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New spinorial particle models and massive HS free fields

- **with S. Fedoruk**
  "Massive twistor particle with spin generated by Souriau-Wess-Zumino term and its quantization"
  Phys.Lett. B733, 309-315 (2014); arXiv:1403.4127 [hep-th] A

- **with J. A. de Azcarraga, S. Fedoruk, J. M. Izquierdo**
  "Two-twistor particle models and free massive higher spin fields" B
  (almost final draft)

Three ways of Lagrangean description of $D=4$ relativistic particles using twistor formalism:

i) using relativistic phase space $(x_{\alpha\dot\beta}, p_{\alpha\dot\beta}, \ldots)$

ii) using hybrid space-time/spinor geometry $(x_{\alpha\dot\beta}, \pi^i_\alpha, \bar\pi^i_{\dot\alpha})$

iii) using twistorial description $\begin{pmatrix} Z^i_A = (\pi^i_\alpha, \omega^{\dot\alpha i}) \\ \bar Z^i_A = (\bar\pi^i_{\dot\alpha}, \omega^{\alpha i}) \end{pmatrix}$ $i = 1 \ldots N$

i) $\xrightarrow[p_{\alpha\dot\beta} = \pi^i_\alpha \bar\pi^i_{\dot\alpha}]{\text{Penrose fourmomentum}}$ ii) $\xrightarrow{\text{Penrose incidence relations}}$ iii)

$$\left.\begin{aligned} \omega^{\dot\alpha i} &= i\, x^{\dot\alpha\beta} \pi^{i}_{\beta} + \ldots\ldots \\ \bar\omega^{\alpha i} &= -i\, \bar\pi^i_{\dot\beta}\, x^{\dot\beta\alpha} + \ldots \end{aligned}\right\}$$ important extra terms!!

Physical cases: h – helicity

a) $m=0$, $h=0$ particles: $N=1$ (one-twistor geometry)

b) $m\neq 0$, $s=0$ particles: $N=2$ (two-twistor geometry)

c) $m\neq 0$, $s\neq 0$ particles: $N=2$ ← our two papers A, B

Problem: how to describe $s \neq 0$ term in relativistic $D=4$ particle Lagrangean?

Our proposal: two ways

A → to add to rel. phase space formulation i) the Souriau - Wess - Zumino term ⇔ Liouville one-form Ω_1 corresponding to Souriau symplectic two-form for massive particles with spin Ω_2

$$\Omega_2 = d\,\Omega_1$$

| mass m
| spin s

$$\Omega_2 = \frac{1}{2m^2}\epsilon_{\mu\nu\rho\tau} W^\rho p^\sigma \left(\frac{1}{m^2} dp^\mu \wedge dp^\nu + \frac{1}{W^2} dW^\mu \wedge dW^\nu\right)$$

$W_\mu p^\mu = 0 \qquad W^2 \equiv W_\mu W^\mu = -m^2 j^2 \qquad p^2 = m^2$ (Souriau ~1970)

$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\tau} p^\nu M^{\rho\tau}$ ← Pauli-Lubanski fourvector

Important issue:

No covariant formula for Ω_1 – it is necessary to introduce some fourvector n_μ determining singularity ("Dirac string"). However if we use formulae of twistor theory

$$p_{\alpha\dot\beta} = \pi_\alpha^i \bar\pi_{\dot\beta}^i \qquad W_{\alpha\dot\beta} = S^r u^r_{\alpha\dot\beta}$$

$r = 1,2,3$

← follows from $N=2$ twistor formulation. $0 = p_\mu W^\mu$ satisfied as identity

where

$$u^a_{\alpha\dot\beta} = \pi_\alpha^i (\tau^a)_{ik} \bar\pi^k_{\dot\beta} \quad \leftarrow \quad u^0_{\alpha\dot\beta} = p_{\alpha\dot\beta}$$

$a = 0,1,2,3$

one can find solution for Ω_1

$$\Omega_1 = -\frac{i}{2M\bar M} S^r (\tau^r)_{ik} \left(\bar M \pi^{\alpha i} d\pi^k_\alpha + M \bar\pi^{\dot\alpha i} d\bar\pi^k_{\dot\alpha} \right)$$

$$\pi^{\alpha i}\pi_{\alpha k} = M \delta^i_k \qquad \bar\pi^{\dot\alpha i}\bar\pi_{\dot\alpha k} = \bar M \delta^i_k \quad \Rightarrow p^{\alpha\dot\beta} p_{\alpha\dot\beta} = 2|M|^2 = m^2$$

M – complexified mass

The actions i), ii) and iii):

$\Omega_1 = F_A dx^A \rightarrow \dot{\Omega}_1 = F_A \underset{\substack{\uparrow \\ \frac{dx^A}{d\tau}}}{\dot{x}^A}$

i) $$S_1 = \int d\tau \left(p_\mu \dot{x}^\mu + \lambda(p_\mu p^\mu - m^2) + \dot{\Omega}_1 + l_1 (p_\mu W^\mu) + l_2 (W_\mu W^\mu + m^2 j^2) \right)$$

ii) $$S_2 = \int d\tau \Big\{ \Pi^k_\alpha \bar{\Pi}^k_{\dot\beta} \dot{x}^{\alpha\dot\beta} + \Lambda \mathcal{M} + \bar{\Lambda} \bar{\mathcal{M}} + \frac{i}{2M\bar{M}} \underset{\substack{\Uparrow \\ \text{spin part}}}{S^r (\tau^r)_{ik} (\bar{M} \Pi^{\alpha i} \dot{\Pi}^k_\alpha + \text{H.c.})} + l(\vec{S}^2 - j^2) \Big\}$$

where

$$\mathcal{M} = \Pi^{\alpha i} \Pi_{\alpha i} - M \qquad \bar{\mathcal{M}} = \bar{\Pi}^{\dot\alpha i} \bar{\Pi}_{\dot\alpha i} - \bar{M}$$ (spinorial mass constraints)

iii) To get twistorial action we modify incidence relations:

$$\omega^{\dot\alpha i} = -i\, x^{\alpha\dot\beta} \bar{\Pi}^i_{\dot\beta} + \frac{i}{2M} S^r (\tau^r)^i{}_j \Pi^{\alpha j}$$
$$\bar{\omega}^{\dot\alpha i} = i\, \Pi^i_\beta x^{\dot\beta\alpha} - \frac{i}{2M} S^r (\tau^r)_i{}^j \bar{\Pi}^{\dot\alpha j}$$

→ leads to free 2-twistor action with constraints

$$S_3 = \int d\tau \{ \Pi^k_\alpha \dot{\omega}^{\alpha k} + H.C. \} + \Lambda \mathcal{M} + \bar{\Lambda} \bar{\mathcal{M}} + \ell (\vec{S}^2 - j^2) + \\ + \Lambda^r (V^r + S^r) + \Lambda^0 V^0$$

V^a $(a=0,1,2,3)$ describe conformal-inv. scalar products of two twistors Z^i_A, $\bar{Z}^{Ai}$

$$V^a = \bar{Z}^{Ai} (\tau^a)_{ij} Z^j_A \qquad G_{AB} = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$$

The variables S^r are endowed with $SU(2)$ PB relations

$$\{S^p, S^r\}_{PB} = \epsilon^{prq} S^q$$

which can be derived if we supplement S_3 with the following Chern-Simons coupling term:

$$\Delta S_3 = \int d\tau\, A^r(S) \dot{S}^r \quad \text{where } \partial^r A^q - \partial^q A^r = -j\, \epsilon^{rqt} \frac{S^t}{|S|^3}$$

Quantization (fixed values of m and j) – constraints:

$$\mathcal{V}^r = V^r + S^r = 0 \qquad V^0 = 0$$

$$\mathcal{M} = 0 \qquad \bar{\mathcal{M}} = 0 \qquad \vec{S}^2 = j^2$$

$$\Updownarrow \qquad \Updownarrow$$

$$F_1 = \bar{M}\mathcal{M} + M\bar{\mathcal{M}}$$

$$F_2 = i(\bar{M}\mathcal{M} - M\bar{\mathcal{M}}) \quad \Rightarrow \quad \begin{array}{l} \mathcal{V}^r, F_1 \text{ – first class} \\ V^0, F_2 \text{ – second class} \end{array}$$

Degrees of freedom:

$(Z^i_A, \bar{Z}^i_A, S^r) \longrightarrow 18$ degrees

4 first class + 2 second class $\longrightarrow$ eliminated $4\cdot 2+2=10$ degrees

Physical degrees: $18-10=8$ ⇐ correct number of degrees

Quantization: we introduce gauge fixing of F_1 gauge transformations and introduce Schrödinger representation of Dirac brackets → e.g. for twistor wave function $\Psi^{(j)}_{j_3}$ where $j = 0, \frac{1}{2}, 1, \frac{3}{2} \ldots$ and $j_3 = (-j, -j+1 \ldots j)$ ← see paper (A)

B Alternative way of adding spin term - to add new terms in the hybrid formulation ii)

our second paper

If $N=2$ one can introduce

$$\underbrace{P_{\alpha\dot\beta} = \Pi_\alpha \bar\Pi_{\dot\beta}}_{N=1} \longrightarrow \underbrace{u^a_{\alpha\dot\beta} = \Pi^i_\alpha (\tau^a)_{ij} \bar\Pi^j_{\dot\beta}}_{N=2} \qquad a=0,1,2,3$$

We introduce additional three ~~comp~~ four-vector coordinates dual to the composite "momenta" $u^a_{\alpha\dot\beta}$

$$x_{\alpha\dot\beta} \longrightarrow (x_{\alpha\dot\beta}, y^r_{\alpha\dot\beta}) \equiv y^a_{\alpha\dot\beta} \qquad (y^0_{\alpha\dot\beta} \equiv x_{\alpha\dot\beta})$$

The hybrid action is the following:

$$S_2 = \int d\tau \{ \Pi^i_\alpha \bar\Pi^i_{\dot\beta} \dot x^{\alpha\dot\beta} + \overbrace{c\, \Pi^i_\alpha (\tau^a)_{ij} \bar\Pi^j_{\dot\beta} \dot y^{r\,\alpha\dot\beta}}^{N=2 \text{ Shirafuji term}}$$

$$+ (\underbrace{f\, \Pi^i_\alpha \dot y^{\alpha i}}_{\text{"Vasiliev term" with extra spinorial coordinates } y_{\alpha i}} + \text{H.c.}) + \Lambda \mathcal{M} + \bar\Lambda \bar{\mathcal{M}} \}$$

c, f - constants

Remark:

If $N=1$ the only possible extension of Shirafuji model was to add tensorial coordinates

$$\pi_\alpha \pi_{\dot\beta} X^{\alpha\dot\beta} + \pi_\alpha \pi_\beta y^{\alpha\beta} + \bar\pi_{\dot\alpha} \bar\pi_{\dot\beta} \bar y^{\dot\alpha\dot\beta}$$

$D=4$ space-time $\Rightarrow$ 10-dimensional tensorial space-time

$$(X_{\alpha\dot\beta}, y_{\alpha\beta}, \bar y_{\dot\alpha\dot\beta}) \Leftrightarrow (X_\mu, Y_{[\mu\nu]}) \quad \text{(BLS 1989)}$$

In AdS case $(R<\infty)$

$$(X_\mu, Y_{[\mu\nu]}) \xrightarrow{R<\infty} Sp(4;R)$$

In general one can consider the particle models on $Sp(2n,R)$ (Vasiliev 1991)

If $N=2$ one can introduce additional vector coordinates (besides the tensorial ones, dual to $\pi^{(i}_{(\alpha} \pi^{j)}_{\beta)}$, $\bar\pi^{(i}_{(\dot\alpha} \bar\pi^{j)}_{\dot\beta)}$).

$\Rightarrow$ similarity to description by (p_μ, q_μ), with $p^2=q^2=m^2$ $pq=0$ (Biedenharn et all, 1988)

Passage from hybrid (ii) to twistor (iii) formulation:

The generalization of incidence relation:

$$\omega^{\alpha i} = -i X^{\alpha\dot\beta}\bar\Pi_{\dot\beta}{}^{i} + c\, y^{r\,\alpha\dot\beta}(\tau^r)^{ij}\bar\Pi_{\dot\beta j} + f\, y^{\alpha i}$$

$$\bar\omega^{\dot\alpha i} = i\,\Pi_\beta^{i} X^{\beta\dot\alpha} + c\, y^{r\beta\dot\alpha}(\tau^r)^{ij}\Pi_{\beta j} + f\,\bar y^{\dot\alpha i}$$

One gets two-twistor action (iii)

$$S_3 = \int d\tau \{ (\Pi_\alpha^i \dot\omega^\alpha_i + \text{H.c.}) + \Lambda \mathcal{M} + \bar\Lambda \bar{\mathcal{M}} + \text{constraints} \}$$

We recall $V^a = \bar Z^{Ai}(\tau^a)_{ij} Z^j_A \quad (a=0,1,2,3)$

The constraints:

1) $c=0,\ f=0$ (2-twistor Shirafuji model): $V^a = 0,\ \mathcal{M}=0\ \bar{\mathcal{M}}=0$

no spin!

2) $f=0$: $V^0 = 0,\ \mathcal{M}=0\ \bar{\mathcal{M}}=0$

3) $c\neq 0$ and $f\neq 0 \Rightarrow$ only $\mathcal{M}=0,\ \bar{\mathcal{M}}=0$

The cases 2) and 3) give the same number of physical degrees of freedom:

2): one first class + two second class : $16-4=12$ degrees of freedom

3): two second class: $16-2\cdot 2=12$ degrees of freedom

From 3) follows that the spinor coordinates span 6-dimensional manifold $SL(2;C)$ $(\pi_\alpha^{\ i}, \bar{\pi}_{\dot{\alpha}}^{\ \dot{i}}) \in SL(2;C)$

After quantization: the wave function $\Psi(g)$ $g \in SL(2;C)$

Split into ~~...~~ fourmomentum and spin part:

$$g = h \cdot v \qquad h = h^+ \in \frac{SL(2;C)}{SU(2)} \qquad v^+ v = 1 \in SU(2)$$

$$p_{\alpha\dot{\beta}} = h_\alpha^{\ \tilde{i}}\, \bar{h}_{\dot{\beta}}^{\ \tilde{i}} \qquad v = v_{\tilde{i}}^{\ i} \qquad i, \tilde{i} \in (1,2)$$

Spin S_r is realized ~~...~~ only on $v \in SU(2)$ after quantization

$$\hat{S}_r = \frac{1}{2}(\sigma_r)_j^{\ k}\, v_{\tilde{i}}^{\ j} \frac{\partial}{\partial v_{\tilde{i}}^{\ k}} \qquad [\hat{S}_r, \hat{S}_p] = \epsilon_{rpt}\, \hat{S}_t$$

<u>General wave function</u> $\left(V_{\tilde{i}}^{\,k} = (V_{\tilde{i}}^{+}, V_{\tilde{i}}^{-}) \right)$

$$\Psi(h_\alpha^{\tilde{i}}, V_{\tilde{i}}^{k}) = \sum_{K,N=0}^{\infty} V_{\tilde{i}_1}^{+} \cdots V_{\tilde{i}_N}^{+} v_{\tilde{j}_1}^{-} \cdots v_{\tilde{j}_K}^{-} f^{(\tilde{i}_1 \ldots \tilde{i}_N, \tilde{j}_1 \ldots \tilde{j}_K)}(p_\mu)$$

on mass-shell

where

$$\hat{W}_\mu \hat{W}^\mu v_{\tilde{i}_1}^{+} \cdots v_{\tilde{i}_N}^{+} v_{\tilde{j}_1}^{-} \cdots v_{\tilde{j}_K}^{-} = -m^2 \cancel{\ } j(j+1) v_{\tilde{i}_1}^{+} \cdots v_{\tilde{i}_N}^{+} v_{\tilde{j}_1}^{-} \cdots v_{\tilde{j}_K}^{-}$$

<u>The spectrum is degenerate</u> – in order to remove multiplicity of spins one should introduce <u>the harmonic condition</u>

can be expressed in term of harmonic derivative D^{++} $\Rightarrow$

$$\frac{\partial}{\partial v_{\tilde{j}}^{-}} \psi^{(+)}(h_\alpha^{\tilde{i}}, V_{\tilde{i}}^{k}) = 0 \quad \Longrightarrow \text{only dependence on } v_{\tilde{i}}^{+}$$

i.e.

$$\psi^{(+)}(h, v) = \sum_{N=0}^{\infty} v_{\tilde{i}_1}^{+} \cdots v_{\tilde{i}_N}^{+} f^{\tilde{i}_1 \ldots \tilde{i}_N}(p_\mu) \Leftarrow \text{spins not occurring more then once!}$$

<u>The Bargmann–Wigner fields</u> obtained by Fourier transform: $p_\mu \to$, $\Pi_\alpha^{-} \to$

$$\phi_{\alpha_1 \ldots \alpha_K \dot{\beta}_{K+1} \ldots \dot{\beta}_{N}}(x) = \int d^6\Pi \cdot e^{-i x_\mu p^\mu} \Pi^{-}_{\alpha_1} \cdots \Pi^{-}_{\alpha_K} \bar{\Pi}^{-}_{\dot{\beta}_{K+1}} \cdots \bar{\Pi}^{-}_{\dot{\beta}_N} \psi^{(+)}(h_\alpha^{\tilde{i}}, V_{\tilde{i}}^{k})$$

FINAL REMARKS:

α) The scheme can be done for D=3 spinning particles with mass – in such a case spin is a scalar

$$S = \frac{1}{2m}\epsilon_{\mu\nu\rho}P^{\mu}M^{\nu\rho}$$

twistors are real with four components ($Sp(4)$ spinors) and the wave function depends on $SL(2;R)$ manifold coordinates – 2 degrees describe D=3 p_μ on mass-shell, and third degree the scalar spin

β) Helicity in D+1 corresponds to D-dimensional spin

symplectic two-form for (D+1)-dimensional massless "helicity term" $\longleftrightarrow$ symplectic two-form for D-dimensional massive spin term (Horvathy talk!)

γ) Additional possible coordinates for two-twistor particle model corresponds to tensorial central coordinates for N=2 SUSY:

$i=1,2$ $\quad \{Q^i_\alpha, \bar{Q}^j_{\dot\beta}\} = P^{ij}_{\alpha\dot\beta} = P^a_{\alpha\dot\beta}(\tau_a)^{ij}$ $\qquad \{Q^i_\alpha, Q^j_\beta\} = \delta^{ij} Z_{(\alpha\beta)}$

in our last paper $\quad a=1,2,3$ – vectorial central charges $\quad$ in old BLS papers (1989 –)