DOUBLE EXTENDED

CUBIC PEAKON EQUATIONS

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Plan:

1.) Comassa-Holm and Degasperis-Procesi equation

2.) Cubic peakon equation

3.) Bi-Hamiltonian structure,

4.) Method of generalizations:

   a.) multi-component C-H equation ,

   b.) multi-peakon cubic.

5.) Supersymmetric Cubic Peakon Equation

6.) Double extended Cubic Peakon Equation

1.) Z. Popowicz
“*A two - Component Generalization of the Degasperis-Procesi Equation*”
J. Phys. A.: Math. Gen 39, 13717-13726 (2006) , arXiv:0604067

2.) Z. Popowicz
"*Double Extended Cubic Peakon Equation*"
arXiv:1407.6141

Camassa - Holm equation

$$u_t - u_{xxt} = \frac{1}{2}\left(-3u^2 + 2uu_{xx} + u_x^2\right)_x$$

$$u_t + \frac{1}{2}\partial\left(u^2 + G \divideontimes (2u^2 + u_x^2)\right) = 0$$

$$f \divideontimes g = \int dy\ f(y)g(x-y), \qquad G(x) = \frac{1}{2}e^{-|x|}$$

Tautological solutions

$$u_t - u_{xxt} = \alpha uu_x + \beta u_x u_{xx} + \gamma uu_{xxx}$$

$$u(x,t) = c_1(t)e^x + c_2(t)e^{-x}$$

$$\alpha + \beta + \gamma = 0$$

From the physical point of view (Novikov due to symmetry) only

A.) Comassa-Holm $\alpha = -3, \; \beta = 2, \; \gamma = 1,$

B.) Degasperis-Procesi $\alpha = -4, \; \beta = 3, \; \gamma = 1$

Peakon Solutions of Camassa-Holm

$$m_t = -um_x - 2mu_x, \qquad m = u - u_{xx}.$$

The scalar spectral problem is

$$\Psi_{xx} = (\frac{1}{4} - \lambda m)\Psi$$

One peakon

$$\begin{aligned} u(x,t) &= p(t)e^{-|x-q(t)|} = p(t)e^{-|x-ct-c_o|} \\ u_x &= -sgn(x-q)u, \quad m = 2\delta(x-q)u \\ p_t &= 0, \quad q_t = 0 \Rightarrow q = ct + c_o, \quad p = c. \end{aligned}$$

Multipeakon solution.

$$u(x,t) = \sum_{j-1}^{N} p_j(t) e^{-|x-q_j(t)|}$$

$$\mathring{p}_j = \sum_{k=1}^{N} p_j p_k sgn(q_j - q_k) e^{-|q_j - q_k|}, \qquad \mathring{q}_j = \sum_{k=1}^{N} p_k e^{-|q_j - q_k|}$$

Degasperis-Procesi

$$\begin{aligned} u_t - u_{t,xx} &= \left(-2u^2 + uu_{xx} + u_x^2\right)_x \\ m_t &= -3u_x m - m_x u, \qquad m = u - u_{xx}. \end{aligned}$$

The scalar spectral problem is

$$\Psi_{xxx} = \Psi_x - \lambda m \Psi.$$

Cubic Peakon Equation

$$(1-\partial_{xx})u_t = W(u, u_x, u_{xx}, u_{xxx})$$

How to find the polynomial W?

Hint: Higher symmetries Novikov

For square and cubic we have 9 equations.

Only 2 cubic equations are interesting.

$$\begin{aligned} m_t &+ (m(u^2-u_x^2))_x = 0 \\ m_t &+ u^2 m_x + 3uu_x m = 0 \end{aligned}$$

Bi-Hamiltonian Structure $$m_t = \mathcal{B}_0 \frac{\delta H_2}{\delta m} = \mathcal{B}_1 \frac{\delta H_1}{\delta m}$$

1.) C-H equation.

$$\begin{aligned} m_t &= -um_x - 2mu_x, \qquad m = u - u_{xx}. \\ \mathcal{B}_0 &= -\partial(1-\partial^2) = -\mathcal{L}, \qquad \mathcal{B}_1 = -(m\partial + \partial m) \\ H_2 &= \frac{1}{2}\int dx\ (u^3 + uu_x^2), \qquad H_1 = \frac{1}{2}\int dx\ (u^2 + u_x^2) \end{aligned}$$

2.) D-P equation.

$$\begin{aligned} m_t &= -um_x - 3mu_x, \\ \mathcal{B}_0 &= \mathcal{L}(4 - \partial_{xx}), \quad \mathcal{B}_1 = (m_x + 3m\partial)\mathcal{L}^{-1}(2m_x + 3m\partial) \\ H_2 &= -\frac{1}{2}\int dx\ u^3, \qquad H_1 = -\frac{1}{2}\int dx\ m \end{aligned}$$

3.) Cubic peakon equation.

$$\begin{aligned}
m_t &= -3uu_x m - u^2 m_x, \\
\mathcal{B}_0 &= -2(3m\partial + 2m_x)(4\partial - \partial^3)^{-1}(3m\partial + m_x) \\
\mathcal{B}_1 &= (1-\partial^2)m^{-1}\partial m^{-1}(1-\partial^2) \\
H_2 &= \frac{1}{4}\int dx\ mu \\
H_1 &= \frac{1}{6}\int dx\ um\partial^{-1}m(\partial^2 - 1)(u^2 m_x + 3uu_x m)
\end{aligned}$$

Multi-component Peakon Equations

Two-component C-H

$$\begin{aligned}\Psi_{xx} &= \left(\frac{1}{4} - \lambda m + \lambda^2\rho^2\right)\Psi \\ \Psi_t &= -\left(\frac{1}{2\lambda} + u\right)\Psi_x + \frac{1}{2}u_x\Psi, \\ m_t &= -2mu_x - m_x u + \rho\rho_x, \qquad \rho_t = -(u\rho)_x\end{aligned}$$

Ivanov and Holm generalization CH(N,K)

$$\begin{aligned}\Psi_{xx} &= \left(\sum_{i=1}^{N} q_i(x,t)\lambda^i + \frac{1}{4}\right)\Psi \\ \Psi_t &= \sum_{j=0}^{K}\left(-u_j(x,t)/\lambda^j\ \partial_x + u_j(x,t)_x/2\right)\Psi\end{aligned}$$

two-component D-P and Cubic equation ? please wait

Two-peakon cubic equation

$$
\begin{aligned}
m_t &= -3u_x vm - uvm_x, \qquad m = u - u_{xx} \\
n_t &= -3v_x un - uvn_x \qquad n = v - v_{xx}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}_1 &= \frac{1}{2}\begin{pmatrix} 3m\partial + 2m_x \\ 3n\partial + 2n_x \end{pmatrix} \hat{\mathcal{L}}^{-1} \begin{pmatrix} 3m\partial + m_x\ , & 3n\partial + n_x \end{pmatrix} \\
&\quad + \frac{3}{2}\begin{pmatrix} m\partial^{-1}m & -m\partial^{-1}n \\ -n\partial^{-1}m & n\partial^{-1}n \end{pmatrix} \\
\mathcal{B}_0 &= \begin{pmatrix} 0 & \partial^2 - 1 \\ 1 - \partial^2 & 0 \end{pmatrix} \\
\hat{\mathcal{L}} &= (\partial^3 - 4\partial)^{-1}, \qquad H_1 = \int dx\ un \\
H_2 &= -\frac{1}{2}\int dx\ (3u^2 v_x n + u^2 v n_x)
\end{aligned}
$$

Supersymmetric $N = 2$ cubic peakon equation as two-component cubic peakon equation

$$\mathcal{D}_1 = \frac{\partial}{\partial\theta_1} - \frac{1}{2}\theta_2\partial_x, \quad \mathcal{D}_2 = \frac{\partial}{\partial\theta_2} - \frac{1}{2}\theta_1\partial_x, \quad \mathcal{D}_1^2 = \mathcal{D}_2^2 = 0$$
$$\{\mathcal{D}_1, \mathcal{D}_2\} = -\partial_x, \quad \mathcal{D}_3 = [\mathcal{D}_1, \mathcal{D}_2] = \partial_x + 2\mathcal{D}_1\mathcal{D}_2,$$

$$\mathcal{J} = \begin{pmatrix} -\mathcal{D}_3\partial_x + 2\partial_x W + 2(\mathcal{D}_1 W)\mathcal{D}_2 + 2(\mathcal{D}_2 W)\mathcal{D}_1 & -\mathcal{J}^{\star}_{2,1} \\ \partial_x V + V_x + 2(\mathcal{D}_1 V)\mathcal{D}_2 + 2(\mathcal{D}_2 V)\mathcal{D}_1 & 0 \end{pmatrix}$$

Dirac reduction when $W = 1$ gives

$$\mathcal{K} = [\partial_x V + V_x + 2(\mathcal{D}_1 V)\mathcal{D}_2 + 2(\mathcal{D}_2 V)\mathcal{D}_1](4\partial_x - \partial_{xxx})^{-1}(2 + \mathcal{D}_3)$$
$$[\partial_x V + 2(\mathcal{D}_1 V)\mathcal{D}_2 + 2(\mathcal{D}_2 V)\mathcal{D}_1]$$

where $(-\mathcal{D}_3 + 2)^{-1} = (4 - \partial_{xx})^{-1}(2 + \mathcal{D}_3)$

$$\begin{aligned} V_t &= \mathcal{K}\frac{\delta H}{\delta V} \qquad V=(1-\mathcal{D}_3)A \\ H &= \frac{1}{2}\int dx d\theta_1 d\theta_2 VA, \end{aligned}$$

$$V_t = V_x A^2 + VA_x A + (\mathcal{D}_2 V)(\mathcal{D}_1 A^2) + (\mathcal{D}_1 V)(\mathcal{D}_2 A^2)$$

Bosonic sector

$$V = v_0 + \theta_2\theta_1 v_1, \qquad A = u + \theta_2\theta_1 a_1, \qquad v_0 = u - 2a_1,$$

$$v_1 = a_1 - \frac{1}{2}u_{xx}, \qquad a_1 = \frac{1}{2}(u-\rho),$$

$$v_1 = \frac{1}{2}(u - u_{xx} - \rho) = \frac{1}{2}(m-\rho).$$

Two-component cubic pekon equation

$$\begin{aligned} \rho_t &= \rho_x u^2 + \rho u u_x, \\ m_t &= 3u_x u m + u^2 m_x - \rho(u\rho)_x. \end{aligned}$$

Hamiltonian Structure, Not Bi

$$\begin{pmatrix} \rho \\ m \end{pmatrix}_t = \hat{\mathcal{K}} \begin{pmatrix} \frac{\delta H}{\delta \rho} \\ \frac{\delta H}{\delta m} \end{pmatrix}, \qquad H = \frac{1}{2}\int dx(mu - \rho^2)$$

$$\hat{\mathcal{K}} = \begin{pmatrix} \rho^{-1}\partial\rho^2\hat{\mathcal{L}}^{-1}\rho^2\partial\rho^{-1} & 3\rho^{-1}\partial\rho^2\hat{\mathcal{L}}^{-1}m^{1/3}\partial m^{2/3} \\ 3m^{2/3}\partial\hat{\mathcal{L}}^{-1}\rho^2\partial\rho^{-1} & -\rho\partial\rho + 9m^{2/3}\partial m^{1/3}\hat{\mathcal{L}}^{-1}m^{1/3}\partial m^{2/3} \end{pmatrix}$$

$\hat{\mathcal{L}} = \partial^3 - 4\partial_x$.

$\hat{\mathcal{K}}$ is a Dirac reduced version of

$$\begin{pmatrix} \partial_{xxx} - 4u\partial - 2u_x & \rho_x - \rho\partial_x & -m_x - 3m\partial \\ -\rho\partial_x - 2\rho_x & 0 & 0 \\ -2m_x - 3m\partial_x & 0 & -\rho\partial_x\rho \end{pmatrix}$$

How to find two-component and two-peakon cubic equation?

$$\rho_t = \rho_x u^2 + \rho u u_x, \qquad m_t = 3u_x u m + u^2 m_x - \rho(u\rho)_x$$
$$m_t = -3u_x v m - uvm_x, \qquad n_t = -3v_x u n - uvn_x$$

Hint: Construct Bi-Hamiltonian structure

$$\mathcal{J} = \begin{pmatrix} -\partial_{xxx} + 4u\partial_x + 2u_x & -\mathcal{J}^\star_{2,1} & -\mathcal{J}^\star_{3,1} & -\mathcal{J}^\star_{4,1} & -\mathcal{J}^\star_{5,1} \\ \rho_1^{-1}\partial\rho_1^2 & 0 & 0 & 0 & 0 \\ 3m_1^{1/3}\partial_x m_1^{2/3} & 0 & \mathcal{J}_{3,3} & 0 & \mathcal{J}_{3,5} \\ \rho_2^{-1}\partial_x\rho_2^2 & 0 & 0 & 0 & 0 \\ 3m_2^{1/3}\partial_x m_2^{2/3} & 0 & -\mathcal{J}^\star_{3,5} & 0 & \mathcal{J}_{5,5} \end{pmatrix}$$

$$
\begin{aligned}
\mathcal{J}_{3,3} &= \lambda_1 m_1 \partial^{-1} m_1 + \lambda_2 m_2 \partial^{-1} m_2 + \lambda_3 (m_1 \partial^{-1} m_2 + m_2 \partial^{-1} m_1) \\
&\qquad + k_1 \rho_1 \partial \rho_1 + k_2 \rho_2 \partial \rho_2 + k_3 (\rho_1 \partial \rho_2 + \rho_2 \partial \rho_1). \\
\mathcal{J}_{3,5} &= \lambda_4 m_1 \partial^{-1} m_1 + \lambda_5 m_2 \partial^{-1} m_2 + \lambda_6 m_1 \partial^{-1} m_2 + \lambda_7 m_2 \partial^{-1} m_1 \\
&+ k_4 \rho_1 \rho_2 \partial + k_5 \rho_{1,x} \rho_2 + k_6 \rho_1 \rho_{2,x} + k_7 \rho_1^2 \partial + k_8 \rho_2^2 \partial \\
&\qquad + k_9 \rho_1 \rho_{1,x} + k_{10} \rho_2 \rho_{2,x} \\
\mathcal{J}_{5,5} &= \lambda_8 m_1 \partial^{-1} m_1 + \lambda_9 m_2 \partial^{-1} m_2 + \lambda_{10} (m_1 \partial^{-1} m_2 + m_2 \partial^{-1} m_1) \\
&\qquad + k_{11} \rho_1 \partial \rho_1 + k_{12} \rho_1 \partial \rho_2 + k_{13} (\rho_1 \partial \rho_2 + \rho_2 \partial \rho_1).
\end{aligned}
$$

Jacobi identity fixes arbitray constants . Next consider Dirac reduction when $u = 1$. It gives us the second Hamiltonian operator

$$
\mathcal{K} = J\big|_{u=1}
$$

We postulate the following first Hamiltonian operator as

$$\mathcal{L} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1-\partial_{xx} \\ -1 & 0 & 0 & 0 \\ 0 & -1+\partial_{xx} & 0 & 0 \end{pmatrix}$$

R E S U L T S

$$\begin{pmatrix} \rho_1 \\ m_1 \\ \rho_2 \\ m_2 \end{pmatrix}_t = \mathcal{L} \begin{pmatrix} H_{0,\rho_1} \\ H_{0,m_1} \\ H_{0,\rho_2} \\ H_{0,m_2} \end{pmatrix} = \mathcal{K} \begin{pmatrix} H_{1,\rho_1} \\ H_{1,m_1} \\ H_{1,\rho_2} \\ H_{1,m_2} \end{pmatrix}.$$

$$\mathcal{K} =$$

$$-\frac{1}{2}\begin{pmatrix} 2\rho_{1,x}+\rho_1\partial \\ 2m_{1,x}+3m_1\partial \\ 2\rho_{2,x}+\rho_2\partial \\ 2m_{2,x}+3m_2\partial \end{pmatrix} (\partial_{xxx}-4\partial_x)^{-1} \begin{pmatrix} -\rho_{1,x}+\rho_1\partial \\ m_{1,x}+3m_1\partial \\ -\rho_{2,x}+\rho_2\partial \\ m_{2,x}+3m_2\partial \end{pmatrix}^T +$$

$$\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \hat{\mathcal{K}}_{2,2} & 0 & \hat{\mathcal{K}}_{2,4} \\ 0 & 0 & 0 & 0 \\ 0 & -\hat{\mathcal{K}}^\star_{2,4} & 0 & \hat{\mathcal{K}}_{4,4} \end{pmatrix}$$

$$\hat{\mathcal{K}}_{2,2} = 3m_2\partial^{-1}m_2, \qquad \hat{\mathcal{K}}_{4,4} = 3m_1\partial^{-1}m_1$$

$$\hat{\mathcal{K}}_{2,4} = -3m_2\partial^{-1}m_1 + \rho_2\partial\rho_1 - \rho_1\partial\rho_2$$

$$H_0 = -\int dx\; m_1(u_{2,x}u_1^2 - u_{1,x}u_1u_2) + m_2(u_{2,x}u_1u_2 - u_{1,x}u_2^2) + \rho_1\rho_2(u_2u_{2,x}+u_1u_{1,x}) + \rho_2\rho_{1,x}(u_1^2+u_2^2)$$

$$H_1 = \frac{1}{2}\int dx\;(m_1u_1 + m_2u_2 + 2\rho_1^2 + 2\rho_2^2)$$

Equation

$$\rho_{1,t} = \rho_{1,x}(u_1^2 + u_2^2) + \rho_1(u_{1,x}u_1 + u_2u_{2,x})$$
$$\rho_{2,t} = \rho_{2,x}(u_1^2 + u_2^2) + \rho_2(u_{1,x}u_1 + u_2u_{2,x})$$

$$\begin{aligned} m_{1,t} &= [m_1(u_1^2 + u_2^2)]_x + m_1(u_{1,x}u_1 + u_{2,x}u_2) - \\ &\quad 3m_2(u_{2,x}u_1 - u_{1,x}u_2) + u_2(\rho_2\rho_{1,x} - \rho_1\rho_{2,x}) \\ m_{2,t} &= [m_2(u_1^2 + u_2^2)]_x + m_2(u_{1,x}u_1 + u_{2,x}u_2) + \\ &\quad 3m_1(u_{2,x}u_1 - u_{1,x}u_2) + u_1(\rho_1\rho_{2,x} - \rho_2\rho_{1,x}) \end{aligned}$$

Reduction

$$m_1 = i(n_1 - n_2)/2, \quad u_1 = i(v_1 - v_2)/2, \quad \rho_1 = ir_1$$
$$m_2 = (n_1 + n_2)/2, \quad u_2 = (v_1 + v_2)/2, \quad \rho_2 = r_2$$

$$\begin{aligned} r_{1,t} &= \frac{1}{2} r_1 (v_1 v_2)_x + r_{1,x} v_1 v_2 \\ r_{2,t} &= \frac{1}{2} r_2 (v_1 v_2)_x + r_{2,x} v_1 v_2 \\ n_{1,t} &= v_1 v_2 n_{1,x} + 3 v_{1,x} v_2 n_1 + v_2 (r_2 r_{1,x} - r_1 r_{2,x}) \\ n_{2,t} &= v_1 v_2 n_{2,x} + 3 v_{2,x} v_1 n_2 - v_1 (r_2 r_{1,x} - r_1 r_{2,x}) \end{aligned}$$

when $r_1 = r_2 = 0$

$$\begin{aligned} n_{1,t} &= v_1 v_2 n_{1,x} + 3 v_{1,x} v_2 n_1 \\ n_{2,t} &= v_1 v_2 n_{2,x} + 3 v_{2,x} v_1 n_2 \end{aligned}$$

when $r_2 = r_1 = r, n_2 = n_1 = n, v_1 = v_2 = v$

$$r_t = rvv_x + r_x v^2, \qquad n_t = v^2 n_x + 3v_x vn$$