sqrt{3}} \\ g(\phi) &= \int_0^\phi \sqrt{1-\frac{4}{9}\left(\cosh\left(\frac{2t}{\sqrt{3}}\right)-2\right)^2 \mathrm{sech}^8\left(\frac{t}{\sqrt{3}}\right)}\,\mathrm{d}t \end{aligned}$$

The orbits of the gauged isometry group on the surface are circles

N=1 SUGRA potentials

N=1 SUGRA coupled to n Wess Zumino multiplets

$$\mathcal{L}^{\mathcal{N}=1}_{SUGRA} = \sqrt{-g}\left[\mathcal{R}[g] + 2\, g^{HK}_{ij^\star}\,\partial_\mu z^i\,\partial^\mu \bar{z}^{j^\star} - 2\,V(z,\bar{z})\right] \;,$$

where

$$\begin{aligned} g_{ij^\star} &= \partial_i\,\partial_{j^\star}\,\mathcal{K} \\ \mathcal{K} &= \overline{\mathcal{K}} = \text{Kähler potential} \end{aligned}$$

$$V \;=\; 4\,e^2\,\exp\left[\mathcal{K}\right]\left(g^{ij^\star}\,\mathcal{D}_i W_h(z)\,\mathcal{D}_{j^\star}\overline{W_h}(\bar{z}) \;-\; 3\,|W_h(z)|^2\right) \;,$$

and

$$\begin{aligned} \mathcal{D}_i\, W &= \partial_i W + \partial_i \mathcal{K}\, W \\ \mathcal{D}_{j^\star}\,\overline{W} &= \partial_{j^\star}\overline{W} + \partial_{j^\star}\mathcal{K}\,\overline{W} \end{aligned}$$

If one multiplet, for instance

$$\mathcal{M}_K = \frac{SU(1,1)}{U(1)} \;\Rightarrow\; \boxed{\mathcal{K} = \log\left[-(z-\bar{z})^3\right]}$$

Integrable SUGRA model N=1

If in supergravity coupled to one Wess Zumino multiplet spanning the SU(1,1) / U(1) Kaehler manifold we introduce the following superpotential

$$\boxed{W_{int} = \lambda z^4 + \mathrm{i}\,\kappa z^3 \,,} \qquad \boxed{\lambda = \frac{6}{\sqrt{5}} \quad ; \quad \kappa = \frac{2\omega}{\sqrt{5}}}$$

we obtain a scalar potential

$$V_{int}(z,\bar{z}) = \frac{12 z^2 \bar{z}^2 \left((4\mathrm{i}\bar{z} + \omega) z^2 - 4\mathrm{i}\bar{z}^2 z + \bar{z}^2 \omega \right)}{5(z-\bar{z})^2}$$

where

$$z = \mathrm{i}\exp[h] + b$$

Truncation to zero axion b=0 is consistent

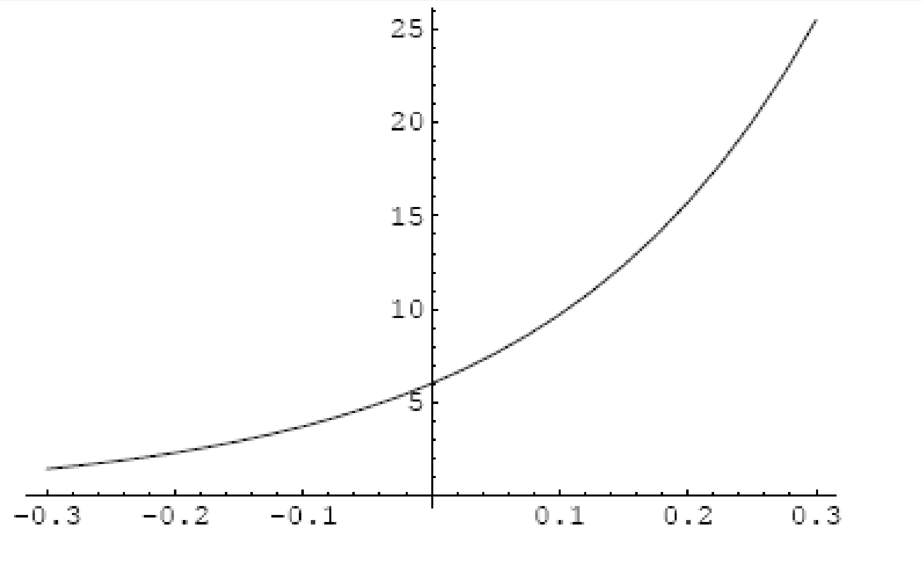
$$\boxed{V_{Int} = \frac{6}{5} e^{4\mathfrak{h}} \left(\omega + 4 e^{\mathfrak{h}} \right)}$$

THIS IS AN INTEGRABLE MODEL

$$\mathfrak{h} = \frac{\varphi}{3}$$

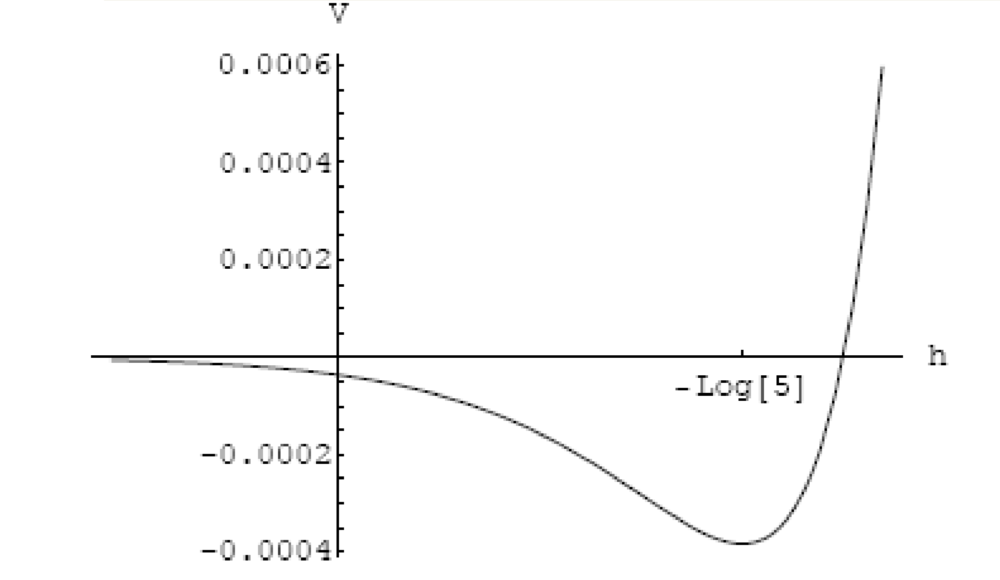
$$\begin{aligned} \mathcal{V}(\varphi) &= C_1 \exp\left[2\gamma\,\varphi\right] + C_2 \exp\left[(\gamma+1)\,\varphi\right] \\ \gamma &= \frac{2}{3} \end{aligned}$$

The form of the potential



Hyperbolic: $\omega > 0$
Runaway potential

Trigonometric $\omega < 0$
Potential with a negative extremum: stable AdS vacuum

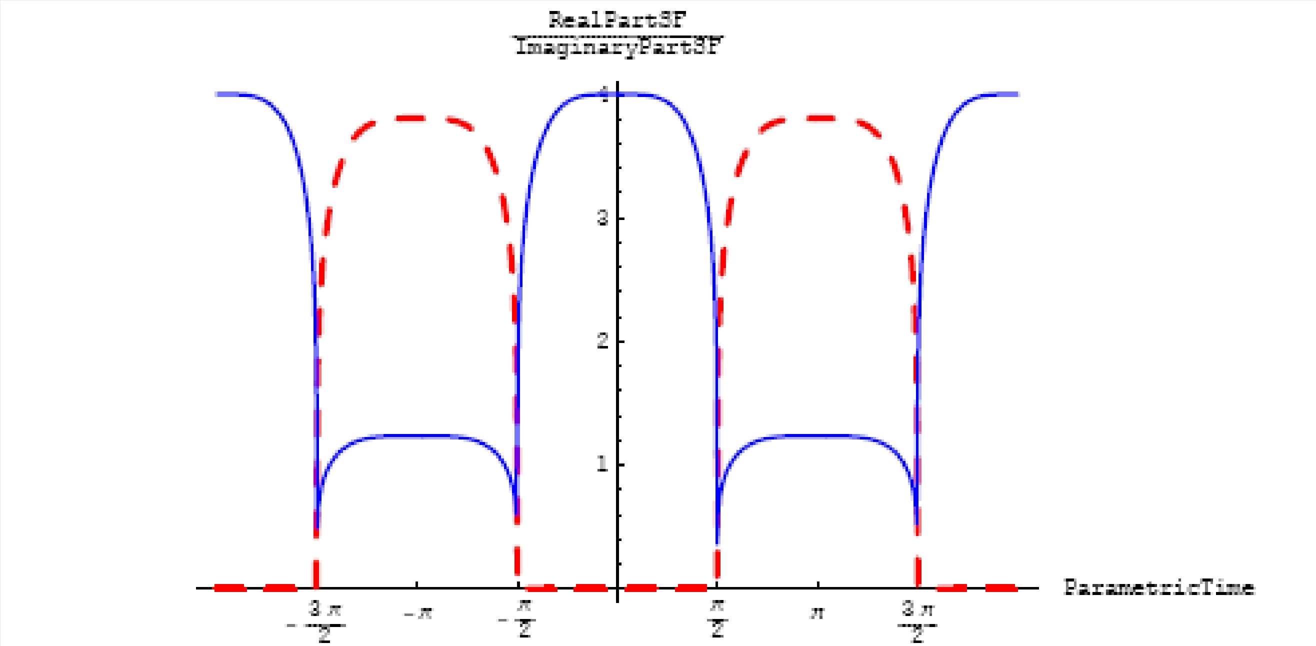


The General Integral in the trigonometric case

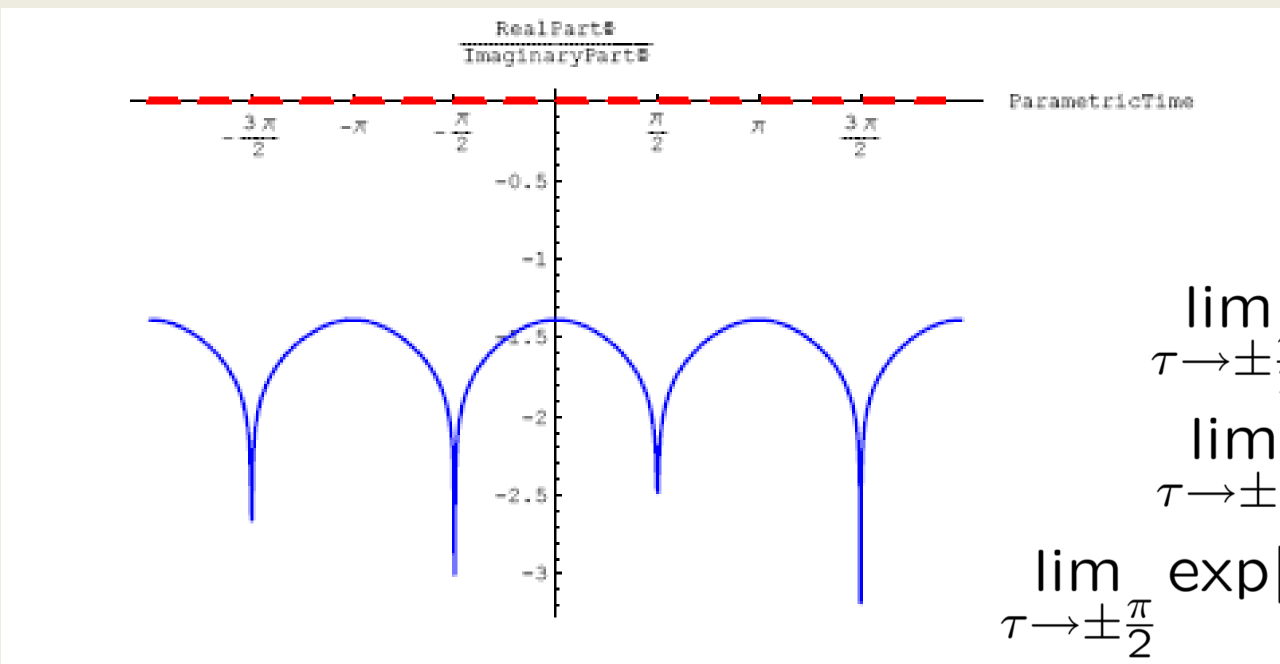
$$
\begin{aligned}
a\left(\tau,\Lambda,Y\right) = \Lambda\,\mathfrak{a}(\tau,Y) &= \Lambda\left[\frac{4}{5}\cos^{\frac{2}{5}}(\tau)\left(\left(\cos\tau\right)^{9/5}\,{}_2F_1\left(\frac{1}{2},\frac{9}{10};\frac{3}{2};\sin^2(\tau)\right)\tan^2(\tau)+5\right)\right. \\
&\quad \left. -Y\cos^{\frac{1}{5}}(\tau)\sin(\tau)\right] \\
\exp\left[\mathcal{B}\left(\tau,Y\right)\right] &= \frac{1}{25}\left(4\cos^2(\tau)^{9/10}\,{}_2F_1\left(\frac{1}{2},\frac{9}{10};\frac{3}{2};\sin^2(\tau)\right)\tan^2(\tau)-\frac{5Y\sin(\tau)}{\cos^{\frac{1}{5}}(\tau)}+20\right)^2 \\
\mathfrak{h}\left(\tau,Y\right) &= -\log\left(\frac{4}{5}\cos^2(\tau)^{9/10}\,{}_2F_1\left(\frac{1}{2},\frac{9}{10};\frac{3}{2};\sin^2(\tau)\right)\tan^2(\tau)-\frac{Y\sin(\tau)}{\cos^{\frac{1}{5}}(\tau)}+4\right)
\end{aligned}
$$

The scalar field tries to set down at the negative extremum but it cannot since there are no spatial flat sections of AdS space!

The result is a BIG CRUNCH. General Mechanism whenever there is a negative extremum of the potential



The simplest solution Y=0

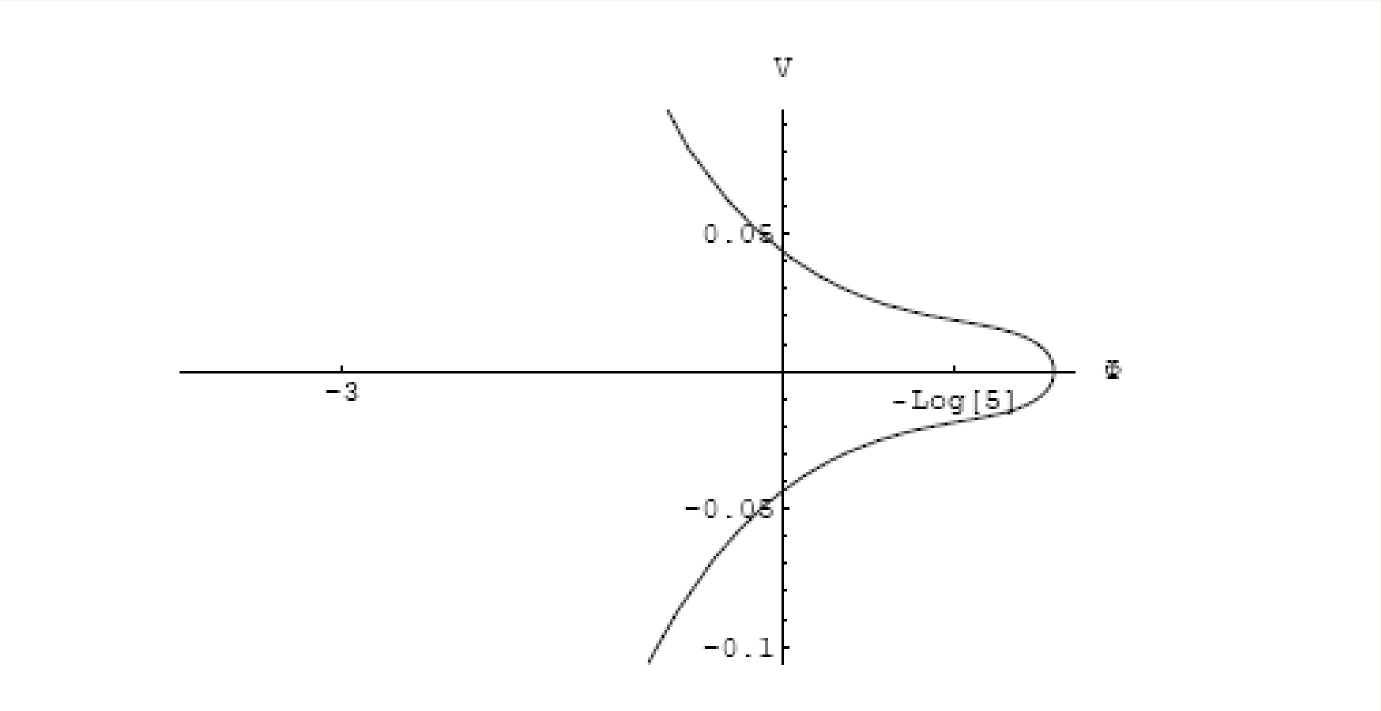


$$\lim_{\tau \to \pm\frac{\pi}{2}} \mathfrak{a}(\tau;0) = 0$$

$$\lim_{\tau \to \pm\frac{\pi}{2}} \mathfrak{h}(\tau,0) = -\infty$$

$$\lim_{\tau \to \pm\frac{\pi}{2}} \exp[\mathcal{B}(\tau,0)] = +\infty$$

Phase portrait of the simplest solution

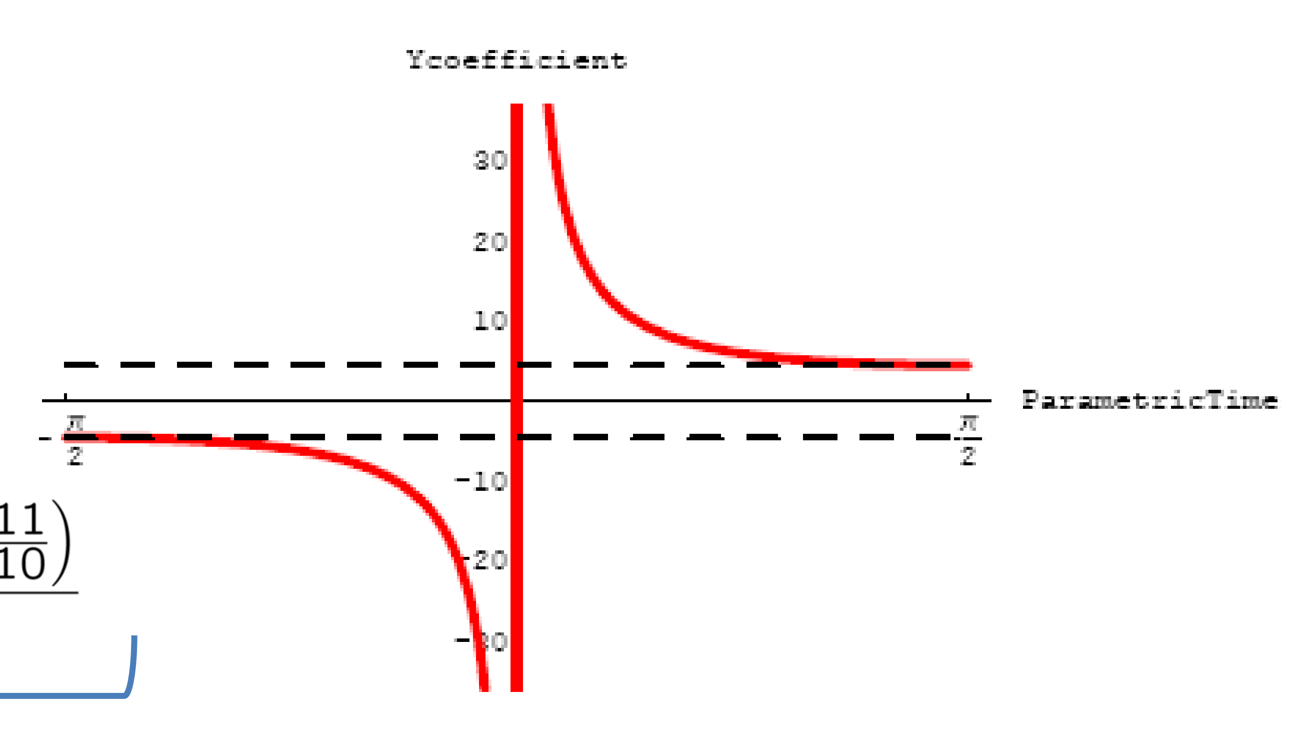


The extremum of the potential is at $\Phi_0 = -\log[5]$. It is reached by the solution however with a non vanishing velocity. There is no fixed point and the trajectory is from infinity to infinity.

Y-deformed solutions

An additional zero of the scale factor occurs for τ_0 such that

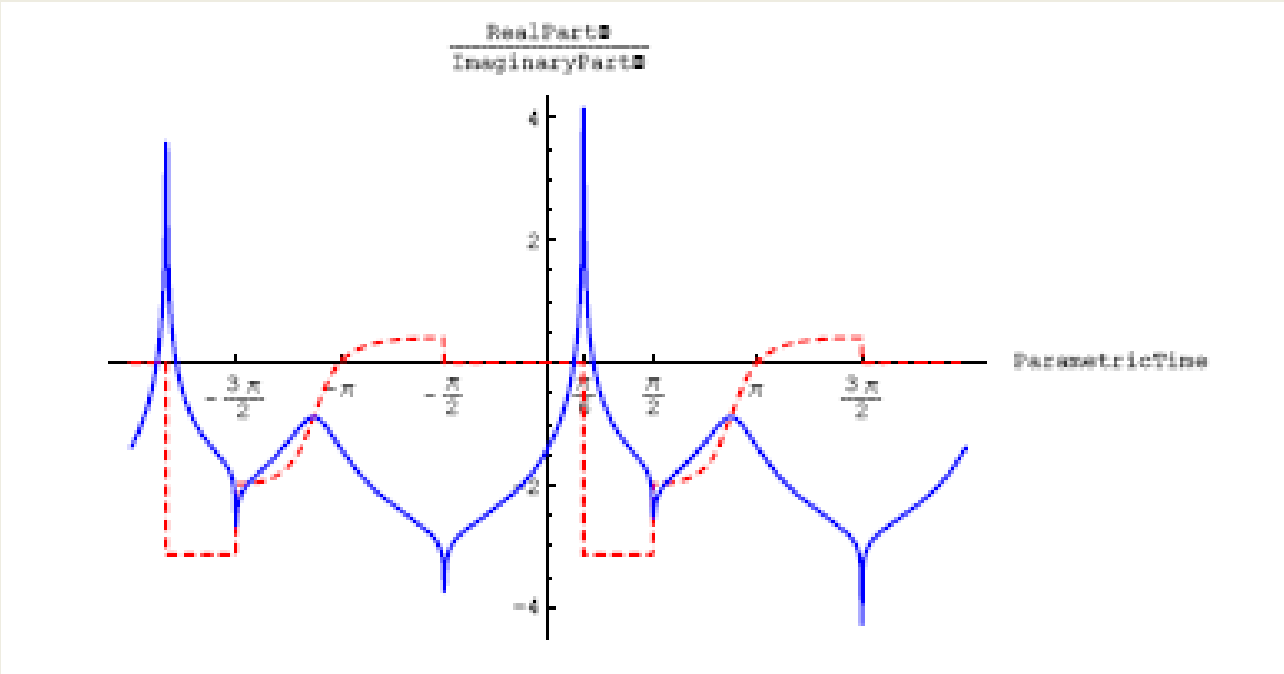
$$Y = \frac{4}{5}\cos^{\frac{1}{5}}(\tau_0)\csc(\tau_0)\left(\cos^2(\tau_0)^{9/10}\,{}_2F_1\left(\frac{1}{2},\frac{9}{10};\frac{3}{2};\sin^2(\tau_0)\right)\tan^2(\tau_0)+5\right) \equiv \mathfrak{f}(\tau_0)$$



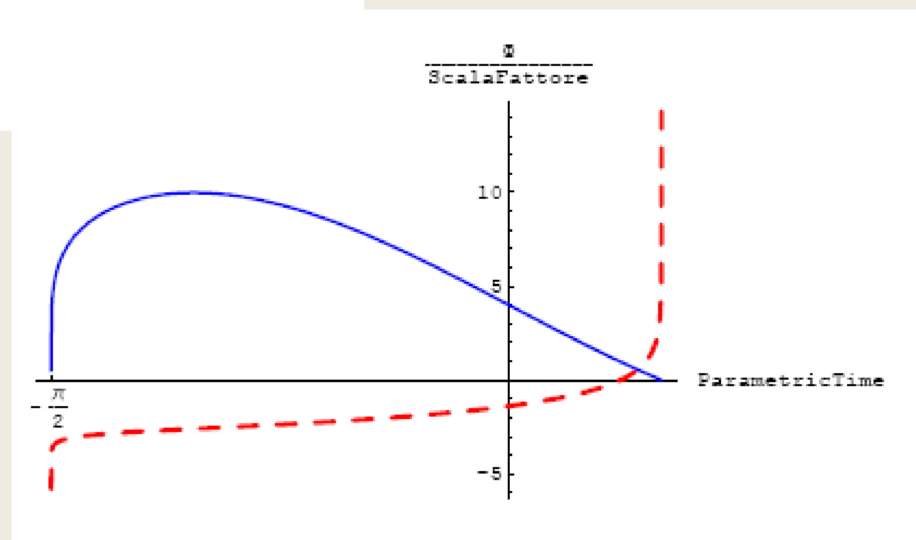
$$|Y| \leq Y_0 \equiv \frac{4\sqrt{\pi}\,\Gamma\left(\frac{11}{10}\right)}{\Gamma\left(\frac{3}{5}\right)}$$

Region of moduli space without early Big Crunch

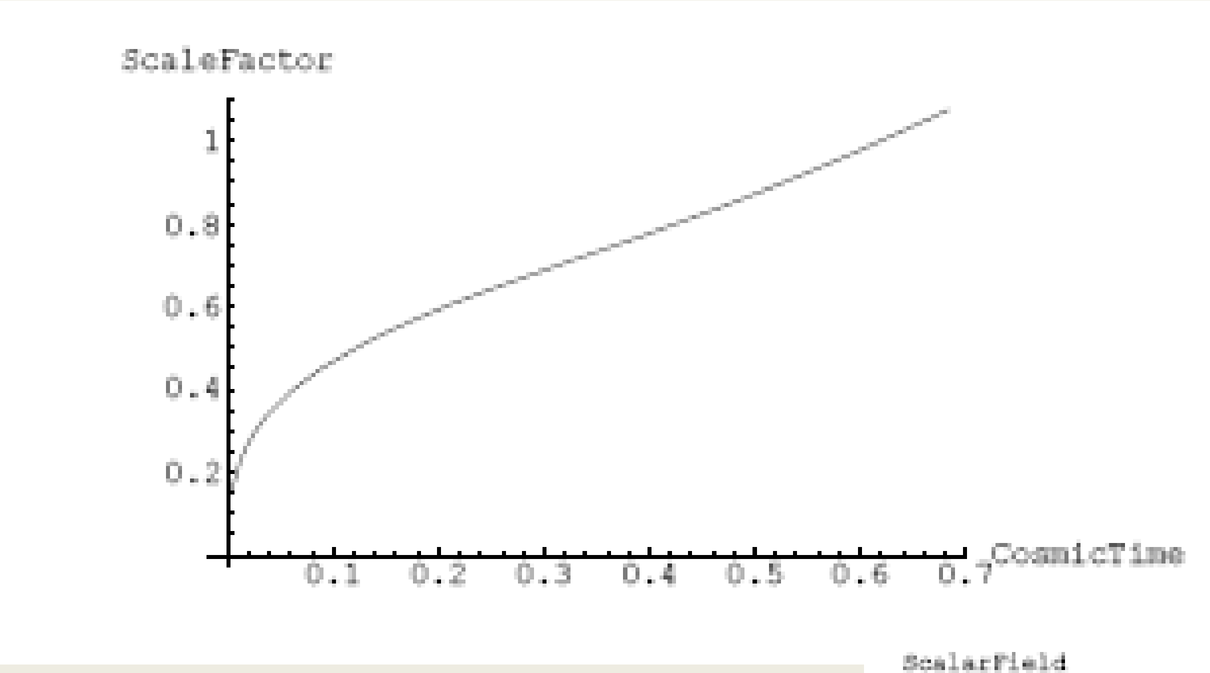
What new happens for Y > Y_0 ?



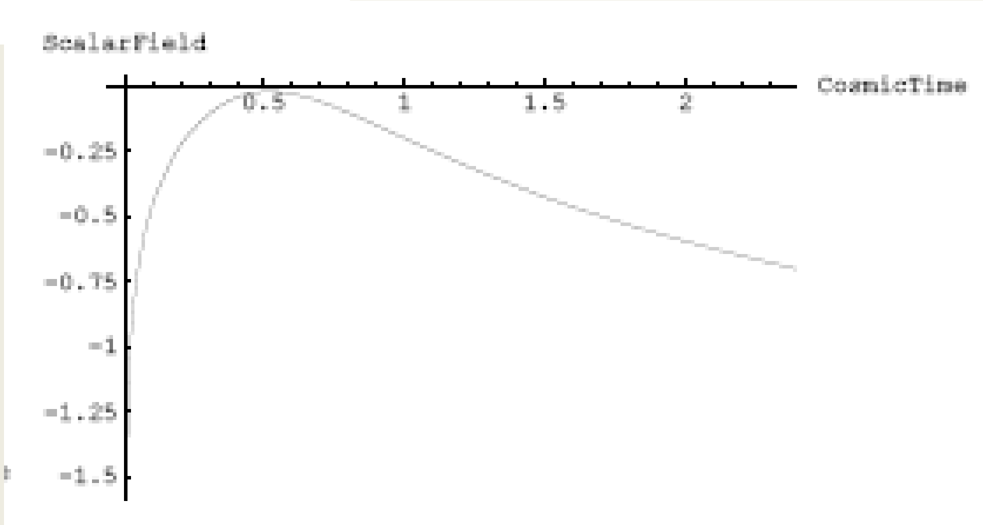
Early Big Bang
and
climbing scalar
from -1 to + 1



Hyperbolic solutions



We do not write the analytic form. It is also given in terms of hypergeometric functions of exponentials



$$a(T_c) \sim T_c^{\frac{1}{3}}$$ at Big Bang

$$\underbrace{a(T_c) \sim (T_c - T_{max})^{\frac{3}{4}}}_{\text{at Big Crunch}}$$

Conclusion

The study of integrable cosmologies within superstring and supergravity scenarios has just only begun.
Integrable cases are rare but do exist and can provide a lot of unexpected information that illuminates also the Physics behind the non integrable cases.

Thank you for attention!