Towards the integrable system closest to SU(3) Yang-Mills quantum mechanics

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- Gribov ambiguity (1978) $\rightarrow$ Attempt of an exact resolution of the non-Abelian Gauss-laws to have an QCD Hamiltonian at low energy (Jackiw+Goldstone, Faddeev, T.D.Lee, t'Hooft)
- Physical quantum Hamiltonian of $SU(2)$ Yang-Mills theory in the "symmetric gauge".
- Physical Hamiltonian of $SU(2)$ Yang-Mills QM of spatially constant fields as the zeroth order in an expansion in the number of spatial derivatives $\equiv$ expansion in $\lambda = g^{-2/3}$
- Calogero-type model as the integrable system closest to $SU(2)$ YM QM: allows for the calculation of the spectrum and other expectation values with high numerical precision in zeroth and higher order in λ (Lorentz-inv and renormalisation in the IR)
- Extension to SU(3) YM QM$\rightarrow$ Gribov-copies and horizons

Aim: Calculation of expectation values in nonperturbative QCD with high numerical precision

The SU(3) Yang-Mills action

$$\mathcal{S}[A] := \int d^4x \left[-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}\right]$$

$$F^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu \ , \quad a = 1,..,8$$

is invariant under the $SU(3)$ gauge transformations $U[\omega(x)] \equiv \exp(i\omega_a \tau_a/2)$

$$A^\omega_{a\mu}(x)\tau_a/2 = U[\omega(x)]\left(A_{a\mu}(x)\tau_a/2 + \frac{i}{g}\partial_\mu\right)U^{-1}[\omega(x)]$$

chromoelectric : $E^a_i \equiv F^a_{i0}$ and chromomagnetic $B^a_i \equiv \frac{1}{2}\epsilon_{ijk}F^a_{jk}$

$\Pi_{ai} = -E_{ai}$ momenta can. conj. to the spatial A_{ai} → canonical Hamiltonian

$$H_C = \int d^3x \left[\frac{1}{2}E^2_{ai} + \frac{1}{2}B^2_{ai}(A) - gA_{a0}\left(D_i(A)_{ab}E_{bi}\right)\right]$$

with the covariant derivative $D_i(A)_{ab} \equiv \delta_{ab}\partial_i - g f_{abc} A_{ci}$

Exploit the time dependence of the gauge transformations to put

$$A_{a0} = 0 \ , \qquad a = 1, .., 8 \qquad \text{(Weyl gauge)}$$

The dynamical variables A_{ai} and $-E_{ai}$, are quantized imposing the equal-time CR
The physical states Φ satisfy

$$\begin{aligned} H\Phi &= \int d^3x \left[\frac{1}{2}E^2_{ai} + \frac{1}{2}B^2_{ai}[A]\right]\Phi = E\Phi \ , \\ G_a(x)\Phi &= \left[D_i(A)_{ab}E_{bi}\right]\Phi = 0 \ , \quad a = 1, .., 8 \ . \end{aligned}$$

The Gauss law operators G_a are the generators of the residual time independent gauge transformations, satisfying $[G_a, H] = 0$ and $[G_a, G_b] = i f_{abc} G_c$.

Angular momentum operators $\quad [J_i, H] = 0$

$$J_i = \int d^3x \left[-\epsilon_{ijk} A_{aj} E_{ak} + \text{orbital parts}\right] \ , \quad i = 1, 2, 3 \ ,$$

The matrix element of an operator O is given in the Cartesian form

$$\langle \Phi' | O | \Phi \rangle \ \propto \int dA \ \Phi'^*(A) \, O \, \Phi(A) \ .$$

The spectrum of Yang-Mills QM of spat.const.gluon fields has first been obtained numerically for the case of $SU(2)$ by Luescher and Muenster (1984), and for $SU(3)$ by P. Weisz and V. Ziemann (1986).

Point trafo to new set of adapted coordinates,
$A_{ai} \to$ 3 gauge angles q_j of orthog. $O(q)$, the pos. definite sym. 3×3 matrix S

$$A_{ai}\left(q, S\right) = O_{ak}\left(q\right) S_{ki} - \frac{1}{2g}\epsilon_{abc}\left(O\left(q\right)\partial_i O^T\left(q\right)\right)_{bc} ,$$

Generalisation of the polar decomposition of A and corresponds to (A. Khvedelidze and H.-P. P. 1999)

$$\chi_i(A) = \epsilon_{ijk}A_{jk} = 0 \quad (\text{"symmetric gauge"}).$$

Preserving the CCR $\to$ old canonical momenta in terms of the new variables

$$-E_{ai}(q, S, p, P) = O_{ak}\left(q\right)\left[P_{ki} + \epsilon_{kil}{}^{*}D^{-1}_{ls}(S)\left(\Omega^{-1}_{sj}(q)p_j + D_n(S)_{sm}P_{mn}\right)\right]$$

$$\Rightarrow \qquad G_a\Phi \equiv O_{ak}(q)\Omega^{-1}_{ki}(q)p_i\Phi = 0 \quad \Leftrightarrow \quad \frac{\delta}{\delta q_i}\Phi = 0 \quad (\text{Abelianisation})$$

$\to$ physical Hamiltonian $H(S, P)$ and physical ang. mom. op. $J_i(S, P)$

$$J_i(S, P) = \int d^3x \left[-2\epsilon_{ijk}S_{mj}P_{mk} + \text{orbital parts}\right]$$

$\to$ S colorless spin 0,2 glueball fields   Reduction: Color $\to$ Spin

The correctly ordered physical quantum Hamiltonian (Christ and Lee 1980) in terms of the physical variables $S_{ik}(\mathbf{x})$ and the can. conj. $P_{ik}(\mathbf{x})$ reads

$$H(S,P) = \frac{1}{2}\mathcal{J}^{-1}\int d^3\mathbf{x}\; P_{ai}\;\mathcal{J}P_{ai} + \frac{1}{2}\int d^3\mathbf{x}\; B^2_{ai}(S)$$
$$-\mathcal{J}^{-1}\int d^3\mathbf{x}\int d^3\mathbf{y}\Big\{\Big(D_i(S)_{ma}P_{im}\Big)_{\mathbf{x}}\;\mathcal{J}\;\langle \mathbf{x}\; a|{}^*D^{-2}(S)|\mathbf{y}\; b\rangle\Big(D_j(S)_{bn}P_{nj}\Big)_{\mathbf{y}}\Big\}$$

with the FP operator

$${}^*D_{kl}(S) \equiv \epsilon_{kmi}D_i(S)_{ml} = \epsilon_{kmi}(\partial_i\delta_{ml} - g\epsilon_{mlk}S_{ki}) \equiv \epsilon_{kli}\partial_i - g\gamma_{kl}(S)$$

Jacobian $\mathcal{J} \equiv \det|{}^*D|$ and chromomag field $B_{ai}(S) \equiv \epsilon_{ijk}\partial_j S_{ak} + \frac{g}{2}\epsilon_{abc}S_{bj}S_{ck}$
The matrix element of a physical operator O is given by

$$\langle\Psi'|O|\Psi\rangle \;\propto \int_{\mathrm{S\ sym.pos.def.}}\prod_{\mathbf{x}}\Big[dS(\mathbf{x})\Big]\mathcal{J}\;\Psi'^*[S]\;O\;\Psi[S]$$

The inverse of the FP operator and hence the physical Hamiltonian can be expanded in the number of spatial derivatives $\equiv$ expansion in $\lambda = g^{-2/3}$
Introduce infinite spatial lattice of granulas $G(\mathbf{n},a)$ at $\mathbf{x} = a\mathbf{n}$ $(\mathbf{n}\in Z^3)$ and averaged

$$S(\mathbf{n}) := \frac{1}{a^3}\int_{G(\mathbf{n},a)} d\mathbf{x}\; S(\mathbf{x}) \quad\rightarrow\quad H = \frac{g^{2/3}}{a}\left[\sum_{\mathbf{n}}\mathcal{H}_0^{QM}(\mathbf{n}) + \lambda\sum_\alpha \mathcal{V}_\alpha^{(\partial)} + \lambda^2\sum_\beta \mathcal{V}_\beta^{(\partial\partial)} + \ldots\right]$$

H.-P. P., Phys. Lett. B **685** (2010) 353-364. "free" H + interact. between granulas

Physical Hamiltonian of $SU(2)$ YM quantum mechanics

$$H_0^{QM} = \frac{g^{2/3}}{a}\sum_{m,n}\Big[-\frac{1}{2}P_{mn}^2 - \frac{i}{2}\left[\gamma_{mn}^{-1}(S) - \delta_{mn}\mathrm{tr}(\gamma^{-1}(S))\right]P_{mn} + \frac{1}{4}\gamma_{mn}^{-2}(S)J_m J_n + \frac{1}{2}\left(B_{mn}^{\mathrm{hom}}(S)\right)^2\Big]$$

commuting with the physical spin operator

$$J_i = 2\epsilon_{ijk}S_{mj}P_{mk}\ , \qquad [H_0^{QM}, J_i] = 0\ ,$$

the homogeneous part of the FP operator $\gamma_{ik}(S) \equiv S_{ik} - \delta_{ik}\mathrm{tr}S$ and of the chromomagn. field $B_{ai}^{\mathrm{hom}}(S) := \frac{g}{2}\epsilon_{ijk}\epsilon_{abc}S_{bj}S_{ck}$.
Matrix elements

$$\langle\Phi_1|\mathcal{O}|\Phi_2\rangle = \int dS\ (\phi_1+\phi_2)(\phi_2+\phi_3)(\phi_3+\phi_1)\Phi_1^*\mathcal{O}\Phi_2\ .$$

with ϕ_1, ϕ_2, ϕ_3 eigenvalues of S,

$$S = R^T\begin{pmatrix}\phi_1 & 0 & 0\\ 0 & \phi_2 & 0\\ 0 & 0 & \phi_3\end{pmatrix}R \quad\rightarrow\quad \gamma(S) = -R^T\begin{pmatrix}\phi_2+\phi_3 & 0 & 0\\ 0 & \phi_1+\phi_3 & 0\\ 0 & 0 & \phi_1+\phi_2\end{pmatrix}R$$

H.-P. P.,Phys.Lett.B **648** (2007) 97-106., incl.ferm. Phys.Lett.B **700** (2011) 265-276.

Intrinsic System of symmetric tensor S

Intrinsic system:

$$S = R^T[\alpha,\beta,\gamma]\begin{pmatrix}\phi_1 & 0 & 0\\ 0 & \phi_2 & 0\\ 0 & 0 & \phi_3\end{pmatrix}R[\alpha,\beta,\gamma]$$

Jacobian: $\sin[\beta]\,(\phi_3-\phi_2)(\phi_3-\phi_1)(\phi_2-\phi_1)$

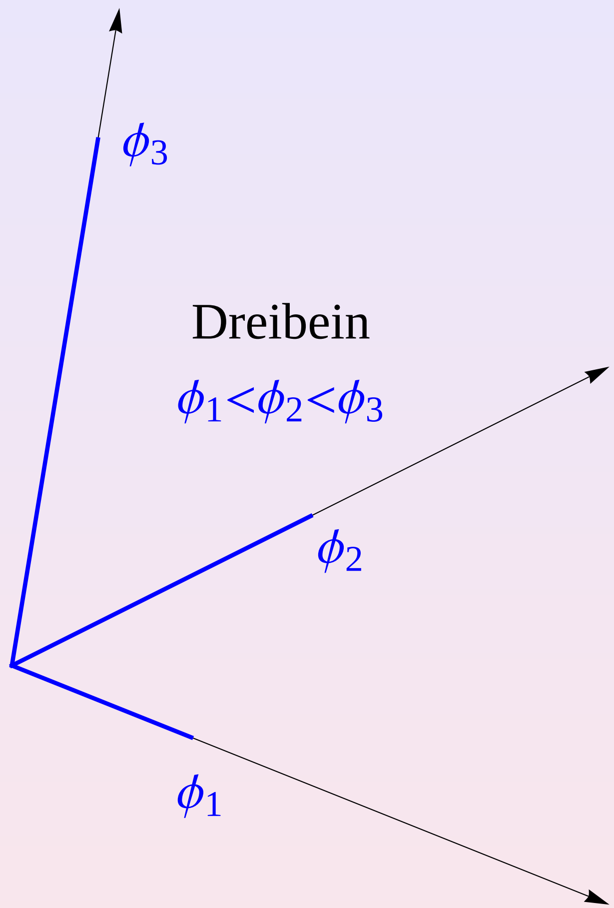
$\rightarrow$ local part of magn.pot.

$$B^2 = g^2[\phi_2^2\phi_3^2+\phi_3^2\phi_1^2+\phi_1^2\phi_2^2]$$

classical zero energy valleys:

$B^2 = 0$: $\quad \phi_1 = \phi_2 = 0, \quad \phi_3$ arbitrary

$\rightarrow$ strings on the quantum level



$$P = R^T[\alpha,\beta,\gamma]\begin{pmatrix}\pi_1 & \mathcal{P}_3 & \mathcal{P}_2\\ \mathcal{P}_3 & \pi_2 & \mathcal{P}_1\\ \mathcal{P}_2 & \mathcal{P}_1 & \pi_3\end{pmatrix}R[\alpha,\beta,\gamma]$$

$$\pi_i := -i\partial/\partial\phi_i \qquad \mathcal{P}_1 := 2^{-1/2}\xi_1/(\phi_2-\phi_3)$$

$$\xi_i := -i\mathcal{M}_{ij}^{-1}\frac{\partial}{\partial\chi_j}\ , \quad \mathcal{M}_{ij} := -\frac{1}{2}\,\epsilon_{jst}\left(R^T\frac{\partial R}{\partial\chi_i}\right)_{st}\ , \quad [\xi_i,\xi_j] = -i\epsilon_{ijk}\xi_k\ .$$

Euler angles $\chi = (\alpha,\beta,\gamma)$: $\mathcal{M}^{-1} = \begin{pmatrix}\sin\gamma & -\cos\gamma/\sin\beta & \cos\gamma\cot\beta\\ \cos\gamma & \sin\gamma/\sin\beta & -\sin\gamma\cot\beta\\ 0 & 0 & 1\end{pmatrix}$.

Put $a \equiv 1 \to g^{2/3}$ dim. of energy. Restore at the end of calculation $g^{2/3} \to g^{2/3}/a$.

$$H_0 = \frac{1}{2}\sum_{ijk}^{\text{cyclic}}\left[-\frac{\partial^2}{\partial\phi_i^2} - \frac{2}{\phi_j^2-\phi_k^2}\left(\phi_j\frac{\partial}{\partial\phi_j} - \phi_k\frac{\partial}{\partial\phi_k}\right) + \xi_i^2\frac{\phi_j^2+\phi_k^2}{(\phi_j^2-\phi_k^2)^2} + g^2\phi_j^2\phi_k^2\right] .$$

and the total spin $\quad J_i = R_{ij}(\chi)\,\xi_j\ , \qquad [J_i, H] = 0 \qquad$ Savvidy, Simonov (1985)

in terms of the intrinsic spin ξ_i satisfying $[J_i, \xi_j] = 0$ and $[\xi_i, \xi_j] = -i\epsilon_{ijk}\xi_k$

$$\langle\Phi_1|\mathcal{O}|\Phi_2\rangle = \int d\alpha\sin\beta d\beta d\gamma\int_{0<\phi_1<\phi_2<\phi_3} d\phi_1 d\phi_2 d\phi_3(\phi_1^2-\phi_2^2)(\phi_2^2-\phi_3^2)(\phi_3^2-\phi_1^2)\Phi_1^*\mathcal{O}\Phi_2 .$$

a) Symmetries of H_0:

- $[J_i, H] = 0 \ \to$ classification of states by Spin $J^2 = \xi^2$ and J_3
- $[\sigma_{ij}, H] = 0 \ \to$ classification in terms of represention of permutation group
- $[T, H] = 0$ and $[P, H] = 0 \ \to$ restrict to real and parity even/odd states

b) Boundary conditions:

- $\Psi|_{\phi_1\downarrow 0} = 0$ or $\partial_{\phi_1}\Psi|_{\phi_1\downarrow 0} = 0$
- wavefn finite for $\phi_1 \to \phi_2$ and $\phi_2 \to \phi_3$.
- $\Psi|_{\phi_3\to\infty} = 0$ sufficiently fast.

c) Virial Theorem: $\langle n|\underline{E}^2|n\rangle = 2\langle n|\underline{B}^2|n\rangle$.

For $\mathbf{J}^2 = \xi^2 = 0$, the Hamiltonian reduces to

$$H_0 = \frac{1}{2} \sum_{\text{cyclic}}^{3} \left[-\frac{\partial^2}{\partial \phi_i^2} - \frac{2}{\phi_i^2 - \phi_j^2} \left(\phi_i \frac{\partial}{\partial \phi_i} - \phi_j \frac{\partial}{\partial \phi_j} \right) + g^2 \phi_j^2 \phi_k^2 \right] \tag{1}$$

$$\langle \Phi_1 | \mathcal{O} | \Phi_2 \rangle = \int_{0<\phi_1<\phi_2<\phi_3} d\phi_1 d\phi_2 d\phi_3 (\phi_1^2 - \phi_2^2)(\phi_2^2 - \phi_3^2)(\phi_3^2 - \phi_1^2) \Phi_1^* \mathcal{O} \Phi_2 \ .$$

Similar as in the Calogero model one can prove that the eigenstates are completely symmetric in the arguments (ϕ_1, ϕ_2, ϕ_3). We therefore can have only the two possible forms, the parity even

$$\Psi^{(+)}(\phi_1, \phi_2, \phi_3) = \widetilde{\Psi}^{(+)}(\phi_1^2, \phi_2^2, \phi_3^2) \ , \tag{2}$$

and the parity odd

$$\Psi^{(-)}(\phi_1, \phi_2, \phi_3) = \phi_1 \phi_2 \phi_3 \widetilde{\Psi}^{(-)}(\phi_1^2, \phi_2^2, \phi_3^2) \ , \tag{3}$$

where the functions $\widetilde{\Psi}^{(\pm)}$ are completely symmetric in the arguments $(\phi_1^2, \phi_2^2, \phi_3^2)$.

Change to the new coordinates (e_1, e_2, e_3), the elementary symmetric combinations

$$e_1 := \phi_1^2 + \phi_2^2 + \phi_3^2 \ , \quad e_2 := \phi_1^2\phi_2^2 + \phi_2^2\phi_3^2 + \phi_3^2\phi_1^2 \ , \quad e_3 := \phi_1^2\phi_2^2\phi_3^2 \ .$$

The corresponding Jacobian

$$\left|\frac{\partial(\phi_1, \phi_2, \phi_3)}{\partial(e_1, e_2, e_3)}\right| = \frac{1}{\sqrt{e_3}\sqrt{\Delta}} \ ,$$

with the square root of the discriminant

$$\Delta \equiv \prod_{i<j}(\phi_i^2 - \phi_j^2)^2 = -27e_3^2 + 18e_1e_2e_3 - 4e_1^3e_3 - 4e_2^3 + e_2^2e_1^2,$$

cancels the original measure $\prod_{i<j}(\phi_i^2 - \phi_j^2)$. Furthermore consider the rescaled

$$s_1 := e_1 \ , \qquad s_2 := 3e_2/e_1^2 \ , \qquad s_3 := 27e_3/e_1^3 \ ,$$

with $|\partial(e_1, e_2, e_3)/\partial(s_1, s_2, s_3)| \propto s_1^5$. Then the spin-0 Schrödinger equ. becomes

$$\left[\frac{1}{6}\, g^2 s_1^2 s_2 - 2s_1 \frac{\partial^2}{\partial s_1^2} - 9\frac{\partial}{\partial s_1} + \frac{1}{2s_1}\left(D^{(0)} - \frac{49}{4}\right)\right]\Psi = E\Psi \ ,$$

with $D^{(0)} := \ D_0^{(0)} + \ D_{-1}^{(0)} + \ D_{-2}^{(0)}$

$$D_0^{(0)} := \left(2\left(2s_2\frac{\partial}{\partial s_2} + 3s_3\frac{\partial}{\partial s_3}\right) + \frac{7}{2}\right)^2 \ ,$$

$$D_{-1}^{(0)} := -4s_3\frac{\partial^2}{\partial s_2^2} - 18\left(2s_3\frac{\partial}{\partial s_3} + 1\right)s_2\frac{\partial}{\partial s_3} \ , \quad D_{-2}^{(0)} := -12\left(s_2\frac{\partial}{\partial s_2} + 4s_3\frac{\partial}{\partial s_3} + 2\right)\frac{\partial}{\partial s_2} \ .$$

Elementary symmetric polynomials (2)

The matrix elements become

$$\langle\Psi'|O|\Psi\rangle \propto \int_0^\infty ds_1\; s_1^{7/2}\left[\int_0^{3/4} ds_2 \int_0^{s_3^{(\mathrm{up})}(s_2)} \frac{ds_3}{\sqrt{s_3}}\;\Psi'^*O\Psi + \int_{3/4}^{1} ds_2 \int_{s_3^{(\mathrm{low})}(s_2)}^{s_3^{(\mathrm{up})}(s_2)} \frac{ds_3}{\sqrt{s_3}}\;\Psi'^*O\Psi\right].$$

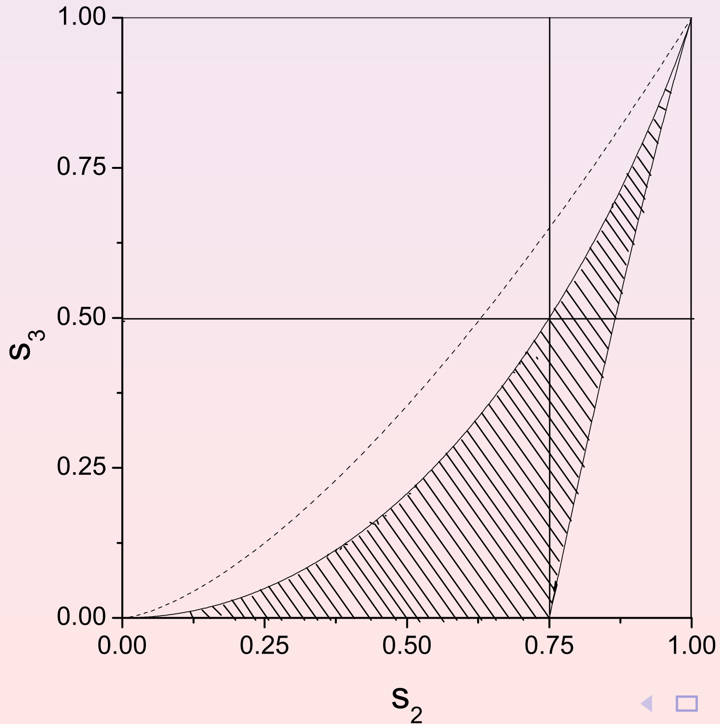
Here the region of integration is given by the (s_2, s_3) satisfying the inequality

$$\tilde{\Delta} \equiv (-s_3^2 + 6s_2s_3 - 4s_3 - 4s_2^3 + 3s_2^2) > 0\;.$$

Denoting the two roots of the equation $\tilde{\Delta}\,(s_3, s_2) = 0$, quadratic in s_3, by

$$s_3^{(\mathrm{up})}(s_2) := \left(1-\sqrt{1-s_2}\right)^2\left(1+2\sqrt{1-s_2}\right) \qquad 0 \le s_2 \le 1\;,$$

$$s_3^{(\mathrm{low})}(s_2) := \left(1+\sqrt{1-s_2}\right)^2\left(1-2\sqrt{1-s_2}\right) \qquad 3/4 \le s_2 \le 1\;.$$



Exact solution of the corresponding harmonic oscillator problem(1)

Consider the corresponding harmonic oscillator potential

$$g^2\left(\phi_1^2\phi_2^2+\phi_2^2\phi_3^2+\phi_3^2\phi_1^2\right)\longrightarrow\omega^2\left(\phi_1^2+\phi_2^2+\phi_3^2\right)\ ,$$

that is $g^2 s_2 s_1^2/6$ replaced by $\omega^2 s_1/2$, the Schrödinger eigenvalue problem

$$\left[\frac{1}{2}\,\omega^2 s_1-2s_1\frac{\partial^2}{\partial s_1^2}-9\frac{\partial}{\partial s_1}+\frac{1}{2s_1}\left(D^{(0)}-\frac{49}{4}\right)\right]\Psi=E\Psi\ ,$$

separates into a density and a deformation problem

$$\Phi_{nk\mu}(s)=R_{nk}(s_1)P_{k\mu}(s_2,s_3)\ .$$

The solutions of the density equations are given by

$$R_{nk}(s_1)=\sqrt{\frac{n!}{\Gamma(n+k+1)}}\,\omega^{\frac{9}{4}}\,(\omega s_1)^{\frac{1}{2}(k-\frac{7}{2})}\,e^{-\omega s_1/2}L_n^k(\omega s_1)\ ,$$

satisfying the orthonormality relations

$$\int_0^\infty ds_1\ s_1^{7/2}\ R_{nk}(s_1)R_{n'k}(s_1)=\delta_{nn'}\ ,$$

with the energy eigenvalues

$$E_{nk}=(2n+1+k)\,\omega\ .$$

Exact solution of the corresponding harmonic oscillator problem (2)

The values of k are determined by the corresponding deformation problem

$$D^{(0)} P_{k\mu}(s_2, s_3) = k^2 \; P_{k\mu}(s_2, s_3) \; .$$

with $D^{(0)} := \; D_0^{(0)} + \; D_{-1}^{(0)} + \; D_{-2}^{(0)}$

$$D_0^{(0)} := \left(2\left(2s_2 \frac{\partial}{\partial s_2} + 3s_3 \frac{\partial}{\partial s_3}\right) + \frac{7}{2}\right)^2 \; ,$$

$$D_{-1}^{(0)} := -4s_3 \frac{\partial^2}{\partial s_2^2} - 18\left(2s_3 \frac{\partial}{\partial s_3} + 1\right) s_2 \frac{\partial}{\partial s_3} \; , \quad D_{-2}^{(0)} := -12\left(s_2 \frac{\partial}{\partial s_2} + 4s_3 \frac{\partial}{\partial s_3} + 2\right) \frac{\partial}{\partial s_2} \; .$$

In the space of monomials $s_2^p \; s_3^q$, $\quad p, q \in N_0$ ordered by increasing $2p + 3q$, the operator $D^{(0)}$ has tridiagonal form with the eigenvalues

$$k = 2(2p + 3q) + \frac{7}{2} \; .$$

The first 3 eigenstates $P_{(p,q)}$, ordered by increasing eigenvalue k, are then

$$k = \; 7/2 \quad : \quad P_{(0,0)} = 1 \; ,$$

$$k = 15/2 \quad : \quad P_{(1,0)} = \frac{11}{2}\sqrt{\frac{13}{15}}\left(s_2 - \frac{6}{11}\right) \; ,$$

$$k = 19/2 \quad : \quad P_{(0,1)} = \frac{221}{126}\sqrt{\frac{209}{10}}\left(s_3 - \frac{9}{17}s_2 + \frac{36}{221}\right) \; ,$$

and so on.

Exact solution of the corresponding harmonic oscillator problem (3)

Orthonormality of the states Φ leads to the corresponding orthonormality relations required for the $P_{k\mu}$,

$$\int_0^{3/4} ds_2 \int_0^{s_3^{(\mathrm{up})}(s_2)} \frac{ds_3}{\sqrt{s_3}} P_{k'\mu'} P_{k\mu} + \int_{3/4}^{1} ds_2 \int_{s_3^{(\mathrm{low})}(s_2)}^{s_3^{(\mathrm{up})}(s_2)} \frac{ds_3}{\sqrt{s_3}} P_{k'\mu'} P_{k\mu} = \frac{12\sqrt{3}}{35} \delta_{k'k}\delta_{\mu'\mu} \, .$$

One can easily check that states with different values of k are orthonormal to each other as it should be. To orthonormalise the states of equal k one can use Gram-Schmidt orthonormalisation, $P_{(p,q)} \to P'_{(p,q)}$.
In summary we obtain the eigenstates

$$\Phi_{npq}(s) = R_{nk}(s_1) P'_{(p,q)}(s_2, s_3) \ ,$$

with the energy eigenvalues

$$E_{n,p,q} = \left(2\,(n + 2p + 3q) + \frac{9}{2} \right) \omega \ ,$$

which are equidistant and depend only on the total number $2(n + 2p + 3q)$ of nodes. The degeneracy is therefore rapidly increasing with energy.

Low energy spectrum from variational calculation

Variational calculation: minimization of the energy functional

$$\mathcal{E}[\Psi] \equiv \frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle} \; .$$

Trial wavefunctions: analytical solutions $\Phi^{(\pm)}_{nlm}$ of corresp. 3-dim harm. osc. problem.
fix frequency ω using the lowest state:

$$\Phi^{(+)}_{000} \;\propto\; \exp\left[-\omega\left(\phi_1^2+\phi_2^2+\phi_3^2\right)\right] \text{ and } \Phi^{(-)}_{000} \;\propto\; \phi_1\phi_2\phi_3 \exp\left[-\omega\left(\phi_1^2+\phi_2^2+\phi_3^2\right)\right]$$

(attention: higher Spin 0 states are not product of 3 Hermite polynomials!)

$$\mathcal{E}[\Phi^{(+)}_{000}] = \langle\Phi^{(+)}_{000}|T + \frac{1}{2}g^2\left(\phi_1^2\phi_2^2+\phi_2^2\phi_3^2+\phi_1^2\phi_3^2\right)|\Phi^{(+)}_{000}\rangle = \frac{9}{4}\omega + \frac{9}{4}\frac{g^2}{\omega^2}$$

$$\rightarrow\; \omega^{(+)} = 2^{3/2}\; g^{2/3} \simeq 1.26\; g^{2/3}$$

$$E_0^{(+)} \leq \mathcal{E}[\Phi^{(+)}_{000}] \simeq 4.25\; g^{2/3}$$

include higher and higher harm. osc. solutions and diagonalize H in truncated space $\rightarrow$ rapid convergence. Including all $174(1041)$ trial states up to $30(60)$ nodes, For the lowest state S_0,

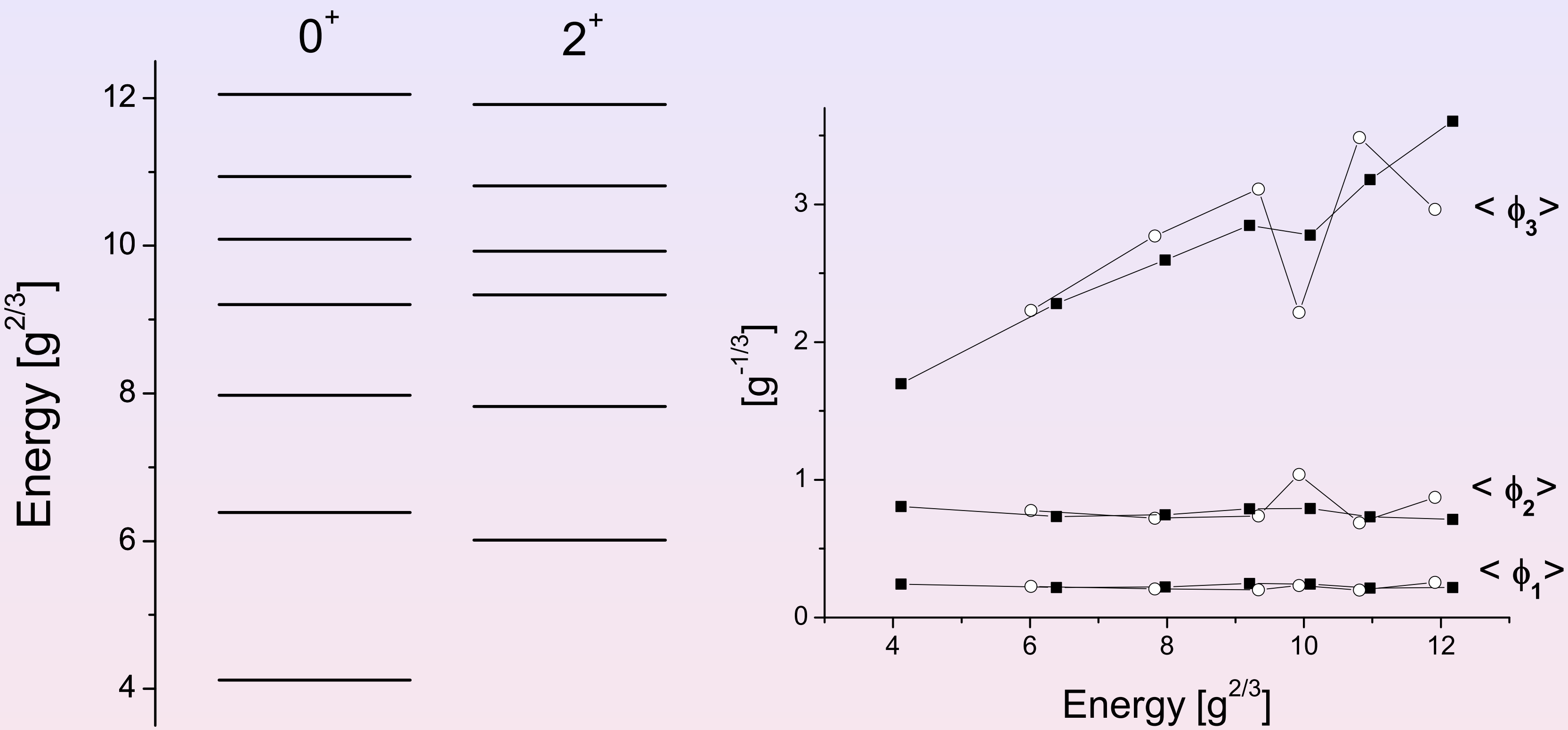
$$E[S_0] = 4.116719740(35)\; g^{2/3} \; ,$$

given explicitly as (up to coefficients < 0.01)

$$S_0 = 0.98\;\Phi_{000} - 0.07\;\Phi_{100} - 0.04\;\Phi_{200} - 0.16\;\Phi_{010} + 0.02\;\Phi_{001} + 0.03\;\Phi_{020}\; ,$$

nearly coincides with the state Ψ_{000},

Energy and $\langle\phi_i\rangle$ expectation values for 0^+ and 2^+



The energy of the first 2^+ state is lower than the first excited 0^+ state!. $\langle\phi_3\rangle$ is raising with increasing excitation, whereas $\langle\phi_1\rangle$ and $\langle\phi_2\rangle$ are practically constant.

H.-P. P., Phys. Lett. B **648** (2007) 97-106.

Higher spin states

For non-vanishing spin we can write the Hamiltonian

$$H = H_0 + \frac{1}{2}\sum_{i=1}^{3} \xi_i^2 V_i \ , \quad V_i = \frac{\phi_j^2 + \phi_k^2}{(\phi_j^2 - \phi_k^2)^2}, \quad \text{i, j, k} \ \ \text{cyclic}$$

with the spin-0 Hamiltonian H_0. The eigenstates can be classified according to their $\mathbf{J}^2$ and J_3 quantum numbers and are superpositions of simultaneous eigenstates of $\mathbf{J}^2$,J_3 and ξ_3

$$|JM\rangle = \sum_{M'=0,1\pm,..,J\pm} \Psi^{(J)}_{MM'}(\phi_1,\phi_2,\phi_3)|JMM'\rangle \ ,$$

with the combinations (for $M' > 0$)

$$|JMM'\pm\rangle := \frac{1}{\sqrt{2}}\left(|JMM'\rangle \pm |JM\, -M'\rangle\right) \ ,$$

of

$$|JMM'\rangle := i^J \sqrt{\frac{2J+1}{8\pi^2}} D^{(J)}_{MM'}(\alpha,\beta,\gamma) \ .$$

The spin-1 Schrödinger equation decays into three equations, one for each member of the cyclic triplet $(\Psi^{(1)}_1, \Psi^{(1)}_2, \Psi^{(1)}_3) = (\Psi^{(1)}_{M1-}, \Psi^{(1)}_{M1+}, \Psi^{(1)}_{M0})$,

$$\left[H_0 - E + \frac{1}{2}(V_2 + V_3)\right] \Psi^{(1)}_1 = 0 \ , \quad \text{and cycl. perm.}$$

One can easily show that no solutions exist which satisfy the necessary boundary conditions.

The spin-2 Schrödinger equation decays into and into three equations, one for each member of the cyclic triplet $(\Psi_1^{(2)}, \Psi_2^{(2)}, \Psi_3^{(2)}) = (\Psi_{M1+}^{(2)}, \Psi_{M1-}^{(2)}, \Psi_{M2-}^{(2)})$,

$$\left[H_0 - E + 2V_1 + \frac{1}{2}(V_2 + V_3)\right]\Psi_1^{(2)} = 0 \ , \quad \text{and cycl. perm.}$$

for which no solutions exist satisfying the correct b. c., and the coupled system

$$\begin{aligned}\left[H_0 - E + \frac{3}{2}(V_1 + V_2)\right]\Psi_{M0}^{(2)} + \frac{\sqrt{3}}{2}(V_1 - V_2)\Psi_{M2+}^{(2)} &= 0\\ \left[H_0 - E + \frac{1}{2}(V_1 + V_2) + 2V_3\right]\Psi_{M2+}^{(2)} + \frac{\sqrt{3}}{2}(V_1 - V_2)\Psi_{M0}^{(2)} &= 0 \ .\end{aligned}$$

The solution of this spin-2 singlet system can be written in the form

$$|2M\rangle = \Psi_1(s)s_1^{-1}Y_M(\phi_1^2, \phi_2^2, \phi_3^2; \alpha, \beta, \gamma) + \Psi_2(s)s_1^{-2}\widetilde{Y}_M(\phi_1^2, \phi_2^2, \phi_3^2; \alpha, \beta, \gamma) \ ,$$

with the cyclic symmetric functions $\Psi_{1,2}(s_1, s_2, s_3)$ and elementary spin-2 fields

$$Y_M = \sqrt{\frac{2}{3}}\left[\left(\phi_3 - \frac{1}{2}(\phi_1 + \phi_2)\right)|2M0\rangle + \frac{\sqrt{3}}{2}(\phi_1 - \phi_2)|2M2+\rangle\right]$$

and its dual $\widetilde{Y}_M := Y_M\big|_{\phi_1 \to \phi_2\phi_3}$. The matrix element of $O(s)$ can be written as

$$\begin{aligned}\langle 2M'|O|2M\rangle &= \frac{2}{3}\Big[\langle\Psi_1'|(1 - s_2)O|\Psi_1\rangle + \frac{1}{6}\langle\Psi_1'|(s_3 - s_2)O|\Psi_2\rangle\\ &\quad + \frac{1}{6}\langle\Psi_2'|(s_3 - s_2)O|\Psi_1\rangle + \frac{1}{9}\langle\Psi_2'|(s_2^2 - s_3)O|\Psi_2\rangle\Big] \ .\end{aligned}$$

in terms of the corresponding spin-0 matrix elements.

Spin 2

Specifying again to parity even states, the vector $\mathbf{\Psi} = (\Psi_1, \Psi_2)$ satisfies the Schrödinger equation of the same form as for spin-0 only with the scalar $D^{(0)}$ replaced with the new matrix operators $D^{(2)}$ with

$$D_0^{(2)} = \begin{pmatrix} \left(2\left(2s_2\frac{\partial}{\partial s_2} + 3s_3\frac{\partial}{\partial s_3} + 1\right) + \frac{7}{2}\right)^2 & 0 \\ 0 & \left(2\left(2s_2\frac{\partial}{\partial s_2} + 3s_3\frac{\partial}{\partial s_3} + 2\right) + \frac{7}{2}\right)^2 \end{pmatrix},$$

$$D_{-1}^{(2)} = \begin{pmatrix} D_{-1}^{(0)} & 8s_3\frac{\partial}{\partial s_3} + 4 \\ 24\frac{\partial}{\partial s_2} & D_{-1}^{(0)} \end{pmatrix}, \qquad D_{-2}^{(2)} = \begin{pmatrix} D_{-2}^{(0)} & 0 \\ 0 & D_{-2}^{(0)} - 24\frac{\partial}{\partial s_2} \end{pmatrix}.$$

Again the corresponding harmonic oscillator Schrödinger Equation separates

$$\mathbf{\Phi}_{nk\mu} = R_{nk}(s_1)\mathbf{P}_{k\mu}(s_2, s_3)\ .$$

The density equations for R_{nk} are the same as in the spin-0 problem The values of k are determined by the corresponding deformation problem

$$D^{(2)}\mathbf{P}_{k\mu}(s_2, s_3) = k^2\mathbf{P}_{k\mu}(s_2, s_3)\ ,$$

using the basis $(s_2^p s_3^q, 0)$ and $(0, s_2^{\tilde{p}} s_3^{\tilde{q}})$ The first 3 eigenstates $P_{(p,q)}$, ordered by increasing eigenvalue k, are then

$$k = 11/2 \quad : \quad \mathbf{P}_{(0,0)} = \sqrt{\frac{33}{10}}\binom{1}{0}\ ,$$

$$k = 15/2 \quad : \quad \mathbf{P}_{(\tilde{0},\tilde{0})} = \frac{39}{2}\sqrt{\frac{11}{35}}\binom{2/13}{1}\ ,$$

$$k = 19/2 \quad : \quad \mathbf{P}_{(1,0)} = \frac{17}{14}\sqrt{\frac{2717}{30}}\binom{s_2 - 6/17}{12/17}\ \text{, and so on.}$$

Use idea of *minimal embedding* of $su(2)$ in $su(3)$ by Kihlberg + Marnelius (1982)

$$\tau_1 := \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \tau_2 := -\lambda_5 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad \tau_3 := \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tau_4 := \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \tau_5 := \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \tau_6 := \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tau_7 := \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \tau_8 := \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The corresponding non-trivial non-vanishing structure constants $[\frac{\tau_a}{2}, \frac{\tau_b}{2}] = i c_{abc} \frac{\tau_c}{2}$, have at least one index $\in \{1, 2, 3\}$

"symmetric gauge" for SU(3) : $\quad \chi_a(A) = \sum_{b=1}^{8} \sum_{i=1}^{3} c_{abi} A_{bi} = 0 \ , \quad a = 1, ..., 8$

H.-P. P., arXiv: 1205.2237 [hep-th] (2012);
H.-P. P., PoS (Confinement X) (2013) 071, arXiv: 1303.3763 [hep-th] (2013);
H.-P. P., EPJ Web of Conferences **71**, 00104 (2014).

Symmetric gauge for SU(3): Unconstrained representation

Carrying out the coordinate transformation (generalized polar decomposition)

$$A_{ak}\left(q_1,..,q_8,\widehat{S}\right) \;=\; O_{a\hat{a}}\left(q\right)\widehat{S}_{\hat{a}k} - \frac{1}{2g}c_{abc}\left(O\left(q\right)\partial_k O^T(q)\right)_{bc},$$

$$\widehat{S}_{\hat{a}k} \equiv \begin{pmatrix} S_{ik} \\ \overline{S_{Ak}} \end{pmatrix} = \left(\begin{array}{ccc} \multicolumn{3}{c}{S_{ik}\ \text{pos. def.}} \\ \hline W_0 & X_3 - W_3 & X_2 + W_2 \\ X_3 + W_3 & W_0 & X_1 - W_1 \\ X_2 - W_2 & X_1 + W_1 & W_0 \\ -\frac{\sqrt{3}}{2}Y_1 - \frac{1}{2}W_1 & \frac{\sqrt{3}}{2}Y_2 - \frac{1}{2}W_2 & W_3 \\ -\frac{\sqrt{3}}{2}W_1 - \frac{1}{2}Y_1 & \frac{\sqrt{3}}{2}W_2 - \frac{1}{2}Y_2 & Y_3 \end{array}\right), \quad c_{\hat{a}\hat{b}k}\widehat{S}_{\hat{b}k} = 0$$

exists and unique : $\widehat{S}_{\hat{a}i}\widehat{S}_{\hat{a}j} = A_{ai}A_{aj}$ (6) $\qquad d_{\hat{a}\hat{b}\hat{c}}\widehat{S}_{\hat{a}i}\widehat{S}_{\hat{b}j}\widehat{S}_{\hat{c}k} = d_{abc}A_{ai}A_{bj}A_{ck}$ (10)

reduced gluons (glueballs): Spin 0,1,2,3 $\qquad$ Reduction: Color $\rightarrow$ Spin

$$V_i := \sqrt{\frac{3}{5}}X_i - \sqrt{\frac{2}{5}}Y_i \qquad \overline{W}_i := \sqrt{\frac{2}{5}}X_i + \sqrt{\frac{3}{5}}Y_i \qquad i = 1,2,3$$

$\{V_1, V_2, V_3\}$ spin-1 field $\qquad \{\overline{W}_1, \overline{W}_2, \overline{W}_3, W_1, W_2, W_3, W_0\}$ spin-3 field

Rotate into the intrinsic frame of submatrix S representing the embedded $su(2)$

$$\widehat{S} = \left(\frac{S}{\overline{S}}\right) = \left(\begin{array}{c|c} R(\alpha,\beta,\gamma) & 0 \\ \hline 0 & D^{(2)}(\alpha,\beta,\gamma) \end{array}\right) \cdot \left(\frac{\text{diag}(\phi_1,\phi_2,\phi_3)}{\overline{S}(\{x_i\},\{y_i\},\{w_i\})}\right) \cdot R^T(\alpha,\beta,\gamma)$$

The magnetic potential $V_{\rm magn}$ has the zero-energy valleys ("constant Abelian fields")

$$B^2 = 0 \quad : \quad \phi_3 \text{ and } y_3 \text{ arbitrary} \quad \wedge \quad \text{all others zero}$$

Represent also the spin-1 and spin-3 fields in their "intrinsic frames", transforming to rotational and intrinsic variables:

$\{V_1, V_2, V_3\} \rightarrow \{\alpha^{(1)}, \beta^{(1)}, v\}$ for the spin-1 field with Jacobian v

$\{\overline{W}_1, \overline{W}_2, \overline{W}_3, W_1, W_2, W_3, W_0\} \rightarrow \{\alpha^{(3)}, \beta^{(3)}, \gamma^{(3)}, w'_1, w'_2, w'_3, w'_0\}$ for the spin-3 field with nontrivial Jacobian $w'^2_0(w'^2_1 - w'^2_2)$.

Alltogether we then have 8 angle-variables $\{\alpha,\beta,\gamma,\alpha^{(1)},\beta^{(1)},\alpha^{(3)},\beta^{(3)},\gamma^{(3)}\}$ and 8 intrinsic variables $\{\phi_1,\phi_2,\phi_3,v,w'_1,w'_2,w'_3,w'_0\}$ with
$\widehat{S}^2 = \phi_1^2 + \phi_2^2 + \phi_3^2 + v^2 + w'^2_1 + w'^2_2 + w'^2_3 + w'^2_0$

Faddeev-Popov operator for ymmetric gauge for SU(3)

$$^{*}D_{\hat a\hat b}(\widehat{S}) = c_{\hat a\hat c i}D_i(S)_{\hat c\hat b} = c_{\hat a\hat c i}\left(\delta_{\hat b\hat c}\partial_i - c_{\hat b\hat c\hat d}\widehat{S}_{\hat d i}\right) = -c_{\hat a\hat c i}c_{\hat b\hat c\hat d}\widehat{S}_{\hat d i} + c_{\hat a\hat b i}\partial_i = \gamma_{\hat a\hat b}(\widehat{S}) + c_{\hat a\hat b i}\partial_i$$

Explicit form of the intrinsic $\widetilde{\gamma}$,

$$\left(\begin{array}{ccc|ccc|cc}
\phi_2+\phi_3 & 0 & 0 & & & & & \\
0 & \phi_3+\phi_1 & 0 & \multicolumn{5}{c}{-2\overline{S}^T(-\frac{3}{2}v, w)} \\
0 & 0 & \phi_1+\phi_2 & & & & & \\
\hline
 & & & 4\phi_1+\phi_2+\phi_3 & 0 & 0 & 0 & 0 \\
 & & & 0 & \phi_1+4\phi_2+\phi_3 & 0 & 0 & 0 \\
\multicolumn{3}{c|}{-2\overline{S}\,(-\frac{3}{2}v, w)} & 0 & 0 & \phi_1+\phi_2+4\phi_3 & 0 & 0 \\
\cline{4-8}
 & & & 0 & 0 & 0 & \phi_1+\phi_2+4\phi_3 & -\sqrt{3}(\phi_1-\phi_2) \\
 & & & 0 & 0 & 0 & -\sqrt{3}(\phi_1-\phi_2) & 3(\phi_1+\phi_2)
\end{array}\right)$$

In contrast to the $SU(2)$ case, transition to the intrinsic system does not completely diagonalize γ.

Symmetric gauge for SU(3): 1 spatial dimension

In one spatial dimension the symmetric gauge for SU(3) reduces to ($\sim$ t'Hooft gauge)

$$A^{(1d)} = \begin{pmatrix} 0 & 0 & A_{13} \\ 0 & 0 & A_{23} \\ 0 & 0 & A_{33} \\ 0 & 0 & A_{43} \\ 0 & 0 & A_{53} \\ 0 & 0 & A_{63} \\ 0 & 0 & A_{73} \\ 0 & 0 & A_{83} \end{pmatrix} \rightarrow S^{(1d)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \phi_3 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_3 \end{pmatrix} \quad (\sim A_3^a \lambda_a = U(\phi_3 \lambda_3 + y_3 \lambda_8) U^+)$$

which consistently reduces the above equs. for S for given A_3 to

$$\phi_3^2 + y_3^2 = A_{a3} A_{a3} \quad \wedge \quad \phi_3^2\, y_3 - 3\, y_3^3 = d_{abc} A_{a3} A_{b3} A_{c3}$$

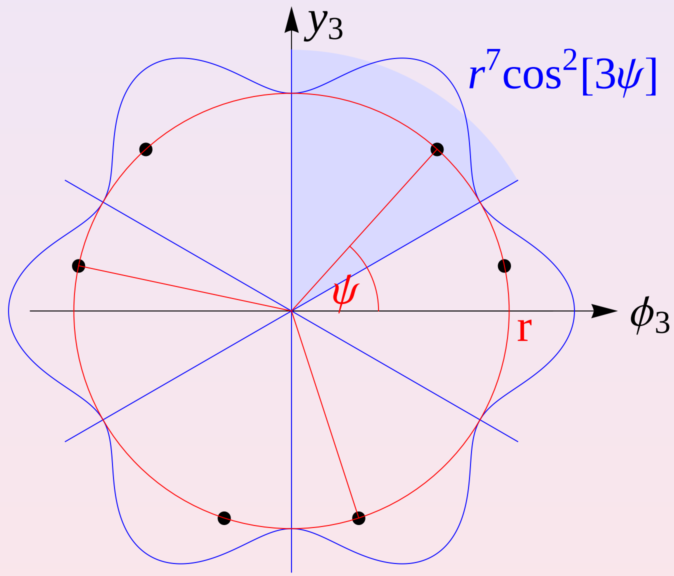
$$\phi_3 = \mathrm{r}\cos[\psi] \qquad y_3 = \mathrm{r}\sin[\psi]$$

$$r^2 = A_3^a A_3^a$$

$$r^3 \sin[3\psi] = \sqrt{3}\, d_{\mathrm{abc}} A_3^a A_3^b A_3^c$$

$$\text{FP-det} \rightarrow r^7 \cos^2[3\psi]$$

$\rightarrow$ 6 Weyl-chambers



with 6 sol. separated by 0-lines of the FP-det $\phi_3^2\,(\phi_3^2 - 3y_3^2)^2$ ("Gribov-horizons").
Exactly 1 sol. exists in the "fundamental domain" (blue area) and we can replace

$$\int_{-\infty}^{+\infty} \prod_{a=1}^{8} dA_{a3} \quad \rightarrow \quad \int_0^\infty d\phi_3 \int_{\phi_3/\sqrt{3}}^\infty dy_3\; \phi_3^2\,(\phi_3^2 - 3y_3^2)^2 \propto \int_0^\infty r^7 dr \int_{\pi/6}^{\pi/2} d\psi \cos^2(3\psi)$$

Symmetric gauge for SU(3): 2 spatial dimensions

For two spatial dimensions, one can show that (putting $W_1 \equiv X_1, W_2 \equiv -X_2$)

$$A^{(2d)} = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & 0 \\ A_{41} & A_{42} & 0 \\ A_{51} & A_{52} & 0 \\ A_{61} & A_{62} & 0 \\ A_{71} & A_{72} & 0 \\ A_{81} & A_{82} & 0 \end{pmatrix} \to \widehat{S}^{(2d)}_{\rm intr} = \left(\begin{array}{ccc} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & 0 \\ \hline 0 & x_3 & 0 \\ x_3 & 0 & 0 \\ 2x_2 & 2x_1 & 0 \\ -\frac{\sqrt{3}}{2}y_1 - \frac{1}{2}x_1 & \frac{\sqrt{3}}{2}y_2 + \frac{1}{2}x_2 & 0 \\ -\frac{\sqrt{3}}{2}y_1 + \frac{1}{2}x_1 & -\frac{\sqrt{3}}{2}y_2 + \frac{1}{2}x_2 & 0 \end{array}\right)$$

consistently reduces the above equs. for S to a system of 7 equs. $(i,j,k = 1,2)$

$$\widehat{S}_{\hat{a}i}\widehat{S}_{\hat{a}j} = A_{ai}A_{aj} \;\; (3) \qquad d_{\hat{a}\hat{b}\hat{c}}\widehat{S}_{\hat{a}i}\widehat{S}_{\hat{b}j}\widehat{S}_{\hat{c}k} = d_{abc}A_{ai}A_{bj}A_{ck} \;\; (4)$$

for 8 physical fields (incl. rot.-angle γ), which, adding as an 8th equ.
$(d_{\hat{a}\hat{b}\hat{c}}\widehat{S}_{\hat{b}1}\widehat{S}_{\hat{c}2})^2 = (d_{abc}A_{b1}A_{c2})^2$, can be solved numerically for randomly gen. $A^{(2d)}$, again yielding solutions separated by horizons. Restricting to a fundamental domain

$$\int_{-\infty}^{+\infty} \prod_{a,b=1}^{8} dA_{a1}dA_{b2} \quad \to \quad \int_0^{2\pi} d\gamma \int_0^{\infty} r^{15} dr \int_{0<\hat{\phi}_1<\hat{\phi}_2<1} d\hat{\phi}_1 d\hat{\phi}_2 (\hat{\phi}_2 - \hat{\phi}_1) \int_{\rm fund.\ domain} d\alpha_1\, d\alpha_2\, d\alpha_3\, d\alpha_4\, \mathcal{J}$$

in terms of the compact variables $\hat{\phi}_i \equiv \phi_i / r$ (i=1,2) and α_k (k=1,..4). Due to the difficulty of the FP-determinant $\mathcal{J}$, I have, however, not yet found a satisfactory description of the fundamental domain.

Table: The spin-0, spin-1,spin-2, and spin-3 contents $\phi^{(0)}$, $v^{(1)}$, $\phi^{(2)}$, and $w^{(3)}$, as well as the values of the Jacobian of the 2-dim solutions

$\phi^{(0)2}$	$\phi^{(2)2}$	x_3^2	$v^{(1)2}$	$w^{(3)2}$	$\phi^{(0)2}+\phi^{(2)2}$	$x_3^2+v^{(1)2}+w^{(3)2}$	$\mathcal{J}$
0.77	0.08	0.10	0.04	0.01	0.85	0.15	1.70
0.33	0.22	0.33	0.04	0.08	0.56	0.44	0.14
0.31	0.01	0.18	0.37	0.13	0.31	0.69	0.20
0.55	0.16	0.06	0.15	0.28	0.71	0.29	- 0.61
0.43	0.01	0.03	0.35	0.18	0.44	0.56	- 0.17
0.17	0.06	0.12	0.61	0.04	0.23	0.77	- 0.33

For 3 dimensions, I have found several solutions of the 16 S-equations $(i,j,k=1,2,3)$

$$\widehat{S}_{\hat{a}i}\widehat{S}_{\hat{a}j} = A_{ai}A_{aj} \ (6) \qquad d_{\hat{a}\hat{b}\hat{c}}\widehat{S}_{\hat{a}i}\widehat{S}_{\hat{b}j}\widehat{S}_{\hat{c}k} = d_{abc}A_{ai}A_{bj}A_{ck} \ (10)$$

for the 16 physical fields numerically for a randomly generated A (see Table).

But to write the corresponding unconstrained integral over a fundamental domain

$$\int_{-\infty}^{+\infty} \prod_{a,b,c=1}^{8} dA_{a1}dA_{b2}dA_{c3} \quad \to \quad \int d\alpha \sin\beta \, d\beta \, d\gamma \int_0^\infty r^{23} dr$$

$$\int_{0<\hat{\phi}_1<\hat{\phi}_2<\hat{\phi}_3<1} d\hat{\phi}_1 d\hat{\phi}_2 d\hat{\phi}_3 \prod_{i<j} (\hat{\phi}_i - \hat{\phi}_j) \int_{\text{fund. domain}} d\alpha_1 ... d\alpha_9 \ \mathcal{J}$$

in terms of the compact variables $\hat{\phi}_i \equiv \phi_i / r$ and α_k is a difficult, but I think solvable, future task.

Symmetric gauge for SU(3): 3 spatial dimensions: example

Table: Spin-0, spin-1, spin-2 and spin-3 contents $\phi^{(0)}$,$v^{(1)}$,$\phi^{(2)}$ and $w^{(3)}$ and values of the Jacobian $\mathcal{J}$ for the 3-dim solutions.

$\phi^{(0)2}$	$\phi^{(2)2}$	$v^{(1)2}$	$w^{(3)2}$	$\phi^{(0)2}+\phi^{(2)2}$	$v^{(1)2}+w^{(3)2}$	$\prod_{i<j}(\phi_i-\phi_j)$	$\mathcal{J}\propto$
0.776	0.086	0.014	0.124	0.862	0.138	0.016	2.800
0.682	0.139	0.026	0.153	0.821	0.179	0.034	1.035
0.629	0.164	0.024	0.183	0.792	0.208	0.046	0.543
0.588	0.147	0.019	0.246	0.734	0.266	0.040	0.285
0.600	0.124	0.013	0.263	0.724	0.276	0.030	0.476
0.544	0.138	0.011	0.307	0.682	0.318	0.035	0.223
0.463	0.196	0.020	0.321	0.659	0.341	0.055	0.015
0.558	0.094	0.055	0.292	0.652	0.348	0.011	0.078
0.470	0.105	0.005	0.420	0.575	0.425	0.023	0.041
0.471	0.054	0.031	0.444	0.525	0.475	0.004	0.036
0.535	0.198	0.041	0.225	0.734	0.266	0.062	-0.062
0.469	0.230	0.083	0.217	0.699	0.301	0.074	-0.254
0.494	0.167	0.040	0.299	0.661	0.338	0.046	-0.178
0.394	0.241	0.041	0.324	0.636	0.364	0.083	-0.094
0.455	0.164	0.011	0.370	0.619	0.381	0.047	-0.034
0.468	0.149	0.132	0.251	0.617	0.383	0.039	-0.286
0.462	0.135	0.030	0.373	0.597	0.403	0.033	-0.016
0.455	0.123	0.081	0.341	0.578	0.422	0.015	-0.298
0.348	0.220	0.111	0.321	0.568	0.432	0.060	-0.312
0.429	0.113	0.010	0.448	0.541	0.459	0.023	-0.035
0.395	0.120	0.093	0.392	0.515	0.485	0.029	-0.138

Conclusions

- Calogero-type model as the integrable system closest to $SU(2)$ YM QM allows for the calculation of the spectrum and other expectation values of low energy $SU(2)$ YM quantum theory with high numerical precision in zeroth and higher order in $\lambda = g^{-2/3}$ (Lorentz-inv and renormalisation in the IR)
- Extension to SU(3) YM QM is very difficult due to the complicated structure of the FP operator (appearance of Gribov-copies and horizons). Restriction of the functional integration over the physical fields to the fundamental domain necessary in order to have an invertible FP operator.