TAMING THE ZOO OF SQM SYSTEMS

based on
A.S., arXiv:1301.7438 [math-phys]

MOTIVATION AND MAIN RESULT

- A great variety of $d = 1$ supermultiplets.
- A zoo (a jungle) of SQM systems.

CONJECTURE: All such systems can be derived from the trivial SQM model describing free dynamics in flat complex space

$$Q = \sqrt{2}\psi_a \pi_a, \; \bar{Q} = \sqrt{2}\bar{\psi}_a \bar{\pi}_a, \; H = \frac{1}{2}\{\bar{Q}, Q\} = \bar{\pi}_a \pi_a \; , \qquad (a = 1, \ldots, d)$$

by 2 operations:

1. Similarity transformations of holomorphic supercharges
2. Hamiltonian reduction

WARM-UP: Witten's SQM

- Take $d = 1$. Put $z = (x + iy)/\sqrt{2}$ and impose the constraint $p_y \Psi = -i\partial\Psi/\partial y = 0$ (Ham. reduction). We obtain

$$Q^{\text{free}} = p_x \psi, \;\; \bar{Q}^{\text{free}} = p_x \bar{\psi} \, .$$

- Perform similarity transformations

$$Q = e^W Q^{\text{free}} e^{-W} \, , \qquad \bar{Q} = e^{-W} \bar{Q}^{\text{free}} e^W \, .$$

We obtain Witten's SQM (1981)

$$Q = \psi[p + iW'(x)] \, , \qquad \bar{Q} = \bar{\psi}[p - iW'(x)]$$

$$H = \frac{1}{2}\left[p^2 + (W')^2 + W''(x)(\bar{\psi}\psi - \psi\bar{\psi})\right] \, .$$

$N = 2$ SIGMA MODELS

1. Dolbeault complex

Consider

$$Q = e^R Q^{\text{free}} e^{-R}, \qquad R = \omega_{ab}\psi_a\bar{\psi}_b$$

• anti-Hermitian ω gives unitary e^R. $\bar{Q}$ and H are then transformed with the same matrix. Just an unitary transformation of H.

• If ω is Hermitian, e^R is not unitary. Gives new nontrivial Hamiltonian.

We derive

$$Q = \sqrt{2}\psi_d \left(e^{\omega}\right)_{dc} \left[\pi_c - i \left(e^{\omega}\right)_{ae} \left(\partial_c e^{-\omega}\right)_{eb} \psi_a \bar{\psi}_b\right] .$$

$$H = \left(e^{\omega^\dagger} e^{\omega}\right)_{ab} \bar{\pi}_a \pi_b + \dots$$

• Nontrivial complex metric.

• The matrices $e^{\pm\omega}, e^{\pm\omega^\dagger}$ can be interpreted as the complex vielbeins,

$$\left(e^{\omega}\right)_{ac} \to e_a^j, \ \left(e^{-\omega}\right)_{ca} \to e_j^a, \ \left(e^{\omega\dagger}\right)_{ca} \to e_{\bar{a}}^{\bar{j}}, \ \left(e^{-\omega\dagger}\right)_{ac} \to e_{\bar{j}}^{\bar{a}}$$

• Supercharges can be rewritten as

$$Q \;=\; \sqrt{2}\psi^j \left(\pi_j + i\Omega_{j,\bar{b}a}\psi_a\bar{\psi}_b\right) \tag{1}$$

where $\Omega_{k,\bar{b}a}$ are *Bismut spin connection* for the metric $h = e^{\omega\dagger}e^{\omega}$ and the vielbein $e = e^{\omega}$.

• We define also

$$\bar{Q} = (\det h)^{-1} Q^{\dagger} \,\det h\,, \tag{2}$$

• The SQM system (1), (2)
(Hull, 99; E.Ivanov+A.S., 2012)
describes the Dolbeault complex on a generic complex manifold.

- An additional transformation

$$Q \to e^G Q e^{-G}$$

(G being a function or rather a section of a bundle on the manifold) gives a *twisted* Dolbeault complex.

- Still extra rotation with the operator

$$e^R = \exp\left\{\mathcal{B}_{jk}\psi^j\psi^k + \mathcal{B}_{jklm}\psi^j\psi^k\psi^l\psi^m + \ldots\right\}$$

gives complex sigma models with torsion.
[S.Fedoruk+ E. Ivanov+ A.S., 2012]

2. De Rham complex

• Start with the free complex system in d dimensions. Set $z^j = (x^j + iy^j)/\sqrt{2}$ and impose the contraints $p_y^j \Psi = 0$. We obtain the free real system

$$Q = p_A \psi_A, \qquad \bar{Q} = p_A \bar{\psi}_A \,.$$

• apply the similarity transformation $Q \to e^R Q e^{-R}$ with

$$R = \omega_{AB} \psi_A \bar{\psi}_B \,.$$

and Hermitian ω_{AB}.

• If ω is real and symmetric, one derives

$$Q = \psi^M \left(p_M - i\Omega_{M,AB}\psi_A\bar{\psi}_B\right) ,$$

where $\Omega_{M,AB}$ are standard spin connections for the (real) vielbeins $e_A^M = (e^\omega)_A^{\ M}$ and the metric $g_{MN} = \left(e^{-2\omega}\right)_{MN}$.

This is the standard de Rham complex.

• For generic Hermitian ω , one derives a quasicomplex sigma model

[E.Ivanov + A.S., 2013]

with the superfield action

$$S = \frac{1}{2}\int dt d\theta d\bar{\theta}\, g_{MN}(X) DX^M \bar{D}X^N .$$

with Hermitian (not necessarily real) g_{MN}.

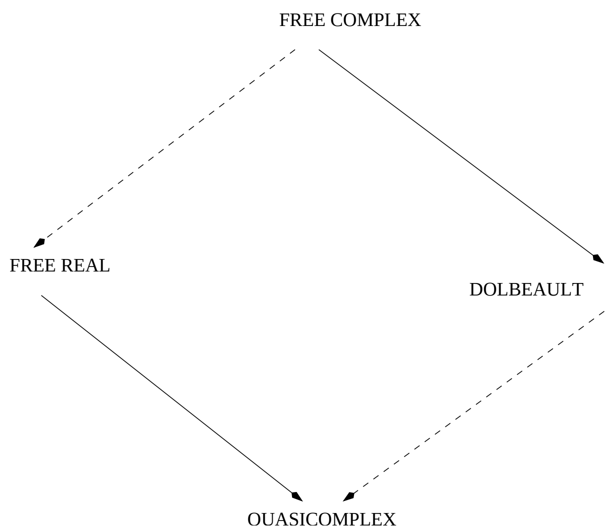

Figure 1: A rhombus of sigma models. The solid arrows stand for a similarity transformation and the dashed arrows — for a Hamiltonian reduction.

One can also obtain this model by Ham. reduction from the Dolbeault model on a manifold with isometries: metric not depending on imaginary parts of z^j.

EXTENDED SUPERSYMMETRIES

They show up for some manifolds.

1. Kähler sigma models

• Definition 1 :

Complex manifolds are even-dimensional manifolds where the antisymmetric complex structure tensor I_{MN} satisfying the conditions

1. $I_M{}^P I_P{}^N = -\delta_M^N$.
2. $\partial_{[M} I_{N]P} \; = \; I_M{}^Q I_N{}^S \partial_{[Q} I_{S]P}$.

can be defined.

• Definition 2 :

Kähler manifolds are complex manifolds where I_{MN} is covariantly constant, $\nabla_P I_{MN} = 0$.

- Theorem:

1. on a Kähler manifold, the supercharges

$$
\begin{aligned}
Q &= \psi^M \left(p_M - i\Omega_{M,AB}\psi_A\bar{\psi}_B\right) , \\
S &= \psi^M I_M^N \left(p_N - i\Omega_{N,AB}\psi_A\bar{\psi}_B\right)
\end{aligned}
\tag{3}
$$

and their conjugates satisfy the extended $N = 4$ supersymmetry algebra.

(extended de Rham complex)

2. The supercharges (3) can be obtained from the flat supercharges

$$Q = p_A\psi_A, \qquad S = p_A I_{AB}\psi_B$$

by *the same* similarity transformation.

2. Hyper-Kähler sigma models

• definition :

A hyper-Kähler manifold is a complex manifold possessing 3 different covariantly constant complex structures $I_{a=1,2,3}$ that satisfy the quaternion algebra

$$I_a I_b = -\delta_{ab} + \epsilon_{abc} I_c \ .$$

The dimension is necessarily $4k$.

• Theorem:

1. On a hyper-Kähler manifold, the supercharges

$$
\begin{aligned}
Q &= \psi^M \left(p_M - i\Omega_{M,AB}\psi_A\bar{\psi}_B\right) , \\
S_a &= \psi^M (I_a)_M^{\ N} \left(p_N - i\Omega_{N,AB}\psi_A\bar{\psi}_B\right)
\end{aligned}
\tag{4}
$$

and their conjugates satisfy the extended $N = 8$ supersymmetry algebra.

2. The supercharges (4) can be obtained from four flat supercharges

$$
Q = p_A\psi_A, \qquad S_a = p_A(I_a)_{AB}\psi_B
$$

by *the same* similarity transformation.

3. *HKT* sigma models.

• remark:

For any complex manifold a special affine connection (called *Bismut connection*) involving totally antisymmetric torsions exists, with respect to which the complex structure tensor is covariantly constant.

• definition :

An HKT manifold is a manifold with 3 complex structures which

a) satisfy the quaternion algebra

b) their Bismut connections coincide.

• Consider the Dolbeault complex on a HKT manifold.

• Theorem:

For a HKT manifold, one can write two complex supercharges which satisfy together with their conjugates the $N = 4$ superalgebra and can be obtained by *the same* similarity transformation from the flat supercharges

$$Q = \sqrt{2}\psi_a^k \pi_a^k, \qquad S = \sqrt{2}\epsilon_{ab}\bar{\psi}_a^k \pi_b^k,$$
$$a = 1, 2; \quad k = 1, \ldots$$

• remark:

In four dimensions, HKT metrics are conformally flat.

$N = 8$ supersymmetric OKT models

- Dolbeault model with 4 complex supercharges.

A similarity transformation making *all four* supercharges flat not explicitly found yet.

REDUCED MODELS

(**3**, **4**, **1**) models

• Take a 4-dim conformally flat HKT model. Assume the conformal factor not to depend on one coordinate t. Impose the constraint $p_t \Psi = 0$.

One obtains a $N = 4$ sigma model with 3-dim conf. flat metric [A.S. , 1987]

$$
\begin{aligned}
Q_\alpha &= f(\sigma_j \bar{\psi})_\alpha \left(p_j - i\partial_j f \, \psi_\beta \bar{\psi}^\beta\right) \\
\bar{Q}^\alpha &= (\psi \sigma_j)^\alpha \left(p_j + i\partial_j f \, \psi_\beta \bar{\psi}^\beta\right)
\end{aligned}
$$

$(\mathbf{5},\ \mathbf{8},\ \mathbf{3})$ models

[Diaconescu, Entin, 1997; A.S., 2002]

• obtained by Ham. reduction from theOKT models.

• the simplest model has the conformal harmonic metric,

$$ds^2 = \left(1 + \frac{C}{r^3}\right) dx_M dx_M, \qquad M = 1, \ldots, 5$$

GAUGE SQM MODELS

• obtained by dimensional reduction from gauge susy theories

under study

FIELD THEORIES

Party line: whatever is true for SQM should be true for field theories (?)

• free massless WZ model

The supercharges are

$$Q_\alpha = \sqrt{2}\int d\mathbf{x}\ \left[\Pi\psi_\alpha + \partial_j\bar{\phi}\,(\sigma_j)_{\alpha\dot{\gamma}}\delta^{\dot{\gamma}\gamma}\psi_\gamma\right]\ ,$$
$$Q_{\dot{\alpha}} = \sqrt{2}\int d\mathbf{x}\ \left[\bar{\Pi}\bar{\psi}_{\dot{\alpha}} + (\partial_j\phi)\bar{\psi}_{\dot{\gamma}}\delta^{\dot{\gamma}\gamma}(\sigma_j)_{\gamma\dot{\alpha}}\right]$$

They satisfy the algebra

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2(\sigma_\mu)_{\alpha\dot{\alpha}}P_\mu = 2\left[\delta_{\alpha\dot{\alpha}}H + (\sigma_j)_{\alpha\dot{\alpha}}P_j\right]\ ,$$

where

$$H = \int d\mathbf{x}\ \left[\bar{\Pi}\Pi + \partial_j\bar{\phi}\partial_j\phi - i\psi\sigma_j\partial_j\bar{\psi}\right]$$

is the Hamiltonian and $\vec{P}$ is the 3-momentum operator.

• Put it in finite box and expand in modes.

$$\phi(\mathbf{x}) = \sum_{\mathbf{n}} \phi_{\mathbf{n}} e^{2\pi i \mathbf{n}\mathbf{x}}, \qquad \psi(\mathbf{x}) = \sum_{\mathbf{n}} \psi_{\mathbf{n}} e^{2\pi i \mathbf{n}\mathbf{x}}$$

$$\bar{\phi}(\mathbf{x}) = \sum_{\mathbf{n}} \bar{\phi}_{\mathbf{n}} e^{-2\pi i \mathbf{n}\mathbf{x}}, \qquad \bar{\psi}(\mathbf{x}) = \sum_{\mathbf{n}} \bar{\psi}_{\mathbf{n}} e^{-2\pi i \mathbf{n}\mathbf{x}}$$

Then it is an SQM system, but $\vec{P}$ plays the role of central charge. Our philosophy does not apply.

• Still one *can* represent the Hamiltonian as the anticommutator $\{\bar{\mathcal{Q}}, \mathcal{Q}\}$ with a nonlocal $\mathcal{Q}$.

H and $\mathcal{Q}$ expressed in modes.

$$H \;=\; \sum_{\mathbf{n}} \left[\bar{\Pi}_{\mathbf{n}} \Pi_{\mathbf{n}} + (2\pi \mathbf{n})^2 \, \bar{\phi}_{\mathbf{n}} \phi_{\mathbf{n}} + 2\pi n_j \psi_{\mathbf{n}} \sigma_j \bar{\psi}_{\mathbf{n}} \right]$$

and

$$\mathcal{Q} = \sum_{\mathbf{n}} \left[\chi^1_{\mathbf{n}} \left(P^1_{\mathbf{n}} + 2i\pi f^1_{\mathbf{n}} \sqrt{\mathbf{n}^2} \right) \right] + \chi^2_{\mathbf{n}} \left(P^2_{\mathbf{n}} - 2i\pi f^2_{\mathbf{n}} \sqrt{\mathbf{n}^2} \right) \quad (5)$$

$P^{1,2}_{\mathbf{n}}/\sqrt{2}$ and $f^{1,2}_{\mathbf{n}}/\sqrt{2}$ being the real and imaginary parts of $\Pi_{\mathbf{n}}$ and $\phi_{\mathbf{n}}$.

$\chi^{1,2}_{\mathbf{n}}$ are fermion eigenvectors of $n_j \sigma_j$.

• (5) can be sim. transformed to the "free supercharge"

$$\mathcal{Q}^{(0)} = \sum_{\mathbf{n}} \left(P^1_{\mathbf{n}} \chi^1_{\mathbf{n}} + P^2_{\mathbf{n}} \chi^2_{\mathbf{n}} \right)$$