Deformed $SU(2|1)$ Superfields and Superconformal Mechanics

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Outline

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$SU(2|1)$ superspace and deformed models of SQM

Recently, we proposed a new type of supersymmetric quantum mechanics based on the worldline realization of the supergroup $SU(2|1)$ in the appropriate $SU(2|1)$, $d = 1$ superspace (E. Ivanov, S. S., 2013).

- The corresponding models are deformations of the standard $\mathcal{N} = 4$ models by the intrinsic mass parameter m.
- In particular, the "Weak Supersymmetry" models (A. Smilga, 2004) were easily reproduced from our superfield approach. They are based on the $SU(2|1)$ multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$.
- The supergroup $SU(2|1)$ has also invariant chiral subspaces which are natural carriers of the chiral superfields $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ for which we also constructed general superfield and component actions.
- The SQM models known as Super Kähler Oscillator (S. Bellucci, A. Nersessian, 2002, 2004) were derived from the $SU(2|1)$ superspace. They are based on the $SU(2|1)$ multiplet $(\mathbf{2}, \mathbf{4}, \mathbf{2})$.

Supercoset

- We consider the $SU(2|1)$ coset space

$$\frac{SU(2|1) \rtimes U(1)_{\text{ext}}}{SU(2) \times U(1)_{\text{int}}} \sim \frac{\{Q^i, \bar{Q}_j, H, F, I^i_j\}}{\{I^i_j, F\}},$$

where the Hamiltonian H is a central charge.

- In this supercoset, the relevant extended superalgebra $\hat{su}(2|1)$ is defined as

$$\{Q^i, \bar{Q}_j\} = 2mI^i_j + 2\delta^i_j H - 2\delta^i_j mF, \qquad \left[I^i_j, I^k_l\right] = \delta^k_j I^i_l - \delta^i_l I^k_j\,,$$
$$\left[I^i_j, \bar{Q}_l\right] = \frac{1}{2}\delta^i_j \bar{Q}_l - \delta^i_l \bar{Q}_j\,, \qquad \left[I^i_j, Q^k\right] = \delta^k_j Q^i - \frac{1}{2}\delta^i_j Q^k,$$
$$\left[F, \bar{Q}_l\right] = -\frac{1}{2}\bar{Q}_l\,, \qquad \left[F, Q^k\right] = \frac{1}{2}Q^k.$$

The generators I^i_j, F form $U(2)$ symmetry. In the limit $m = 0$, this algebra become the standard $\mathcal{N} = 4$ Poincaré superalgebra.

- The superspace coordinates $\{t, \theta_i, \bar{\theta}^j\}$ are identified with the coset generators parameters. An element of this supercoset can be conveniently parametrized as

$$g = \exp\left(i\tilde{\theta}_i Q^i - i\bar{\tilde{\theta}}^j \bar{Q}_j\right)\exp\left(itH\right), \qquad \tilde{\theta}_i = \left[1 - \frac{2m}{3}\,\bar{\theta}^k\theta_k\right]\theta_i\,.$$

$SU(2|1)$ superspace realization

- Transformation properties under Q, $\bar{Q}$:

$$\delta\theta_i = \epsilon_i + 2m\left(\bar{\epsilon}^k\theta_k\right)\theta_i\,, \qquad \delta\bar{\theta}^j = \bar{\epsilon}^i + 2m\left(\bar{\theta}^k\epsilon_k\right)\bar{\theta}^i,$$
$$\delta t = i\left[\bar{\epsilon}^k\theta_k - \bar{\theta}^k\epsilon_k\right].$$

- Invariant integration measure:

$$\mu = dt d^2\theta d^2\bar{\theta}\left[1 + 2m\,\bar{\theta}^k\theta_k\right], \quad \delta\mu = 0\,.$$

- In terms of these coordinates, supercharges are realized as

$$Q^i = \frac{\partial}{\partial\theta_i} - 2m\,\bar{\theta}^i\bar{\theta}^k\frac{\partial}{\partial\bar{\theta}^k} + i\bar{\theta}^i\partial_t\,, \qquad \bar{Q}_j = \frac{\partial}{\partial\bar{\theta}^j} + 2m\,\theta_j\theta_k\frac{\partial}{\partial\theta_k} + i\theta_j\partial_t\,.$$

and bosonic generators are realized as

$$I^i_j = \left(\bar{\theta}^i\frac{\partial}{\partial\bar{\theta}^j} - \theta_j\frac{\partial}{\partial\theta_i}\right) - \frac{\delta^i_j}{2}\left(\bar{\theta}^k\frac{\partial}{\partial\bar{\theta}^k} - \theta_k\frac{\partial}{\partial\theta_k}\right),$$
$$H = i\partial_t\,, \qquad F = \frac{1}{2}\left(\bar{\theta}^k\frac{\partial}{\partial\bar{\theta}^k} - \theta_k\frac{\partial}{\partial\theta_k}\right).$$

Superconformal symmetry

- One can extend this symmetry by defining the new supercharges

$$S^i = e^{-2iamt}\Big[\Big(1-2\,(a-1)\,m\,\bar\theta^k\theta_k-(a-1)^2\,m^2\,(\theta)^2\,(\bar\theta\,)^2\Big)\frac{\partial}{\partial\theta_i} + 2\,(2a-1)\,m\,\bar\theta^i\theta_k\frac{\partial}{\partial\theta_k}+i\bar\theta^i\Big(1+2\,(a-1)\,m\,\bar\theta^k\theta_k\Big)\,\partial_t\Big],$$

$$\bar S_j = e^{2iamt}\Big[\Big(1-2\,(a-1)\,m\,\bar\theta^k\theta_k-(a-1)^2\,m^2\,(\theta)^2\,(\bar\theta\,)^2\Big)\frac{\partial}{\partial\bar\theta^j} - 2\,(2a-1)\,m\,\theta_j\bar\theta^k\frac{\partial}{\partial\bar\theta^k}+i\theta_j\Big(1+2\,(a-1)\,m\,\bar\theta^k\theta_k\Big)\,\partial_t\Big].$$

- They form the extended $su(2|1)$ superalgebra closed on the same bosonic generators I^k_j, F as

$$\{S^i,\bar S_j\} = -2mI^i_j + 2\delta^i_j H - 2\,(4a-1)\,\delta^i_j mF.$$

- These new supercharges extend the superalgebra $\hat{su}(2|1)$ to the $d=1$ superconformal algebra $D\,(2,1;\alpha)$.

Superconformal symmetry

- Transformation properties under S, $\bar{S}$:

$$\begin{aligned}
\delta\theta_i &= \Big(1 - 2\,(a-1)\,m\,\bar{\theta}^k\theta_k - (a-1)^2\,m^2\,(\theta)^2\,(\bar{\theta}\,)^2\Big)\,\varepsilon_i\,e^{-2iamt} \\
&\quad + 2\,(2a-1)\,m\,\varepsilon_k\bar{\theta}^k\theta_i\,e^{-2iamt}, \\
\delta\bar{\theta}^i &= \Big(1 - 2\,(a-1)\,m\,\bar{\theta}^k\theta_k - (a-1)^2\,m^2\,(\theta)^2\,(\bar{\theta}\,)^2\Big)\,\bar{\varepsilon}^i\,e^{2iamt} \\
&\quad - 2\,(2a-1)\,m\,\bar{\varepsilon}^k\theta_k\bar{\theta}^i\,e^{2iamt}, \\
\delta t &= i\left(\bar{\varepsilon}^k\theta_k e^{2iamt} + \varepsilon_k\bar{\theta}^k e^{-2iamt}\right)\left[1 + 2\,(a-1)\,m\,\bar{\theta}^k\theta_k\right].
\end{aligned}$$

- The measure $d\mu$ is not invariant under these transformations:

$$\delta_\varepsilon(d\mu) = 4am\left[1 - 2am\,\bar{\theta}^k\theta_k\right]\left(\bar{\varepsilon}^i\theta_i\,e^{2iamt} - \varepsilon_i\bar{\theta}^i\,e^{-2iamt}\right)d\mu\,.$$

- Performing some substitutions of superspace, one can show that the superconformal generators can be constructed in terms of the pair of deformed supercharges $S(m) \equiv Q(-m)$ and $Q(m)$.
- In a similar way the superconformal algebra $su(2,2|1)$ can be represented as the closure of two $OSp(1,4)$ supergroups (E. Ivanov, A. Sorin, 1980)

Superconformal symmetry

Their anticommutators with Q^i, $\bar{Q}_k$

$$\{S^i, \bar{Q}_j\} = 2\delta^i_j T, \qquad \{Q^i, \bar{S}_j\} = 2\delta^i_j \bar{T},$$
$$\{Q^i, S^k\} = -2m\,(2a-1)\,\varepsilon^{ik} C, \qquad \{\bar{Q}_j, \bar{S}_k\} = 2m\,(2a-1)\,\varepsilon_{jk} \bar{C},$$

give the new bosonic generators:

$$T = e^{-2iamt}\left[i\left(1 + (a-1)\,m^2\,(\theta)^2\,(\bar{\theta}\,)^2\right)\partial_t + 2am\left(1 - 2\,(a-1)\,m\,\bar{\theta}^k\theta_k\right)\theta_i\,\frac{\partial}{\partial\theta_i}\right],$$
$$\bar{T} = e^{2iamt}\left[i\left(1 + (a-1)\,m^2\,(\theta)^2\,(\bar{\theta}\,)^2\right)\partial_t - 2am\left(1 - 2\,(a-1)\,m\,\bar{\theta}^k\theta_k\right)\bar{\theta}^i\,\frac{\partial}{\partial\bar{\theta}^i}\right],$$
$$C = e^{-2iamt}\varepsilon_{jl}\left(1 + 2\,(a-1)\,m\,\bar{\theta}^k\theta_k\right)\bar{\theta}^j\,\frac{\partial}{\partial\theta_l}\,,$$
$$\bar{C} = e^{2iamt}\varepsilon^{jl}\left(1 + 2\,(a-1)\,m\,\bar{\theta}^k\theta_k\right)\theta_j\,\frac{\partial}{\partial\bar{\theta}^l}\,.$$

Superalgebra

The whole superalgebra is given by the following (anti)commutators:

$$\{Q^i, \bar{Q}_j\} = 2mI^i_j + 2\delta^i_j H - 2\delta^i_j mF,$$

$$\{S^i, \bar{S}_j\} = -2mI^i_j + 2\delta^i_j H - 2\,(4a-1)\,\delta^i_j mF,$$

$$\{S^i, \bar{Q}_j\} = 2\delta^i_j T, \qquad \{Q^i, \bar{S}_j\} = 2\delta^i_j \bar{T},$$

$$\{Q^i, S^k\} = -2m\,(2a-1)\,\varepsilon^{ik} C, \qquad \{\bar{Q}_j, \bar{S}_k\} = 2m\,(2a-1)\,\varepsilon_{jk}\bar{C},$$

$$\left[I^i_j, \bar{Q}_l\right] = \frac{1}{2}\,\delta^i_j \bar{Q}_l - \delta^i_l \bar{Q}_j\,, \quad \left[I^i_j, Q^k\right] = \delta^k_j Q^i - \frac{1}{2}\,\delta^i_j Q^k, \quad \left[I^i_j, I^k_l\right] = \delta^k_j I^i_l - \delta^i_l I^k_j\,,$$

$$\left[I^i_j, \bar{S}_l\right] = \frac{1}{2}\,\delta^i_j \bar{S}_l - \delta^i_l \bar{S}_j\,, \quad \left[I^i_j, S^k\right] = \delta^k_j S^i - \frac{1}{2}\,\delta^i_j S^k,$$

$$\left[C, \bar{C}\right] = 2F, \qquad [F, C] = C, \qquad \left[F, \bar{C}\right] = -\bar{C},$$

$$\left[F, \bar{Q}_l\right] = -\frac{1}{2}\,\bar{Q}_l\,, \quad \left[F, Q^k\right] = \frac{1}{2}\,Q^k, \quad \left[F, \bar{S}_l\right] = -\frac{1}{2}\,\bar{S}_l\,, \quad \left[F, S^k\right] = \frac{1}{2}\,S^k,$$

$$\left[C, \bar{Q}_j\right] = -S_j\,, \quad \left[C, \bar{S}_j\right] = -Q_j\,, \quad \left[\bar{C}, Q^i\right] = -\bar{S}^i, \quad \left[\bar{C}, S^i\right] = -\bar{Q}^i,$$

$$\left[T, \bar{T}\right] = -4am\,(H - 2amF)\,, \qquad [H, T] = 2amT, \qquad \left[H, \bar{T}\right] = -2am\bar{T},$$

$$\left[T, Q^i\right] = -2amS^i, \quad \left[T, \bar{S}_j\right] = -2am\bar{Q}_j, \quad \left[\bar{T}, \bar{Q}_j\right] = 2am\bar{S}_j, \quad \left[\bar{T}, S^i\right] = 2amQ^i,$$

$$\left[H, \bar{S}_l\right] = -2am\bar{S}_l\,, \quad \left[H, S^k\right] = 2amS^k, \quad \left[H, \bar{C}\right] = -2am\bar{C}\,, \quad [H, C] = 2amC\,.$$

Superalgebra

Defining new generators as the linear combinations

$$\varepsilon^{ik}Q_{1k1'} = -\frac{1}{2}\left(S^i + Q^i\right), \qquad Q_{1j2'} = -\frac{1}{2}\left(\bar{S}_j + \bar{Q}_j\right),$$

$$\varepsilon^{ik}Q_{2k1'} = \frac{i}{\mu}\left(Q^i - S^i\right), \qquad Q_{2j2'} = -\frac{i}{\mu}\left(\bar{Q}_j - \bar{S}_j\right),$$

$$T_{22} = \frac{2}{\mu^2}\left[H - \mu F - \frac{1}{2}\left(T + \bar{T}\right)\right], \qquad T_{11} = \frac{1}{2}\left[H - \mu F + \frac{1}{2}\left(T + \bar{T}\right)\right],$$

$$T_{12} = T_{21} = \frac{i}{2\mu}\left(T - \bar{T}\right), \qquad \mu = 2am,$$

$$I_{1'1'} = -iC, \quad I_{2'2'} = i\bar{C}, \quad I_{1'2'} = I_{2'1'} = -iF, \qquad J^i_j = -iI^i_j,$$

the superalgebra can be identified as the superconformal algebra $D(2,1;\alpha)$

$$\{Q_{\alpha ii'}, Q_{\beta jj'}\} = 2\left(\epsilon_{ij}\epsilon_{i'j'}T_{\alpha\beta} + \alpha\,\epsilon_{\alpha\beta}\epsilon_{i'j'}J_{ij} - (1+\alpha)\,\epsilon_{\alpha\beta}\epsilon_{ij}I_{i'j'}\right),$$

$$[T_{\alpha\beta}, Q_{\gamma ii'}] = -i\,\epsilon_{\gamma(\alpha}Q_{\beta)ii'}, \qquad [T_{\alpha\beta}, T_{\gamma\delta}] = i\left(\epsilon_{\alpha\gamma}T_{\beta\delta} + \epsilon_{\beta\delta}T_{\alpha\gamma}\right),$$

$$[J_{ij}, Q_{\alpha ki'}] = -i\,\epsilon_{k(i}Q_{\alpha j)i'}, \qquad [J_{ij}, J_{kl}] = i\left(\epsilon_{ik}J_{jl} + \epsilon_{jl}J_{ik}\right),$$

$$[I_{i'j'}, Q_{\alpha ik'}] = -i\,\epsilon_{k'(i'}Q_{\alpha ij')}, \qquad [I_{i'j'}, I_{k'l'}] = i\left(\epsilon_{i'k'}I_{j'l'} + \epsilon_{j'l'}I_{i'k'}\right).$$

Superalgebra

- The number α appears in superalgebra as

$$a = -\frac{1}{2\alpha}\,, \qquad m = -\alpha\mu\,.$$

- Any dependence on μ of superalgebra relations vanishes naturally after passing to new generators. Then the parameter μ is some deformation parameter of the superspace realization of $D\,(2,1;\alpha)$.

- In the degenerate cases $\alpha = 0, -1$ we may retain only eight fermionic generators $Q_{\alpha i i'}$ and six bosonic generators $T_{\alpha\beta}$, J_{ij} forming $psu(1,1|2)$ superalgebra without central charge. $SU(2)$ symmetry produced by the generators $I_{i'j'}$ is not required. In this case it is possible to extend this $psu(1,1|2)$ superalgebra by additional central charges.

- Sending $\mu \to 0$ leads to the limit $m = 0$. In this limit $\mu = 0$, the superconformal algebra is preserved and the deformed superconformal models reduce to the standard superconformal mechanics models.

Superalgebra

- Another feature of the deformation parameter m is that it vanishes in the limit $\alpha = 0$ where $\mu \neq 0$. The superconformal algebra does not depend on μ, but its realization contains dependence on μ.
- For $\alpha = 0$ the $su(2|1)$ superalgebra becomes the standard Poincaré superalgebra and the internal $SU(2)_{\rm int}$ subgroup generators I^i_k become automorphism generators of Poincaré supergroup and the supergroup $PSU(1,1|2)$.
- This is a consequence of the fact that the $su(2|1)$ superalgebra cannot be embedded into the superconformal algebra $D\,(2,1;\alpha)$ in same time for $\alpha = 0$ and $\alpha = -1$:

$$\begin{aligned}
&\alpha \neq 0\,, && SU(2)_{\rm int} \subset SU(2|1) \subset D\,(2,1;\alpha)\,,\\
&\alpha = -1\,, && SU(2)_{\rm int} \subset SU(2|1) \subset PSU(1,1|2) \subset D\,(2,1;-1)\,,\\
&\alpha = 0\,, && D\,(2,1;\alpha) \cong PSU(1,1|2) \rtimes SU(2)_{\rm int}.
\end{aligned}$$

- Indeed the superconformal group $D\,(2,1;\alpha)$ has 2 equivalent subgroups $SU(2)$, but in sense of the symmetry $SU(2|1)$ they are not equivalent.

Covariant derivatives

The covariant derivatives $\mathcal{D}^i$, $\bar{\mathcal{D}}_j$, $\mathcal{D}_t$ are written as

$$\begin{aligned}
\mathcal{D}^i &= \left(1 + m\,\bar{\theta}^k\theta_k - \frac{3m^2}{4}\left(\bar{\theta}^k\theta_k\right)^2\right)\frac{\partial}{\partial\theta_i} - m\,\bar{\theta}^i\theta_j\frac{\partial}{\partial\theta_j} - i\bar{\theta}^i\partial_t \\
&\quad + m\,\bar{\theta}^i\tilde{F} - m\,\bar{\theta}^j\left[1 - m\,\bar{\theta}^k\theta_k\right]\tilde{I}^i_j, \\
\bar{\mathcal{D}}_j &= -\left(1 + m\,\bar{\theta}^k\theta_k - \frac{3m^2}{4}\left(\bar{\theta}^k\theta_k\right)^2\right)\frac{\partial}{\partial\bar{\theta}^j} + m\,\bar{\theta}^k\theta_j\frac{\partial}{\partial\bar{\theta}^k} + i\theta_j\partial_t \\
&\quad - m\,\theta_j\tilde{F} + m\,\theta_k\left[1 - m\,\bar{\theta}^k\theta_k\right]\tilde{I}^k_j, \\
\mathcal{D}_t &= i\partial_t\,.
\end{aligned}$$

Here, $\tilde{I}^k_j$, $\tilde{F}$ are matrix generators of the $U(2)$ representation by which the given superfield is rotated with respect to its external indices. They form the algebra which mimics $\hat{su}(2|1)$:

$$\{\mathcal{D}^i, \bar{\mathcal{D}}_j\} = 2m\left(\tilde{I}^i_j - \delta^i_j\tilde{F}\right) + 2i\delta^i_j\mathcal{D}_t\,, \qquad \left[\tilde{I}^i_j, \tilde{I}^k_l\right] = \delta^i_l\tilde{I}^k_j - \delta^k_j\tilde{I}^i_l\,,$$

$$\left[\tilde{I}^i_j, \bar{\mathcal{D}}_l\right] = \delta^i_l\bar{\mathcal{D}}_j - \frac{1}{2}\delta^i_j\bar{\mathcal{D}}_l\,, \qquad \left[\tilde{I}^i_j, \mathcal{D}^k\right] = \frac{1}{2}\delta^i_j\mathcal{D}^k - \delta^k_j\mathcal{D}^i,$$

$$\left[\tilde{F}, \bar{\mathcal{D}}_l\right] = \frac{1}{2}\bar{\mathcal{D}}_l\,, \qquad \left[\tilde{F}, \mathcal{D}^k\right] = -\frac{1}{2}\mathcal{D}^k.$$

The multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$

- The multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ is described by the real neutral superfield G satisfying

$$\varepsilon^{lj}\bar{\mathcal{D}}_l\,\bar{\mathcal{D}}_j\,G = \varepsilon_{lj}\mathcal{D}^l\,\mathcal{D}^j\,G = 0\,, \qquad \left[\mathcal{D}^i,\bar{\mathcal{D}}_i\right]G = 4mG\,.$$

- They are solved by

$$\begin{aligned} G \quad = \quad & \left[1 - m\,\bar{\theta}^k\theta_k + m^2\,(\theta)^2\,(\bar{\theta}\,)^2\right]x + \frac{\ddot{x}}{4}\,(\theta)^2\,(\bar{\theta}\,)^2 - i\,\bar{\theta}^k\theta_k\left(\theta_i\,\dot{\psi}^i + \bar{\theta}^j\,\dot{\bar{\psi}}_j\right) \\ & + \left[1 - 2m\,\bar{\theta}^k\theta_k\right]\left(\theta_i\,\psi^i - \bar{\theta}^j\,\bar{\psi}_j\right) + \bar{\theta}^j\theta_i\,B^i_j\,, \quad B^k_k = 0\,. \end{aligned}$$

- As the most important requirement, the constraints must be covariant under superconformal symmetry.

- Requiring it, one can restore the supercharges S, $\bar{S}$ and the bosonic generators C, $\bar{C}$, T, $\bar{T}$. Actually, these additional generators for the multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ are extended by weight terms.

Weak supersymmetry

- Invariant action

$$\mathcal{L} = -\int d^2\theta\, d^2\bar{\theta}\left[1 + 2m\,\bar{\theta}^k\theta_k\right] f(G)\,, \quad S = \int dt\,\mathcal{L}\,.$$

- After doing θ integral and performing some substitutions ($f(x) \to V(y)$), we have obtained the Lagrangian

$$\begin{aligned}\mathcal{L} \;=\; & \frac{\dot{y}^2}{2} + \frac{i}{2}\left(\bar{\zeta}_i\dot{\zeta}^i - \dot{\bar{\zeta}}_i\zeta^i\right) + \tilde{B}^i_j\tilde{B}^j_i - \frac{\tilde{B}^j_i\left[V'(y)-1\right]}{V(y)}\left(\delta^i_j\bar{\zeta}_k\zeta^k - 2\bar{\zeta}_j\zeta^i\right)\\ & - \frac{V''(y)V(y) + \left[2V'(y)-3\right]\left[V'(y)-1\right]}{4V^2(y)}\,(\zeta)^2\,(\bar{\zeta})^2\\ & + m\,\bar{\zeta}_i\zeta^i V'(y) - am\,\bar{\zeta}_i\zeta^i - \frac{m^2}{2}V^2(y).\end{aligned}$$

- This Lagrangian corresponds to the off-shell formulation of the general SQM model with “weak” $\mathcal{N}=4$ supersymmetry (A. Smilga, 2004).

Superconformal action

- Some (**1**, **4**, **3**) models obtained from the $SU(2|1)$ superspace possess superconformal symmetry with respect to the supergroup $D(2,1;\alpha)$.
- The superconformal Lagrangian

$$\mathcal{L} = -\int d^2\theta\, d^2\bar{\theta}\left[1 + 2m\,\bar{\theta}^k\theta_k\right] f(G)\,,$$

where

$$f(G) = \frac{1}{8\,(1+\alpha)}\, G^{-\frac{1}{\alpha}} \quad \text{for } \alpha \neq -1\,,$$
$$f(G) = \frac{1}{8}\, G \ln G \quad \text{for } \alpha = -1\,.$$

- The superfield G transformation,

$$\delta_\varepsilon G = -2m\left[1 - 2am\,\bar{\theta}^k\theta_k\right]\left(\bar{\varepsilon}^i\theta_i\, e^{2iamt} - \varepsilon_i\bar{\theta}^i\, e^{-2iamt}\right) G\,,$$

compensate the S, $\bar{S}$ transformations of the $SU(2|1)$ invariant measure $d\mu$ in the superconformal action.

- In the limit $m = 0$, the action become the action of the standard $\mathcal{N} = 4$ superconformal mechanics based on the (**1**, **4**, **3**) multiplet.

Superconformal action

- Passing to the function $V(y)$, one can write that

$$V(y) = -\frac{y}{2\alpha}\,, \qquad V'(y) = -\frac{1}{2\alpha}\,, \qquad \frac{V'(y)-1}{V(y)} = \frac{1+2\alpha}{y}\,.$$

- The superconformal Lagrangian

$$\mathcal{L}_{conf} = \frac{\dot{y}^2}{2} + \frac{i}{2}\left(\bar{\zeta}_i\dot{\zeta}^i - \dot{\bar{\zeta}}_i\zeta^i\right) + \tilde{B}^i_j\tilde{B}^j_i - \frac{\tilde{B}^j_i\,[1+2\alpha]}{y}\left(\delta^i_j\bar{\zeta}_k\zeta^k - 2\bar{\zeta}_j\zeta^i\right)$$
$$- \frac{[1+3\alpha]\,[1+2\alpha]}{2y^2}\,(\zeta)^2\,(\bar{\zeta})^2 - \frac{m^2}{8\alpha^2}\,y^2, \qquad a = -\frac{1}{2\alpha}\,.$$

- The only difference from the standard $\mathcal{N}=4$ superconformal mechanics is the existence of the oscillator term.
- Now, one can say that we deal with the superconformal mechanics induced by *trigonometric* transformations (N. L. Holanda, F. Toppan, 2014).

Superconformal action

- The transformations corresponding to Q, $\bar{Q}$:

$$
\begin{aligned}
\delta y &= \bar{\epsilon}^k\bar{\zeta}_k\, e^{-iamt} - \epsilon_k\zeta^k\, e^{iamt}, \\
\delta\zeta^i &= i\bar{\epsilon}^i\dot{y}\, e^{-iamt} - am\,\bar{\epsilon}^i y\, e^{-iamt} + 2\,\bar{\epsilon}^k\tilde{B}^i_k\, e^{-iamt} \\
&\quad + \zeta^i\left(\bar{\epsilon}^k\bar{\zeta}_k\, e^{-iamt} - \epsilon_k\zeta^k\, e^{iamt}\right)\frac{a-1}{ay}, \\
\delta\tilde{B}^i_j &= -i\left(\epsilon_j\dot{\zeta}^i\, e^{iamt} + \bar{\epsilon}^i\dot{\bar{\zeta}}_j\, e^{-iamt} - \frac{\delta^i_j}{2}\left[\epsilon_k\dot{\zeta}^k\, e^{iamt} + \bar{\epsilon}^k\dot{\bar{\zeta}}_k\, e^{-iamt}\right]\right) \\
&\quad + m\,[1-a]\left(\bar{\epsilon}^i\bar{\zeta}_j\, e^{-iamt} - \epsilon_j\zeta^i\, e^{iamt} - \frac{\delta^i_j}{2}\left[\bar{\epsilon}^k\bar{\zeta}_k\, e^{-iamt} - \epsilon_k\zeta^k\, e^{iamt}\right]\right) \\
&\quad + \tilde{B}^i_j\left(\bar{\epsilon}^k\bar{\zeta}_k\, e^{-iamt} - \epsilon_k\zeta^k\, e^{iamt}\right)\frac{a-1}{ay} \\
&\quad + i\dot{y}\left(\epsilon_j\zeta^i\, e^{iamt} + \bar{\epsilon}^i\bar{\zeta}_j\, e^{-iamt} - \frac{\delta^i_j}{2}\left[\epsilon_k\zeta^k\, e^{iamt} + \bar{\epsilon}^k\bar{\zeta}_k\, e^{-iamt}\right]\right)\frac{a-1}{ay}.
\end{aligned}
$$

- To obtain the hidden supersymmetry transformations corresponding to S, $\bar{S}$, one need to change $m \to -m$ in the $SU(2|1)$ super transformations of component fields.

$\alpha = -1$

- Actually, the $(\mathbf{1},\mathbf{4},\mathbf{3})$ constraints can be generalized as

$$\varepsilon^{lj}\bar{\mathcal{D}}_l\,\bar{\mathcal{D}}_j\,\tilde{G} = \varepsilon_{lj}\mathcal{D}^l\,\mathcal{D}^j\,\tilde{G} = 0\,, \qquad \left[\mathcal{D}^i, \bar{\mathcal{D}}_i\right]\tilde{G} = 4m\,\tilde{G} - 4c\,.$$

- The superconformal group reduces to $PSU(1,1|2) \rtimes U(1)$ and the constraints are covariant only with respect this symmetry where $\alpha = -1$.
- In this case, the corresponding superconformal superfield action is

$$S_{conf} = -\frac{1}{8}\int d^2\theta\, d^2\bar{\theta}\left[1 + 2m\,\bar{\theta}^k\theta_k\right]\tilde{G}\ln\tilde{G}.$$

- In the limit $m = 0$, the parameter c does not vanish and the constraints acquire the form

$$\varepsilon^{lj}\bar{\mathcal{D}}_l\,\bar{\mathcal{D}}_j\,\tilde{G} = \varepsilon_{lj}\mathcal{D}^l\,\mathcal{D}^j\,\tilde{G} = 0\,, \qquad \left[\mathcal{D}^i, \bar{\mathcal{D}}_i\right]\tilde{G} = 4c\,.$$

$\alpha = -1$

- The superconformal Lagrangian is written as

$$\mathcal{L}_{conf} = \frac{\dot{y}^2}{2} + \frac{i}{2}\left(\bar{\zeta}_i\dot{\zeta}^i - \dot{\bar{\zeta}}_i\zeta^i\right) + \tilde{B}^i_j\tilde{B}^j_i + \frac{\tilde{B}^j_i}{y}\left(\delta^i_j\bar{\zeta}_k\zeta^k - 2\bar{\zeta}_j\zeta^i\right) - \frac{1}{y^2}(\zeta)^2(\bar{\zeta})^2 + \frac{c}{2y^2}\bar{\zeta}_i\zeta^i - \frac{m^2y^2}{8} + \frac{cm}{4} - \frac{c^2}{8y^2}\,.$$

- Here, c is responsible for appearance of new potential terms and it occurs in $V(y)$ as

$$V(y) = \frac{y}{2} - \frac{c}{2my}\,.$$

- The relevant on-shell Lagrangian

$$\mathcal{L}_{conf} = \frac{\dot{y}^2}{2} + \frac{i}{2}\left(\bar{\zeta}_i\dot{\zeta}^i - \dot{\bar{\zeta}}_i\zeta^i\right) - \frac{1}{4y^2}(\zeta)^2(\bar{\zeta})^2 + \frac{c}{2y^2}\bar{\zeta}_i\zeta^i - \frac{m^2y^2}{8} + \frac{cm}{4} - \frac{c^2}{8y^2}\,,$$

as a superconformal Lagrangian ($\alpha = -1$) was found (S. Bellucci, S. Krivonos, 2009).

The multiplet $(\mathbf{2}, \mathbf{4}, \mathbf{2})$

- The standard form of the chiral and antichiral conditions is defined as
$$\text{(a)}\ \ \bar{\mathcal{D}}_i\Phi = 0\,, \qquad \text{(b)}\ \ \mathcal{D}^i\bar{\Phi} = 0\,.$$
which means the existence of the left and right chiral subspaces (t_L, θ_i), $(t_R, \bar{\theta}^i)$
- The left one is given by
$$t_L = t + i\,\bar{\theta}^k\theta_k - \frac{i}{2}\,m\,(\theta)^2\,(\bar{\theta}\,)^2\,, \qquad c.c..$$
and it is closed under the $SU(2|1)$ transformations
$$\delta\theta_i = \epsilon_i + 2m\,\bar{\epsilon}^k\theta_k\theta_i\,, \qquad \delta t_L = 2i\,\bar{\epsilon}^k\theta_k\,.$$
- The same coordinates are closed under the second $SU(2|1)$ transformations generated by S, $\bar{S}$ only for $\alpha = -1$:
$$\delta\theta_i = \varepsilon_i e^{-imt_L}, \qquad \delta t_L = 2i\,\bar{\varepsilon}^k\theta_k e^{imt_L}.$$
- It means that only for the supergroup $PSU(1,1|2) \rtimes U(1)$ chiral subspaces are closed and covariance of constraints is preserved. We need to take into account that the generators C, $\bar{C}$ transform the chiral subspace not invariantly and they also break covariance of these constraints.

The chiral superfield

One can impose that the complex superfield Φ possesses a fixed $U(1)$ charge

$$\tilde{F}\Phi = 2\kappa\,\Phi.$$

The general solution reads:

$$\Phi\big(t,\theta,\bar{\theta}\,\big) = \Big[1 + 2m\,\bar{\theta}^k\theta_k\Big]^{-\kappa}\,\Phi_L(t_L,\theta\,),$$

$$\Phi_L(t_L,\theta\,) = z + \sqrt{2}\,\theta_i\xi^i e^{\frac{i}{2}mt_L} + (\theta)^2 Be^{imt_L}, \qquad \overline{(\xi^i)} = \bar{\xi}_i\,.$$

The odd transformations induce the following off-shell transformations of the component fields:

$$\begin{aligned}
\delta z &= -\sqrt{2}\,\epsilon_k\xi^k e^{\frac{i}{2}mt} - \sqrt{2}\,\varepsilon_k\xi^k e^{-\frac{i}{2}mt},\\
\delta\xi^i &= \sqrt{2}\,\bar{\epsilon}^i\left[i\dot{z} - 2\kappa mz\right]e^{-\frac{i}{2}mt} - \sqrt{2}\,\epsilon^i Be^{\frac{i}{2}mt}\\
&\quad + \sqrt{2}\,\bar{\varepsilon}^i\left[i\dot{z} + 2\kappa mz\right]e^{\frac{i}{2}mt} - \sqrt{2}\,\varepsilon^i Be^{-\frac{i}{2}mt},\\
\delta B &= -\sqrt{2}\,\bar{\epsilon}_k\left[i\dot{\xi}^k - m\left(2\kappa - \frac{1}{2}\right)\xi^k\right]e^{-\frac{i}{2}mt} - \sqrt{2}\,\bar{\varepsilon}_k\left[i\dot{\xi}^k + \left(2\kappa - \frac{1}{2}\right)m\,\xi^k\right]e^{\frac{i}{2}mt}.
\end{aligned}$$

Correspondingly, the superfield Φ has the transformations

$$\delta\Phi = 2\kappa m\left(\bar{\epsilon}^i\theta_i + \epsilon_i\bar{\theta}^i\right)\Phi - 2\kappa m\left(3\bar{\varepsilon}^i\theta_i\,e^{imt} - \varepsilon_i\bar{\theta}^i\,e^{-imt}\right)\Big[1 - m\,\bar{\theta}^k\theta_k\Big]\,\Phi\,.$$

Superconformal action

- The general $SU(2|1)$ action is defined as

$$S_{\text{kin}} = \frac{1}{4}\int d\mu\, f\left(\Phi, \bar{\Phi}\right).$$

where $f\left(\Phi, \bar{\Phi}\right)$ is Kähler potential. The action is superconformally invariant only when we define the Kähler potential as $f\left(z, \bar{z}\right) = \left(z\bar{z}\right)^{\frac{1}{4\kappa}}$.

- Then the off-shell Lagrangian is written as

$$\begin{aligned}\mathcal{L}_{\text{kin}} \;=\;& \frac{\left(z\bar{z}\right)^{\frac{1}{4\kappa}-1}}{(4\kappa)^2}\left[\dot{\bar{z}}\dot{z} + \frac{i}{2}\left(\bar{\xi}_i\dot{\xi}^i - \dot{\bar{\xi}}_i\xi^i\right) + \bar{B}B\right] + \frac{(4\kappa-1)^2}{4(4\kappa)^4}\left(z\bar{z}\right)^{\frac{1}{4\kappa}-2}\left(\xi\right)^2\left(\bar{\xi}\right)^2 \\ &+ \frac{4\kappa-1}{(4\kappa)^3}\left(z\bar{z}\right)^{\frac{1}{4\kappa}-2}\left[\frac{i}{2}\,\bar{\xi}^k\xi_k\left(\dot{\bar{z}}z - \dot{z}\bar{z}\right) + \frac{1}{2}\left(\xi\right)^2\bar{B}\bar{z} + \frac{1}{2}\left(\bar{\xi}\right)^2 B z\right] \\ &- \frac{m^2}{4}\left(z\bar{z}\right)^{\frac{1}{4\kappa}}.\end{aligned}$$

- The bosonic on-shell Lagrangian is very simple:

$$\mathcal{L}_{\text{kin}} = \frac{\left(z\bar{z}\right)^{\frac{1}{4\kappa}-1}}{(4\kappa)^2}\dot{\bar{z}}\dot{z} - \frac{m^2}{4}\left(z\bar{z}\right)^{\frac{1}{4\kappa}}.$$

Superalgebra

- The simplest case $\kappa = 1/4$ has the relevant Lagrangian

$$\mathcal{L}^{bos}_{(\kappa=1/4)} \quad = \quad \dot{\bar{z}}\dot{z} + \frac{i}{2}\left(\bar{\xi}_i\dot{\xi}^i - \dot{\bar{\xi}}_i\xi^i\right) + \bar{B}B - \frac{m^2}{4}\, z\bar{z}\,.$$

- The supersymmetric transformations are closed on the superalgebra $psu(1,1|2)$ extended by additional central charge:

$$\{Q_{\alpha ii'}, Q_{\beta jj'}\} = 2\left(\epsilon_{ij}\epsilon_{i'j'}T_{\alpha\beta} - \epsilon_{\alpha\beta}\epsilon_{i'j'}J_{ij} - 2i\,\epsilon_{\alpha\beta}\epsilon_{ij}c_{i'j'}\kappa\right),$$
$$[T_{\alpha\beta}, Q_{\gamma ii'}] = -i\,\epsilon_{\gamma(\alpha}Q_{\beta)ii'}\,, \qquad [T_{\alpha\beta}, T_{\gamma\delta}] = i\left(\epsilon_{\alpha\gamma}T_{\beta\delta} + \epsilon_{\beta\delta}T_{\alpha\gamma}\right),$$
$$[J_{ij}, Q_{\alpha ki'}] = -i\,\epsilon_{k(i}Q_{\alpha j)i'}\,, \qquad [J_{ij}, J_{kl}] = i\left(\epsilon_{ik}J_{jl} + \epsilon_{jl}J_{ik}\right),$$
$$c_{1'2'} = c_{2'1'} = 1\,, \qquad c_{1'1'} = c_{2'2'} = 0\,.$$

- Any superconformal actions with $\kappa = 0$ can not be constructed. It means that deformed $\mathcal{N} = 4$, $d = 1$ actions of superconformal mechanics exist when the superconformal superalgebra is centrally extended.

- In the limit $m = 0$, the superconformal action becomes the standard $\mathcal{N} = 4$ superconformal and its superconformal algebra again is the centrally extended superalgebra $su(1,1|2)$.

Summary and Outlook

- We considered a new type of $\mathcal{N}=4$ supersymmetric mechanics which is based on the supergroup $SU(2|1)$. It is a deformation of the standard $\mathcal{N}=4$ mechanics by a mass parameter m.
- There exists another type of superconformal models (N. L. Holanda, F. Toppan, 2014) that can be obtained from the $SU(2|1)$ superfield approach. We constructed superspace realization of this type superconformal mechanics.
- Exploiting $SU(2|1)$ superfields, we constructed superconformal models for the multiplets $(\mathbf{1},\mathbf{4},\mathbf{3})$ and $(\mathbf{2},\mathbf{4},\mathbf{2})$. The $(\mathbf{1},\mathbf{4},\mathbf{3})$ superconformal models exist for $\alpha \neq 0$, while the $(\mathbf{2},\mathbf{4},\mathbf{2})$ superconformal models exist only for $\alpha=-1$ when superalgebra is extended by central charges.
- One can generalize the superconformal models $(\mathbf{2},\mathbf{4},\mathbf{2})$ to the superalgebra $psu(1,1|2)$ extended by 3 central charges. It requires a generalization of the chiral constraints as it was shown for Super Kähler Oscillator (E. Ivanov, S. S., 2013).
- It is interesting to construct superconformal models corresponding to other $SU(2|1)$ counterparts of the basic $\mathcal{N}=4$ off-shell multiplets.

Thank you for your attention!