Superintegrability of Calogero model with oscillator and Coulomb potentials and of their generalizations to (pseudo)spheres:

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- Oscillator and Coulomb systems (and their free particle limit) and their generalizations to spheres and hyperboloids are most known and most fundamental superintegrable systems. It seems, that all other superintegrable systems can be obtained from these models by appropriate reductions
- Rational Calogero model and rational Calogero model with oscillator potential are also superintegrable systems.
- Is rational Calogero model with Coulomb potential(s)[Khare'96] superintegrable as well?
- Is it possible to generalize these systems to spheres and hyperboloids preserving superintegrability property?

At SIS'13 in Hannover I demonstrated that it is a case

Here I will present some new formulae, explicitly demonstrating superintegrability of these models.

Hidden symmetries in action-angle variables

Let we have Hamiltonian system formulated in action angle variables, with the Hamiltonian

$$H = H(k_1 I_1 + k_2 I_2, I_3, \ldots, I_N)$$

where $k_{1,2}$ are integers.
Then the system possesses hidden symmetry given by the function

$$I = A(I)\cos(k_1\Phi_2 - k_2\Phi_1 + \alpha(I))$$

Maximally superintegrable systems

$$H = H(k_1 I_1 + k_2 I_2 + \ldots + k_N I_N),$$

Additional constants of motion

$$I_{ij} = A_{ij} \cos(k_j \Phi_i - k_i \Phi_j + \alpha_{ij}(I)), \qquad i, j = 1, \ldots, N.$$

Integrable deformations of N-dimensional oscillator and Coulomb models

$$\mathrm{I\!R}^{\mathrm{N}}: \ \mathcal{H} = \frac{\mathrm{p_r^2}}{2} + \frac{\mathcal{I}_{\mathrm{N-1}}(\mathrm{I_i})}{\mathrm{r}^2} + \mathrm{V(r)}, \quad \mathrm{V(r)} = \begin{cases} \omega^2 r^2/2 \\ -\gamma/r \end{cases}$$

$$S^N: \ \mathcal{H} = \frac{p_\chi^2}{2r_0^2} + \frac{\mathcal{I}_{N-1}}{r_0^2 \sin^2\chi} + V(\tan\chi), \quad V(r) = \begin{cases} r_0^2\omega^2 \tan^2\chi/2 \\ -(\gamma\cot\chi)/r_0 \end{cases}$$

$$H^N: \quad \mathcal{H} = \frac{p_\chi^2}{2r_0^2} + \frac{\mathcal{I}_{N-1}}{r_0^2 \sinh^2\chi} + V(\tanh\chi), \quad V(r) = \begin{cases} r_0^2\omega^2 \tanh^2\chi/2 \\ -(\gamma\coth\chi)/r_0 \end{cases}$$

"Classical spectra" of deformed oscillator and Coulomb systems

Oscillator

$$\mathcal{H}_{osc} = \begin{cases} \omega(2I_r + \sqrt{2\mathcal{I}}) & \text{for} \quad \mathbb{R}^N \\ \frac{1}{2}(2I_\chi + \sqrt{2\mathcal{I}} + \omega)^2 - \frac{\omega^2}{2} & \text{for} \quad S^N \\ -\frac{1}{2}(2I_\chi + \sqrt{2\mathcal{I}} - \omega)^2 + \frac{\omega^2}{2} & \text{for} \quad H^N \end{cases}$$

Coulomb

$$\mathcal{H}_C = \begin{cases} -\gamma^2/2(I_r + \sqrt{2\mathcal{I}})^2 & \text{for} \quad \mathbb{R}^N \\ -\gamma^2/2(I_\chi + \sqrt{2\mathcal{I}})^2 + (I_\chi + \sqrt{2\mathcal{I}})^2/2 & \text{for} \quad S^N \\ -\gamma^2/2\left(I_\chi + \sqrt{2\mathcal{I}}\right)^2 - (I_\chi + \sqrt{2\mathcal{I}})^2/2 & \text{for} \quad H^N \end{cases}$$

Superintegrability of deformed o oscillator and Coulomb systems requires:

$$\mathcal{I}_{Sph} = \frac{1}{2}(\sum_i k_i I_i + \mathrm{const})^2, \qquad k_i \in \mathcal{N}$$

Spherical part of rational Calogero model associated with Coxeter root system belongs to this class of Hamiltonians

Spherical part of Calogero model(s)

- The study of spherical part of Calogero model at classical level has been done in our papers with T.Hakobyan, S.Krivonos, O.Lechtenfeld, A.Saghatelian, V.Yeghikyan.
- Spherical parts of rational Calogero models associated with root systems are maximally superintegrable systems.
- Quantum mechanics of the spherical part of rational Calogero model was studied by M.Feigin, O.Lechtenfeld, A.Polychronakos (2013).

Quantum and classical spectra

Quantum

$$E(n_r, l) \qquad \to \qquad E(n_r, \mathcal{E}_{n_1,\dots,n_{N-1}}),$$

where $\mathcal{E}_{n_1,\dots,n_{N-1}}$ is spectrum os spherical part.

"Classical"

$$\mathcal{H}(I_r, I_1 + I_2 + \dots I_{N-1}) \qquad \to \qquad \mathcal{H}(I_r, \mathcal{I}(I_1, \dots, I_{N-1})),$$

where $\mathcal{I}(I_1, \dots, I_{N-1})$ is spherical part written in action variables.

"Spherical part of conventional CM: Quantum spectrum "

$$\mathcal{E}_{sphCal} = \frac{1}{2} q \left(q + \hbar(N-2) \right),$$

where

$$q \equiv \frac{N(N-1)}{2} g + \hbar k_1 + 3\hbar k_3 + \ldots + N\hbar k_N,$$

For the arbitrary root system

$$q = \sum_{\alpha \in \mathcal{R}_+} g_\alpha + \hbar \sum_{i=2}^{N} d_i k_i, \qquad \text{and} \qquad k_i = 0, 1, 2, \ldots \ .$$

Here, $d_1{=}2, d_2, \ldots, d_N$ are the degrees of the basic homogeneous W-invariant polynomials $\sigma_1{=}r^2, \sigma_2, \ldots, \sigma_N$.

"Spherical part of conventional CM: Classical spectrum"

$$\mathcal{I} = \frac{1}{2}\left(\sum_{i=2}^{N} d_i I_i + \sum_{\alpha\in\mathcal{R}_+} g_\alpha\right)^2 .$$

In particular, the angular part of rational A_{N-1} Calogero model is given by

$$\mathcal{I} = \frac{1}{2}\Big(\frac{N(N-1)}{2}g + I_1 + 3I_3 + \ldots + NI_N\Big)^2 ,$$

Calogero-Coulomb and Calogero oscillator models on R^n

$$\mathcal{H}_{CC} = \frac{\mathbf{p}^2}{2} + \sum_i \frac{g^2}{(x_i - x_j)^2} + \frac{\omega^2 x^2}{2}$$

Well-known system.

$$\mathcal{H}_{CC} = \frac{\mathbf{p}^2}{2} + \sum_i \frac{g^2}{(x_i - x_j)^2} - \frac{\gamma}{|\mathbf{x}|}$$

Less known system suggested by A. Khare (1996).

MAXIMAL SUPERINTEGRABILITY !!!

Generalization to spheres and hyperboloids

$$V^{(p)s}_{Coulomb} = \sum_{\alpha\in\Delta_+} \frac{g^2_\alpha(\alpha\cdot\alpha)r_0^2}{2(\alpha\cdot\mathbf{x})^2} - \frac{\gamma}{r_0}\frac{x_0}{|\mathbf{x}|}, \qquad x_0^2 \pm \mathbf{x}^2 = r_0^2,$$

$$V^{(p)s}_{osc} = \sum_{\alpha\in\Delta_+} \frac{g^2_\alpha(\alpha\cdot\alpha)r_0^2}{2(\alpha\cdot\mathbf{x})^2} + \frac{\omega^2 r_0^2}{2}\frac{\mathbf{x}^2}{x_0^2}, \qquad x_0^2 \pm \mathbf{x}^2 = r_0^2.$$

The upper sign corresponds to the sphere, and lower sign – to the hyperboloid.

MAXIMAL SUPERINTEGRABILITY !!!

Superintegrability: Matrix model approach

$$\mathcal{H}_{\mathrm{mat}} = \frac{1}{2}\mathrm{Tr}\mathbf{P}^2 - \gamma\left(\mathrm{Tr}\mathbf{X}^2\right)^{-\frac{1}{2}} = \frac{1}{2}\sum_a P_a^2 - \frac{\gamma}{r}, \qquad r^2 = \sum_a X_a^2,$$

Constants of motion

$$\mathbf{L} = \mathbf{L}_{12} = \mathbf{X}\wedge\mathbf{P}, \qquad \mathbf{A} == \mathrm{Tr}_2\mathbf{L}_{12}\mathbf{P}_2 - \frac{\gamma}{r}\mathbf{X}_1$$

Reduction to CM needs to select the $SU(N)$ invariant quantities

$$L_a = \mathrm{Tr}\mathbf{L}^{2a}, \qquad \mathcal{A}_a = \mathrm{Tr}\,\mathbf{A}^a$$

Explicitly

$$\mathbf{A} = \left(2\mathcal{H}_{\mathrm{cou}} + \frac{\gamma}{r}\right)\mathbf{X} - rp_r\mathbf{P},$$

Superintegrability: Dunkl operators

- Spherical parts of CM are $(N-1)$ dimensional superintegrable systems, with $2N-3$ functionally independent constants of motion defined by

$$L_a = \sum_{i,j:i<j} L_{ij}^{2a}, \qquad L_{ij} \equiv x_i D_j - x_j D_i$$

with D_i be Dunkl operators, and $a = 1, \ldots N$.

- One more constant of motion is given by Hamiltonian $\mathcal{H}_{CalCoul}$
- The rest constant of motion is given by the trace of "Runge-Lenz vector"

$$A_1 = \sum_i A_i = \sum_i \left(\sum_j \{L_{ij}, D_j\} - \gamma \frac{x_i}{r} \right)$$

We found whole set, $2N-1$, constants of motion

Thank you for your attention!