A Dynamical Formulation of Scattering Theory, Inverse Scattering, and Unidirectional Invisibility

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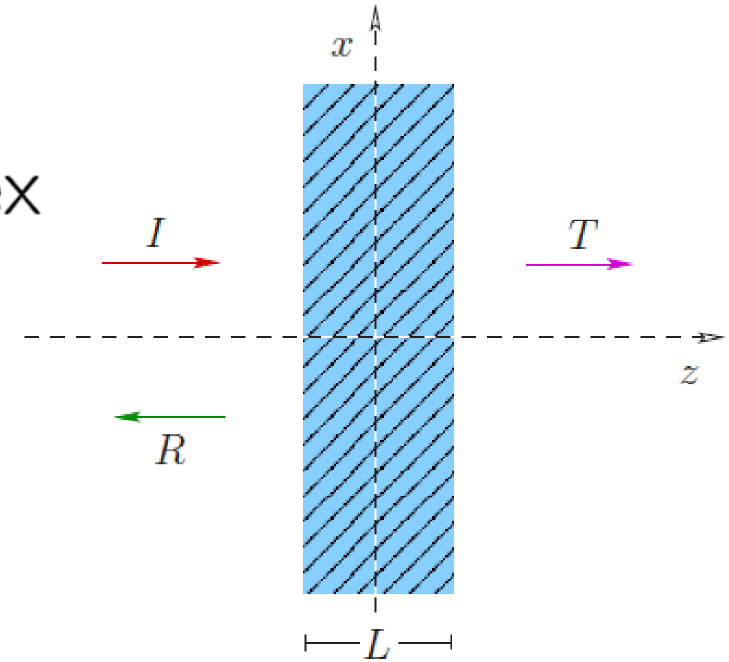
Outline:

- **Introduction:** Transfer Matrix, Spectral Singularities, Unidirectional Invisibility
- **Transfer Matrix as a Non-Unitary S-Matrix**
- **Dynamical Equation for Transfer Matrix**
- **Adiabatic Approximations, Semiclassical Scattering & Geometric Phases**
- **Local Inverse Scattering**

EM Waves in Media with Planar Symmetry:

$$\mathfrak{n}^2(z)\partial_t^2\vec{E} - c^2\partial_z^2\vec{E} = 0$$

$\mathfrak{n} := \eta + i\kappa$: Refractive index



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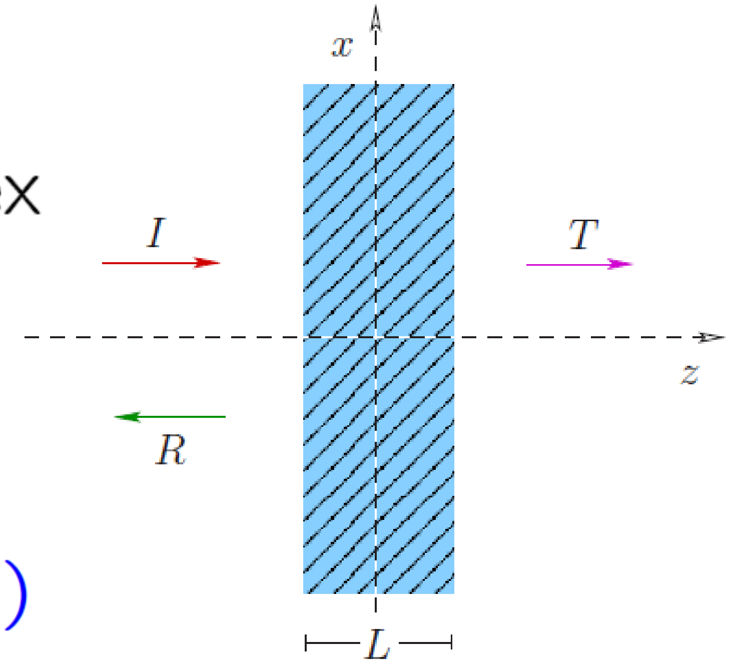

$$\vec{E}(z,t) = E\, e^{-i\omega t}\,\psi(z)\,\hat{\imath}$$

$$-\psi''(z) + v(z)\psi(z) = k^2\psi(z)$$

$$v(z) := k^2[1 - \mathfrak{n}^2(z)]$$

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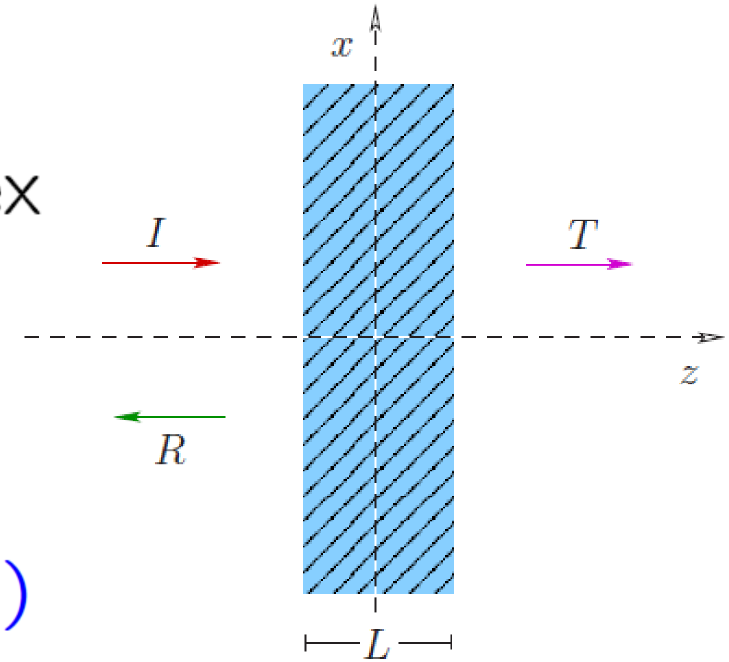

$$\vec{E}(z,t) = E\, e^{-i\omega t}\,\psi(z)\,\hat{\imath}$$

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For $|z| > \frac{L}{2}$, $\mathfrak{n}(z) = 1 \;\Rightarrow\; v(z) = 0$

$v(z)$ is a complex scattering potential.

One-Dimensional Scattering Theory

- Time-Indep. Schrödinger Eq.: $-\psi(x)'' + v(x)\psi(x) = k^2\psi(x)$
- $v : \mathbb{R} \to \mathbb{C}$ is a possibly k-dependent potential &

$$v(x) \to 0 \quad \text{for} \quad x \to \pm\infty.$$

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- Asymptotic solutions:

$$\psi(x) = A_\pm e^{ikx} + B_\pm e^{-ikx} \quad \text{for} \quad x \to \pm\infty.$$

- Transfer matrix: $\begin{bmatrix} A_+ \\ B_+ \end{bmatrix} = \begin{bmatrix} M_{11}(k) & M_{12}(k) \\ M_{21}(k) & M_{22}(k) \end{bmatrix} \begin{bmatrix} A_- \\ B_- \end{bmatrix}.$

- $\det \mathbf{M} = 1.$

- Scattering from the left and right:

$$\psi^{\text{left}}(x) = \begin{cases} e^{ikx} + R^l e^{-ikx} & \text{for} \quad x \to -\infty \\ T^l e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

$$\psi^{\text{right}}(x) = \begin{cases} T^r e^{-ikx} & \text{for} \quad x \to -\infty \\ e^{-ikx} + R^r e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

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$$T^l = T^r =: T = \frac{1}{M_{22}}$$

$$R^l = -\frac{M_{21}}{M_{22}}, \qquad R^r = \frac{M_{12}}{M_{22}}$$

- Bound states & resonances are zeros of $M_{22}(k)$.

Spectral Singularities & Unidirectional Invisibility

$$R^l = -\frac{M_{21}}{M_{22}}, \qquad R^r = \frac{M_{12}}{M_{22}}, \qquad T = \frac{1}{M_{22}}$$

- Spectral singularities are real zeros of $M_{22}(k)$.

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$\Rightarrow$ infinite amplification of incident waves.

- Physically they correspond to scattering states that behave like resonances: Zero-width resonances.

A. M., PRL **102**, 220402 (2009) & PRA **83**, 045801 (2011)
A. M., PRL **110**, 260402 (2013) & PRA **87**, 063838 (2013)

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Optical Spectral Singularity $\Rightarrow$ **Lasing at threshold gain**

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Time-reversed SS: $M_{11}(k) = 0$ for a real k (antilasing)

Chong et al, PRL **105**, 053901 (2010)
Wan et al, Science **331**, 889 (2011)

Spectral Singularities & Unidirectional Invisibility

$$\mathbf{M} = \begin{bmatrix} T - \dfrac{R^l R^r}{T} & \dfrac{R^r}{T} \\ -\dfrac{R^l}{T} & \dfrac{1}{T} \end{bmatrix}$$

Unidir. Reflectionlessness: $R^l = 0 \neq R^r$ or $R^r = 0 \neq R^l$

Only one of M_{12} and M_{21} is zero.

Unidir. Invisibility: $R^l = 0 \neq R^r$ or $R^r = 0 \neq R^l$ & $T = 1$

Only one of M_{12} and M_{21} is zero & $M_{11} = M_{22} = 1$.

Lin et al, PRL **106**, 213901 (2011)
Regensburger et al, Nature **488**, 167 (2012)

If $v(x)$ is a real potential,

$$|R^r| = |R^l|, \qquad |R^{l/r}|^2 + |T|^2 = 1$$

$\Rightarrow$ Spectral singularities and unidirectional reflectionlessness & invisibility cannot happen for a real potential.

Composition Property of M

Let v_1 and v_2 be scattering potentials such that

$$v_1(x) = 0 \quad \text{for} \quad x > a,$$

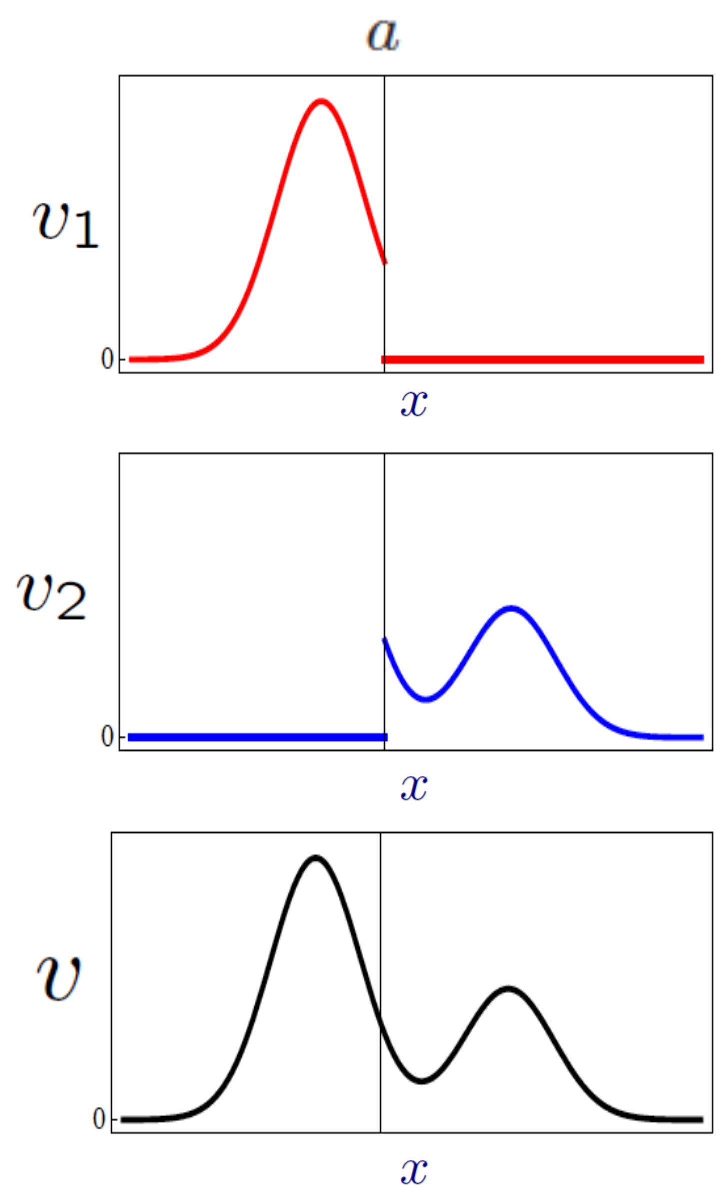
$$v_2(x) = 0 \quad \text{for} \quad x < a$$

$$v(x) = v_1(x) + v_2(x).$$

M_1: Transfer matrix of v_1

M_2: Transfer matrix of v_2

M: Transfer matrix of $v = v_1 + v_2$



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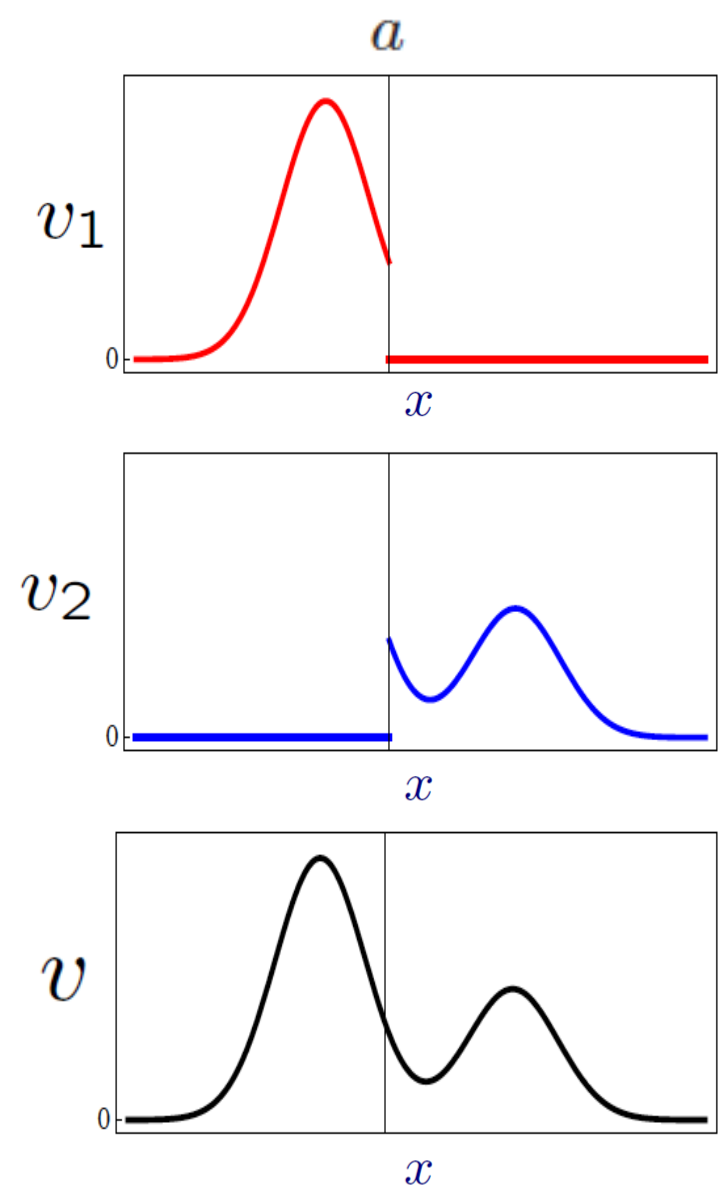
Then $\mathbf{M} = \mathbf{M}_2\,\mathbf{M}_1$.



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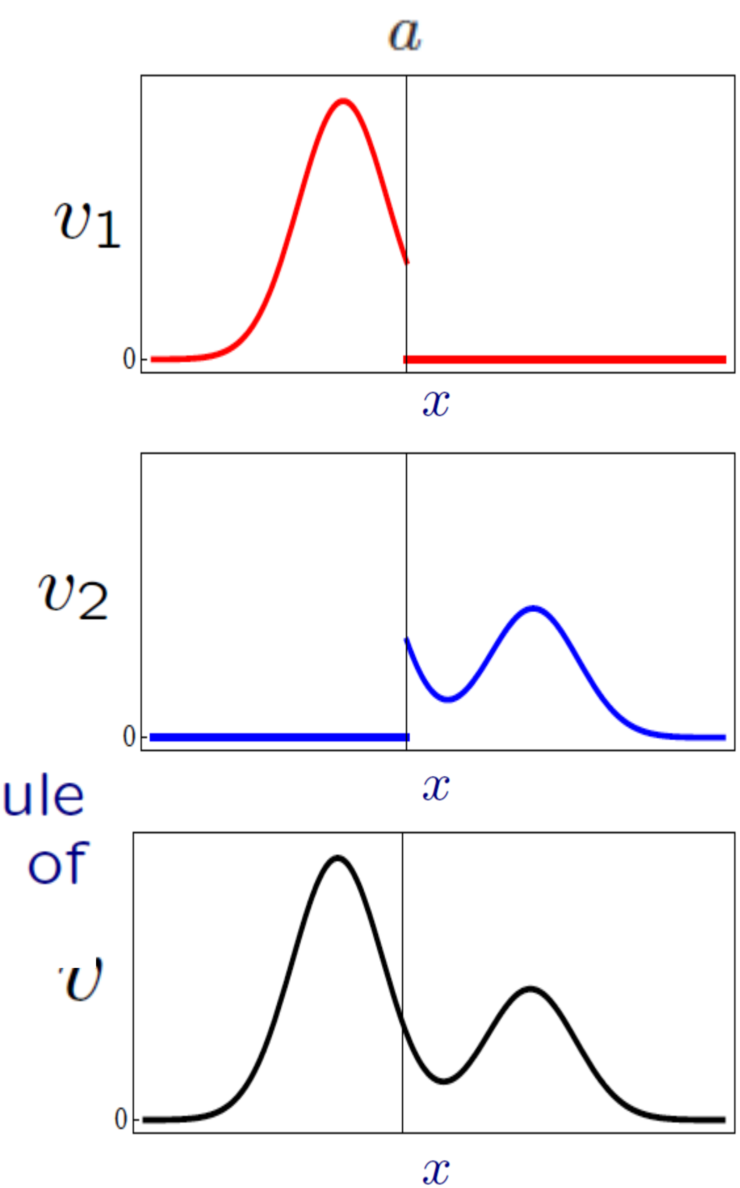
Then $\mathbf{M} = \mathbf{M}_2 \, \mathbf{M}_1$.

This resembles the composition rule for the evolution operator $U(t, t_0)$ of a quantum system:

$$U(t_2, t_0) = U(t_2, t_1) U(t_1, t_0)$$

for $t_0 \leq t_1 \leq t_2$.



Time-indep. Schrödinger Eq.: $-\psi''(x) + v(x)\psi(x) = k^2\psi(x)$

$$i\dot{\Psi}(\tau) = \mathbf{H}(\tau)\Psi(\tau)$$

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$$\tau := kx, \qquad \phi(\tau) := \psi(\tau/k), \qquad \dot{\phi}(\tau) := \frac{d\phi(\tau)}{d\tau}, \qquad w(\tau) := \frac{v(\frac{\tau}{k})}{2k^2}$$

$$\Psi(\tau) \;\; := \;\; \frac{1}{2}\begin{bmatrix} \phi(\tau) - i\dot{\phi}(\tau) \\ \phi(\tau) + i\dot{\phi}(\tau) \end{bmatrix}, \quad \mathbf{N} := \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = i\sigma_2 + \sigma_3$$

$$\mathbf{H}(\tau) \;\; := \;\; \begin{bmatrix} w(\tau) - 1 & w(\tau) \\ -w(\tau) & -w(\tau) + 1 \end{bmatrix} = -\sigma_3 + w(\tau)\mathbf{N}$$

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Time-indep. Schrödinger Eq.: $-\psi''(x) + v(x)\psi(x) = k^2\psi(x)$

$$\mathbf{H}(\tau) \;:=\; \begin{bmatrix} w(\tau)-1 & w(\tau) \\ -w(\tau) & -w(\tau)+1 \end{bmatrix}$$

$$i\dot{\Psi}(\tau) = \mathbf{H}(\tau)\Psi(\tau)$$

- $\mathbf{H}(\tau)$ is a 2-level non-Hermitian Hamiltonian.
- $\mathbf{H}(\tau)$ is σ_3-pseudo-Hermitian, if v is real; $\mathbf{H}(\tau)^\dagger = \sigma_3\mathbf{H}(\tau)\sigma_3^{-1}$.
- Eigenvalues of $\mathbf{H}(\tau) = \pm\mathfrak{n}(\tau)$, $\mathfrak{n} := \sqrt{1-2w} = \sqrt{1-v/k^2}$.
- Classical turning points are exceptional points of $\mathbf{H}(\tau)$.

$$\mathbf{H}(\tau) := -\sigma_3 + w(\tau)\mathbf{N}, \qquad \mathbf{N} := \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

- Evolution operator: $\mathbf{U}(\tau, \tau_0) := \mathscr{T} e^{-i\int_{\tau_0}^{\tau} \mathbf{H}(t)dt}$;

$$i\dot{\mathbf{U}}(\tau, \tau_0) = \mathbf{H}(\tau)\mathbf{U}(\tau, \tau_0), \qquad \mathbf{U}(\tau_0, \tau_0) = 1$$

$$\Psi(\tau) = \mathbf{U}(\tau, \tau_0)\Psi(\tau_0)$$

- Free particle: $\mathbf{H}(\tau) = -\sigma_3$, $\mathbf{U}(\tau, \tau_0) = \mathbf{U}_0(\tau - \tau_0)$.

$$\mathbf{U}_0(\tau) := e^{i\tau\sigma_3}$$

$$\mathbf{H}(\tau) := -\sigma_3 + w(\tau)\mathbf{N}, \qquad \mathbf{N} := \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

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Theorem: The S-matrix of $\mathbf{H}(\tau)$ is the transfer matrix of v;

$$\mathbf{M} = \mathbf{U}_0(+\infty)^{-1}\mathbf{U}(+\infty,-\infty)\mathbf{U}_0(-\infty).$$

A. M. PRA. **89**, 012709 (2014)

- Free particle: $\mathbf{H}(\tau) = -\sigma_3$, $\mathbf{U}(\tau,\tau_0) = \mathbf{U}_0(\tau-\tau_0)$.
- Interaction-picture Hamiltonian: $\Psi(\tau) \to U_0(\tau)^{-1}\Psi(\tau)$

$$\begin{aligned}\mathscr{H}(\tau) &:= \mathbf{U}_0(\tau)^{-1}\mathbf{H}(\tau)\mathbf{U}_0(\tau) - i\mathbf{U}_0(\tau)^{-1}\dot{\mathbf{U}}_0(\tau)\\ &= w(\tau)\left[\begin{array}{cc} 1 & e^{-2i\tau} \\ -e^{2i\tau} & -1 \end{array}\right]\end{aligned}$$

- $\mathscr{H}(\tau)$ is non-diagonalizable matrix.
- Spectrum of $\mathscr{H}(\tau)$ is $\{0\}$.
- $\mathscr{H}(\tau)$ is σ_3-pseudo-normal, i.e., $[\mathscr{H}(\tau), \mathscr{H}(\tau)^\sharp] = 0$, where $\mathscr{H}^\sharp := \sigma_3^{-1}\mathscr{H}^\dagger\sigma_3$.
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Theorem: Let $\mathscr{U}(\tau,\tau_0)$ be the Interaction-picture evolution operator. Then $\mathbf{M} = \mathscr{U}(+\infty,-\infty)$.

Motivation: $\exists$ a dynamical eq. for $\mathscr{U}(\tau, \tau_0)$.

$i\dot{\mathscr{U}}(\tau, \tau_0) = \mathscr{H}(\tau)\mathscr{U}(\tau, \tau_0)$ & $\mathscr{U}(\tau_0, \tau_0) = 1$

Can we find a dynamical eq. for $\mathbf{M}$?

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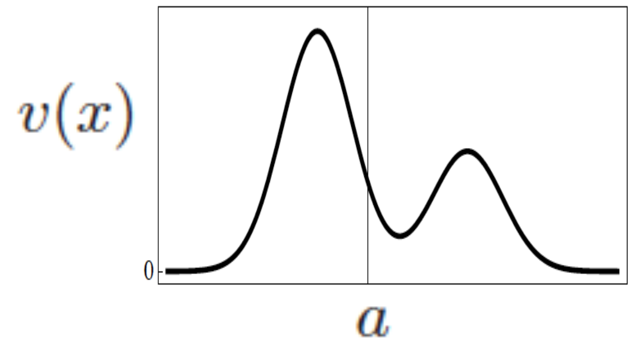
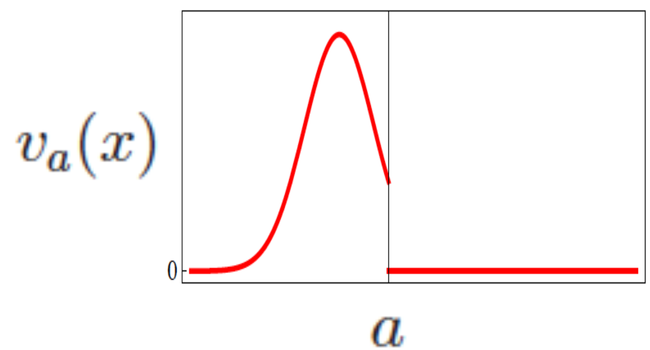


For each $a\in\mathbb{R}$, let $\alpha := ak$,

$$v_a(x) := \begin{cases} v(x) & \text{for} \quad x\le a \\ 0 & \text{for} \quad x>a \end{cases},$$

and $\mathbf{M}(\alpha)$:= transfer matrix of v_a.



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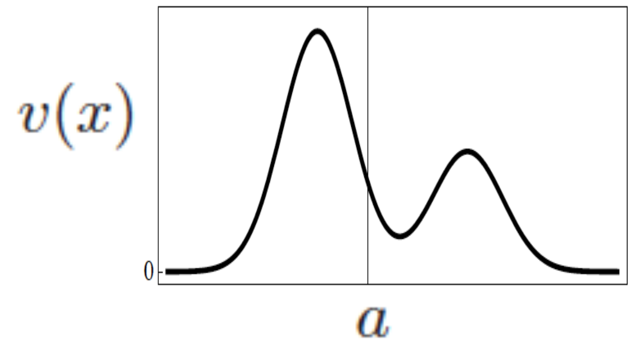

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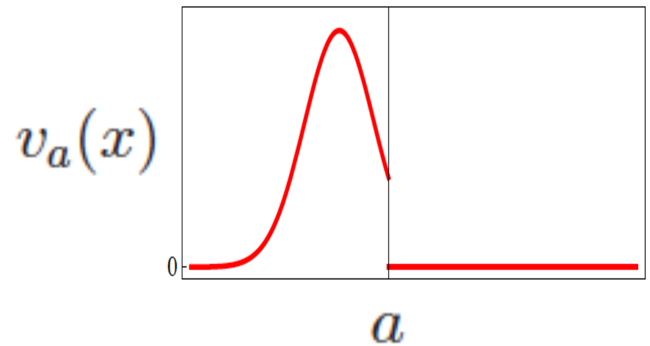

Then $\mathbf{M}(\alpha)=\mathscr{U}(\alpha,-\infty)$. Therefore,

$$i\partial_\alpha\mathbf{M}(\alpha)=\mathscr{H}(\alpha)\mathbf{M}(\alpha),\qquad \mathbf{M}(-\infty)=1.$$

We also have $\mathbf{M}=\mathbf{M}(\infty)$.

$i\partial_\alpha \mathrm{M}(\alpha) = \mathscr{H}(\alpha)\mathrm{M}(\alpha)$ & $\det \mathrm{M} \neq 0 \Rightarrow$

$$i[\partial_\alpha \mathrm{M}(\alpha)]\mathrm{M}(\alpha)^{-1} = \mathscr{H}(\alpha) = w(\alpha) \begin{bmatrix} 1 & e^{-2i\alpha} \\ -e^{2i\alpha} & -1 \end{bmatrix}$$

Recall $w(\alpha) = v(a)/2k^2$.

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Eg: Square Barrier Potential of height $\mathfrak{z} \in \mathbb{C}$,

$$v(x) = v_L(x) := \begin{cases} \mathfrak{z} & \text{for } x \in [0, L] \\ 0 & \text{for } x \notin [0, L] \end{cases}$$

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$$M_{11}(\alpha) = \left[\cos(\mathfrak{n}\,\alpha) + i(\mathfrak{n}^2 + 1)\sin(\mathfrak{n}\,\alpha)/2\,\mathfrak{n}\right] e^{-i\alpha},$$

$$M_{12}(\alpha) = i(\mathfrak{n}^2 - 1)\sin(\mathfrak{n}\,\alpha)e^{-i\alpha}/2\,\mathfrak{n},$$

$$M_{21}(\alpha) = M_{12}(-\alpha), \qquad M_{22}(\alpha) = M_{11}(-\alpha),$$

$$\mathfrak{n} := \sqrt{1 - \mathfrak{z}/k^2}$$

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$$\mathbf{M} = \begin{bmatrix} T - \frac{R^l R^r}{T} & \frac{R^r}{T} \\ -\frac{R^l}{T} & \frac{1}{T} \end{bmatrix}$$

$R^{l/r}(\alpha) := R^{l/r}$ for v_a & $T(\alpha) :=: T$ for $v_a \Rightarrow$

$R^{l/r}(\alpha)$ and $T(\alpha)$ satisfy dynamical eqs.

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$$\begin{aligned} R^r(z) &= \frac{S(z)}{S'(z)} - z, \\ R^l(z) &= -\int_{z_-}^{z} d\zeta \, \frac{S''(\zeta)}{S(\zeta)S'(\zeta)^2}, \\ T(z) &= \frac{1}{S'(z)}. \end{aligned}$$

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$$z^2S''(z)+\left[\frac{\breve{v}(z)}{4k^2}\right]S(z)=0$$

$$S(z_-)=z_-,\qquad S'(z_-)=1$$

$$\breve{v}(z):=v(i\ln z/2k)=v(a)$$

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T(z) &= \frac{1}{S'(z)}.
\end{aligned}
$$

$$z^2 S''(z) + \left[\frac{\breve{v}(z)}{4k^2}\right] S(z) = 0$$

$$S(z_-) = z_-, \qquad S'(z_-) = 1$$

$$\breve{v}(z) := v(i \ln z/2k) = v(a)$$

Can use these for inverse scattering.

$$\mathbf{M} = \begin{bmatrix} T - \dfrac{R^l R^r}{T} & \dfrac{R^r}{T} \\ -\dfrac{R^l}{T} & \dfrac{1}{T} \end{bmatrix}$$

A finite-range potential with a SS at $k = k_0$:

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Choose $S(z) = \dfrac{z^2 - 2z_+ z + 1}{2(1 - z_+)}$, $\tau_- = 0$, $\tau_+ = kL$, $k_0 L \notin 2\pi\mathbb{Z}$.

$$\mathfrak{n}^2(x) = 1 - \frac{v(x)}{k^2} = \begin{cases} 1 + 8\left[e^{4ik_0 x} - 2e^{-2ik_0(L-x)} + 1\right]^{-1} & x \in [0, L] \\ 1 & x \notin [0, L] \end{cases}$$

A. M. Ann. Phys. **341**, 77 (2014)

Finite-range right-invisible potential at $k = k_0$:

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$$\mathfrak{n}^2(x) = \begin{cases} 1 + n f_\alpha(x) & x \in [0, L_n] \\ 1 & x \notin [0, L_n] \end{cases}$$

$$v(x) = v_{\alpha,n}(x) := \begin{cases} -k^2 n f_\alpha(x) & x \in [0, L_n] \\ 0 & x \notin [0, L_n] \end{cases}$$

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At $k = k_0$: $\qquad R^r = 0, \quad T = 1$

For $\alpha > -\frac{1}{4}$: $\qquad R^l = \frac{-8\pi i n\alpha}{(\alpha+1)^2}$

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For given $R = \rho\, e^{i\varphi} \in \mathbb{C}$, choose $\alpha \in [0,1)$, $n \in \mathbb{Z}^+$, $m \in \mathbb{Z}$ & $d_m \in \mathbb{R}$, such that

$$\frac{8\pi n\alpha}{(\alpha+1)^2} = \rho, \qquad d_m = \frac{(4m-1)\pi - 2\varphi}{4k_0}.$$

Let $v^r_R(x) := v_{\alpha,n}(x + d_m)$ & $v^l_R(x) := v^r_{-R^*}(x)^*$.

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Let $v^r_R(x) := v_{\alpha,n}(x + d_m)$ & $v^l_R(x) := v^r_{-R^*}(x)^*$. Then,

- $v^r_R(x)$ is right-invisible with $R^l = R$.
- $v^l_R(x)$ is left-invisible with $R^r = R$.
- Both vanish outside $[-d_m, L_n - d_m]$.

$\Rightarrow$ a **model** for **general unidirectional invisibility**.

Perturbative Expansion for M

$$\mathscr{H}(\tau) = w(\tau) \begin{bmatrix} 1 & e^{-2i\tau} \\ -e^{2i\tau} & -1 \end{bmatrix} \qquad \tau = kx, \qquad w(\tau) = \frac{v(x)}{2k^2}$$

$$\begin{aligned} \mathbf{M} &= \mathscr{U}(+\infty, -\infty) = \mathscr{T}\, e^{-i\int_{-\infty}^{\infty} d\tau\, \mathscr{H}(\tau)} \\ &= 1 - i \int_{-\infty}^{\infty} d\tau_1 \mathscr{H}(\tau_1) - \int_{-\infty}^{\infty} d\tau_2 \int_{-\infty}^{\tau_2} d\tau_1 \mathscr{H}(\tau_2)\mathscr{H}(\tau_1) + \cdots \\ &=: 1 + \sum_{\ell=1}^{\infty} \mathbf{M}^{(\ell)} \end{aligned}$$

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&=: 1 + \sum_{\ell=1}^{\infty} \mathbf{M}^{(\ell)}
\end{aligned}$$

$$\mathbf{M}^{(1)} = \frac{-i}{2k}\begin{bmatrix} \tilde{v}(0) & \tilde{v}(2k) \\ -\tilde{v}(-2k) & -\tilde{v}(0) \end{bmatrix},$$

$$\mathbf{M}^{(2)} = \frac{-1}{4k^2}\begin{bmatrix} \tilde{v}(0,0) - \tilde{v}(-2k,2k) & \tilde{v}(2k,0) - \tilde{v}(0,2k) \\ \tilde{v}(-2k,0) - \tilde{v}(0,-2k) & \tilde{v}(0,0) - \tilde{v}(2k,-2k) \end{bmatrix}$$

$$\tilde{f}(k_1,\cdots,k_\ell) := \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_\ell\; e^{-i(k_1x_1+\cdots+k_\ell x_\ell)} f(x_1,\cdots,x_\ell)$$

$$v(x_1,x_2) := v(x_2)\theta(x_2 - x_1)v(x_1)$$

Perturbative Unidirectional Invisibility

$$v(x) = \begin{cases} \mathfrak{z}\, f(x) & \text{for} \quad x \in [0, L], \\ 0 & \text{for} \quad x \notin [0, L]. \end{cases}$$

$$R^l = \mathcal{O}(\mathfrak{z}^2), \qquad R^r = \mathcal{O}(\mathfrak{z}), \qquad T = 1 + \mathcal{O}(\mathfrak{z}^2).$$

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Example: $f(x) = e^{iKx}$, $k = \dfrac{K}{2} = \dfrac{2\pi m}{L}$, & $m \in \mathbb{Z}^+$.

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$$R^l = \frac{\tilde{v}(-2k)}{2ik - \tilde{v}(0)} + \mathcal{O}(\mathfrak{z}^2),$$

$$R^r = \frac{\tilde{v}(2k)}{2ik - \tilde{v}(0)} + \mathcal{O}(\mathfrak{z}^2),$$

$$T = \frac{2ik}{2ik - \tilde{v}(0)} + \mathcal{O}(\mathfrak{z}^2).$$

$\Rightarrow$ Complete characterization of pert. unidir. invisibility

$\Rightarrow$ Multimode Perturbative Unidirectional Invisibility

$$v(x) = \begin{cases} \mathfrak{z}\, f(x) & \text{for} \quad x \in [0, L], \\ 0 & \text{for} \quad x \notin [0, L]. \end{cases}$$

Example: $$f(x) = \frac{\mathfrak{a}\, e^{2iKx}}{1 - \mathfrak{a}\, e^{2iKx}} + \frac{\mathfrak{b}\, e^{-iKx}}{1 - \mathfrak{b}\, e^{-2iKx}}$$

$$|\mathfrak{a}| < 1, \qquad |\mathfrak{b}| < 1, \qquad K = \frac{2\pi}{L}$$

Perturbatively invisible from **left**: $k = nK,\ n = 1, 2, 3, \cdots$

Perturbatively invisible from **right**: $k = \left(n + \frac{1}{2}\right) K.$

Perturbative Inverse Scattering:

$$\mathbf{M}^{(1)} = \frac{-i}{2k}\begin{bmatrix} \tilde{v}(0) & \tilde{v}(2k) \\ -\tilde{v}(-2k) & -\tilde{v}(0) \end{bmatrix}$$

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$$v(x) = 2\frac{d}{dx}\check{M}^{(1)}_{12}(2x) = 2\frac{d}{dx}\check{M}^{(1)}_{21}(-2x)$$

$$\check{f}(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx} f(k)$$

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Theorem: It is the first Born Approximation of the scattering data that determines the form of the potential.

A. M. PRA. **89**, 012709 (2014)

Adiabatic & WKB Approximations

Use adiabatic approx. to solve $i\dot{\Psi}(\tau) = \mathbf{H}(\tau)\Psi(t)$:

$$\tau := kx$$

$$w(\tau) := \frac{v(\tau/k)}{2k^2}$$

$$\mathbf{H}(\tau) := \begin{bmatrix} w(\tau) - 1 & w(\tau) \\ -w(\tau) & -w(\tau) + 1 \end{bmatrix}$$

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$$\mathbf{H}(\tau)\Psi_\pm(\tau) = E_\pm(\tau)\Psi_\pm(\tau)$$

$$E_\pm(\tau) := \pm\mathfrak{n}(\tau)$$

$$\mathfrak{n}(\tau) := \sqrt{1 - \frac{v(\tau/k)}{k^2}}$$

$$\Psi_\pm(\tau) := \frac{1}{2}\begin{bmatrix} 1 \mp \mathfrak{n}(\tau) \\ 1 \pm \mathfrak{n}(\tau) \end{bmatrix}$$

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Biorthonormal dual of $\Psi_\pm(\tau)$: $\Phi_\pm(\tau) := \frac{1}{2\mathfrak{n}(\tau)^*}\begin{bmatrix} \mathfrak{n}(\tau)^* \mp 1 \\ \mathfrak{n}(\tau)^* \pm 1 \end{bmatrix}$

$$\langle\Phi_i(\tau)|\Psi_j(\tau)\rangle = \delta_{ij}, \qquad \sum_{j=\pm} |\Psi_j(\tau)\rangle\langle\Phi_j(\tau)| = 1$$

$$\mathbf{H}(\tau)\Psi_{\pm}(\tau) = E_{\pm}(\tau)\Psi_{\pm}(\tau)$$

Adiabatic approximation:

$$\Psi_{\pm}(\tau_0) \longrightarrow \Psi(\tau) \approx e^{i\delta_{\pm}(\tau)} e^{i\gamma_{\pm}(\tau)} \Psi_{\pm}(\tau)$$

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$$\Psi_\pm(\tau_0) \longrightarrow \Psi(\tau) \approx e^{i\delta_\pm(\tau)} e^{i\gamma_\pm(\tau)} \Psi_\pm(\tau)$$

$$\begin{aligned}
\delta_\pm(\tau) &= -\int_{\tau_0}^{\tau} E_\pm(\tau')d\tau' = \mp \int_{\tau_0}^{\tau} \mathfrak{n}(\tau')d\tau' \\
\gamma_\pm(\tau) &= i\int_{\tau_0}^{\tau} \langle \Phi_\pm(\tau')|\dot{\Psi}_\pm(\tau')\rangle d\tau' = i\int_{\mathfrak{n}(\tau_0)}^{\mathfrak{n}(\tau)} \langle \Phi_\pm|d\Psi_\pm\rangle
\end{aligned}$$

Adiabaticity Condition:

$$\left|\frac{\langle \Phi_\pm(\tau)|\dot{\Psi}_\mp(\tau)\rangle}{E_+(\tau) - E_-(\tau)}\right| \ll 1$$

Garrison & Wright, PLA **128**, 177 (1988)

Adiabaticity Condition:

$$\left|\frac{\langle\Phi_\pm(\tau)|\dot{\Psi}_\mp(\tau)\rangle}{E_+(\tau)-E_-(\tau)}\right| \ll 1 \quad\Leftrightarrow\quad \left|\frac{\dot{\mathfrak{n}}(\tau)}{4\mathfrak{n}(\tau)^2}\right| \ll 1 \quad\Leftrightarrow\quad \frac{|v'(x)|}{8|k^2-v(x)|^{3/2}} \ll 1$$

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$$\Psi(\tau)\approx e^{i\delta_\pm(\tau)}e^{i\gamma_\pm(\tau)}\Psi_\pm(\tau)$$

$$\delta_\pm(\tau)=\mp\int_{\tau_0}^{\tau}\mathfrak{n}(\tau')d\tau'=\mp\int_{x_0}^{x}\sqrt{k^2-v(x')}dx'$$

$$e^{i\gamma_\pm(\tau)}=\sqrt{\frac{\mathfrak{n}(\tau_0)}{\mathfrak{n}(\tau)}} \qquad \Psi_\pm(\tau):=\frac{1}{2}\begin{bmatrix}1\mp\mathfrak{n}(\tau)\\ 1\pm\mathfrak{n}(\tau)\end{bmatrix}$$

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$$\Psi(\tau) \approx e^{i\delta_\pm(\tau)} e^{i\gamma_\pm(\tau)} \Psi_\pm(\tau)$$

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Recall: $\Psi(\tau) := \frac{1}{2}\begin{bmatrix} \phi(\tau)-i\dot{\phi}(\tau) \\ \phi(\tau)+i\dot{\phi}(\tau)\end{bmatrix}$ & $\phi(\tau) := \psi(\tau/k)$

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$$\psi(x) \approx \sqrt{\frac{\mathfrak{n}(\tau_0)}{\mathfrak{n}(\tau)}}\, e^{\mp i\int_{\tau_0}^{\tau} E_\pm(\tau')d\tau'} = \frac{N_0}{[k^2 - v(x)]^{1/4}}\, e^{\mp i\int_{x_0}^{x}\sqrt{k^2-v(x')}dx'}$$

$\Rightarrow$ **WKB Approximation = Adiabatic Approximation**

- **Semiclassical expression for transfer matrix**
- **Higher-order semiclassical scattering**

A. M. JPA 47, 125301 (2014) & 345302 (2014)

Local Inverse Scattering

Problem: Given a positive real number k_0 and complex numbers $R_0^{l/r}$ and T_0 ($\neq 0$), find a scattering potential $v(x)$ whose reflection and transmission amplitudes at $k = k_0$ are given by $R^{l/r} = R_0^{l/r}$ and $T = T_0$.

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Solution/Theorem: $v(x)$ can be written as the **sum of at most six unidirectionally invisible finite-range potentials**,

$$v(x) = v_1(x) + v_2(x) + \cdots + v_6(x).$$

- $v_i(x)$ have mutually disjoint supports.
- $v_i(x)$ can be selected from the class $\{v_R^r(x), v_R^l(x)\}$.

[arXiv:1407.1760, to appear in Phys. Rev. A]

Application: Design of **bidirectionally reflectionless phase-shifting amplifier**

Example: Choose $T_0 = \sqrt{2}i$. Then $v_0(x)$ doubles the intensity ($|T_0|^2 = 2$) and produces a $\pi/2$ phase shift in the transmitted wave.

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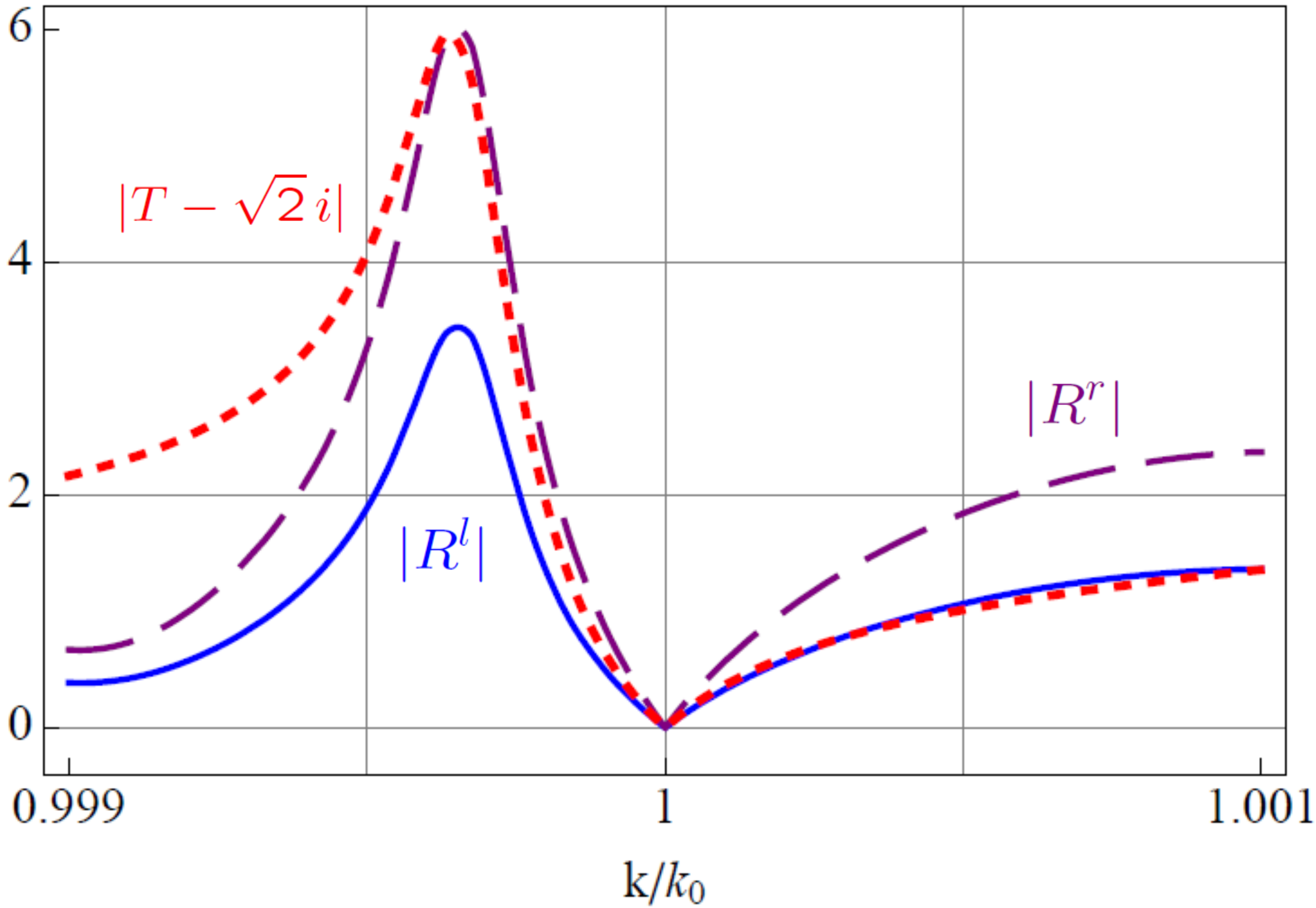
Explicit model: $v_0(x) = v_1(x) + v_2(x) + v_3(x) + v_4(x)$ with

$$v_j(x) := \begin{cases} v_{\alpha_j,n}(x + d_j) & \text{for} \quad j = 1, 3, \\ v_{\alpha_j,n}(x + d_j)^* & \text{for} \quad j = 2, 4, \end{cases}$$

$k_0 = 2\pi/\mu\text{m}$, $n = 300$, so that $L_n = 150\ \mu\text{m}$, and

$$\begin{aligned} \alpha_1 &= 1.57798 \times 10^{-4}, & d_1 &= 300.625\ \mu\text{m}, \\ \alpha_2 &= 1.93283 \times 10^{-4}, & d_2 &= 150.299\ \mu\text{m}, \\ \alpha_3 &= 1.11565 \times 10^{-4}, & d_3 &= 0.00000\ \mu\text{m}, \\ \alpha_4 &= 2.73409 \times 10^{-4}, & d_4 &= -150.326\ \mu\text{m}. \end{aligned}$$



6
4
2
0
0.999
1
1.001
k/k_0
$|T-\sqrt{2}\,i|$
$|R^l|$
$|R^r|$

Summary:

- $\mathbf{M} = S$-matrix for a two-level non-Hermitian Hamiltonian which is pseudo-Hermitian for a real potential.
- $\mathbf{M} =$ Asymptotic value of the evolution operator for a two-level pseudo-normal Hamiltonian.
- Dynamical equations for $\mathbf{M} \Rightarrow$ optical potential design
- Perturbative Unidirectional Invisibility & inverse scattering
- Adiabatic approximation $\Leftrightarrow$ WKB approximation
- Pre-exponential part of the WKB wave functions is actually a complex geometric phase
- **Explicit model** for unidirectional invisibility
- Unidirectionally invisible potentials are **local building blocks** of all scattering potentials
- **Applications:** Design of reflectionless amplifiers, absorbers, phase-shifters, threshold lasers & anti-lasers.

References:

- arXiv:1310.0592 [Ann. Phys. (NY), **341**, 77 (2014)]
- arXiv:1311.1619 [Phys. Rev. A **89**, 012709 (2014)]
- arXiv:1401.4315 [J. Phys. A **47**, 125301 (2014)]
- arXiv:1402.6458 [J. Phys. A, **47**, 345302 (2014)]
- arXiv:1407.1760 [Phys. Rev. A, to appear.]

Thank you for your attention.