(Conformal) supersymmetries of curved superspace

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Based on:
SMK, arXiv:1212.6179;
SMK, U. Lindström, M. Roček, I. Sachs
& G. Tartaglino-Mazzucchelli, arXiv:1312.4267;
SMK, J. Novak & G. Tartaglino-Mazzucchelli, arXiv:1406.0727.

Outline

1. Supersymmetric backgrounds: Introductory comments
2. (Conformal) symmetries of curved spacetime
3. (Conformal) symmetries of curved superspace
4. Supersymmetric backgrounds in $d = 3$, $\mathcal{N} = 2$ supergravity

Supersymmetric backgrounds in supergravity

Supersymmetric solutions of supergravity

M. Duff *et al.* (1981,1982)

. . . .

P. van Nieuwenhuizen & N. Warner (1984)

M. Duff, B. Nilsson & C. Pope, *Kaluza-Klein Supergravity* (PR, 1986)

Concept of Killing spinors

All supersymmetric solutions of minimal (gauged) supergravity in 5D

J. Gauntlett, J. Gutowski, C. Hull, S. Pakis & H. Reall (2003)

J. Gauntlett & J. Gutowski (2003)

Superspace formalism to determine (super)symmetric backgrounds in off-shell supergravity

I. Buchbinder & SMK, *Ideas and Methods of Supersymmetry and Supergravity or a Walk Through Superspace*, IOP, Bristol, 1995 (1998)

Superspace formalism is universal, for it is geometric and may be generalized to supersymmetric backgrounds associated with any off-shell supergravity theory formulated in superspace.

Applications of the formalism:

- Rigid supersymmetric field theories in 5D $\mathcal{N} = 1$ AdS superspace
  SMK & G. Tartaglino-Mazzucchelli (2007)
- Rigid supersymmetric field theories in 4D $\mathcal{N} = 2$ AdS superspace
  SMK & G. Tartaglino-Mazzucchelli (2008)
  D. Butter & SMK (2011)
  D. Butter, SMK, U. Lindström & G. Tartaglino-Mazzucchelli (2012)
- Rigid supersymmetric field theories in 3D (p, q) AdS superspaces
  SMK, & G. Tartaglino-Mazzucchelli (2012)
  SMK, U. Lindström & G. Tartaglino-Mazzucchelli (2012)
  D. Butter, SMK & G. Tartaglino-Mazzucchelli (2012)

Recent developments

Exact results (partition functions, Wilson loops etc.)
in rigid supersymmetric field theories on curved backgrounds
(e.g., S^3, S^4, $S^3 \times S^1$ etc.) using localization techniques

V. Pestun (2007, 2009)
A. Kapustin, B. Willett & I. Yaakov (2010)
D. Jafferis (2010)
. . . .

Necessary technical ingredients:

- Curved space $\mathcal{M}$ should admit some unbroken rigid supersymmetry (supersymmetric background);
- Rigid supersymmetric field theory on $\mathcal{M}$ should be off-shell.

These developments have inspired much interest in the construction and classification of supersymmetric backgrounds that correspond to off-shell supergravity formulations.

Classification of supersymmetric backgrounds in off-shell supergravity

Component approaches

G. Festuccia and N. Seiberg (2011)
B. Jia and E. Sharpe (2011)
H. Samtleben and D. Tsimpis (2012)
C. Klare, A. Tomasiello and A. Zaffaroni (2012)
T. Dumitrescu, G. Festuccia and N. Seiberg (2012)
D. Cassani, C. Klare, D. Martelli, A. Tomasiello and A. Zaffaroni (2012)
T. Dumitrescu and G. Festuccia (2012)
A. Kehagias and J. Russo (2012)

Such results naturally follow from the superspace approach

4D, $\mathcal{N} = 1$ — SMK (2012)
3D $\mathcal{N} = 2$ — SMK, U. Lindström, M. Roček, I. Sachs & G. Tartaglino-Mazzucchelli (2013)
5D $\mathcal{N} = 1$ — SMK, J. Novak & G. Tartaglino-Mazzucchelli (2014)

Superspace approach is more powerful for constructing the most general rigid supersymmetric field theories.

(Conformal) symmetries of curved spacetime

(Conformal) symmetries of a curved superspace may be defined similarly to those corresponding to a curved spacetime within the Weyl-invariant formulation for gravity.

S. Deser (1970)
P. Dirac (1973)

Three formulations for gravity in d dimensions:

- Metric formulation;
- Vielbein formulation;
- Weyl-invariant formulation.

We briefly recall the metric and vielbein approaches and then concentrate in more detail of the Weyl-invariant formulation.

The metric and vielbein formulations for gravity

Metric formulation

Gauge field: metric $g_{mn}(x)$

Gauge transformation: $\delta g_{mn} = \nabla_m \xi_n + \nabla_n \xi_m$

$\xi = \xi^m(x)\partial_m$ a vector field generating an infinitesimal diffeomorphism.

Vielbein formulation

Gauge field: vielbein $e_m{}^a(x)$, $e := \det(e_m{}^a) \neq 0$

The metric is a composite field $g_{mn} = e_m{}^a e_n{}^b \eta_{ab}$

Gauge transformation: $\delta \nabla_a = [\xi^b \nabla_b + \frac{1}{2} K^{bc} M_{bc}, \nabla_a]$

Gauge parameters: $\xi^a(x) = \xi^m e_m{}^a(x)$ and $K^{ab}(x) = -K^{ba}(x)$

Covariant derivatives (M_{bc} the Lorentz generators)

$$\nabla_a = e_a{}^m \partial_m + \frac{1}{2}\omega_a{}^{bc} M_{bc} , \qquad [\nabla_a, \nabla_b] = \frac{1}{2} R_{ab}{}^{cd} M_{cd}$$

$e_a{}^m$ the inverse vielbein, $e_a{}^m e_m{}^b = \delta_a{}^b$;
$\omega_a{}^{bc}[e]$ the Lorentz connection.

Weyl transformations

Weyl transformations

The torsion-free constraint

$$T_{ab}{}^{c} = 0 \quad \Longleftrightarrow \quad [\nabla_a, \nabla_b] \equiv T_{ab}{}^{c}\nabla_c + \frac{1}{2}R_{ab}{}^{cd}M_{cd} = \frac{1}{2}R_{ab}{}^{cd}M_{cd}$$

is invariant under Weyl (local scale) transformations

$$\nabla_a \to \nabla'_a = \mathrm{e}^{\sigma}\Big(\nabla_a + (\nabla^b\sigma)M_{ba}\Big) ,$$

with the parameter $\sigma(x)$ being completely arbitrary.

$$e_a{}^m \to \mathrm{e}^{\sigma}e_a{}^m , \qquad e_m{}^a \to \mathrm{e}^{-\sigma}e_m{}^a , \qquad g_{mn} \to \mathrm{e}^{-2\sigma}g_{mn}$$

Weyl transformations are gauge symmetries of conformal gravity.
Einstein gravity possesses no Weyl invariance.

Weyl-invariant formulation for Einstein gravity

Weyl-invariant formulation for Einstein gravity

Gauge fields: vielbein $e_m{}^a(x)$, $e := \det(e_m{}^a) \neq 0$
& conformal compensator $\varphi(x)$, $\varphi \neq 0$

Gauge transformations ($\mathcal{K} := \xi^b \nabla_b + \frac{1}{2} K^{bc} M_{bc}$)

$$\delta \nabla_a = [\xi^b \nabla_b + \frac{1}{2} K^{bc} M_{bc}, \nabla_a] + \sigma \nabla_a + (\nabla^b \sigma) M_{ba} \equiv (\delta_{\mathcal{K}} + \delta_\sigma) \nabla_a ,$$

$$\delta \varphi = \xi^b \nabla_b \varphi + \frac{1}{2}(d-2)\sigma\varphi \equiv (\delta_{\mathcal{K}} + \delta_\sigma)\varphi$$

Gauge-invariant gravity action

$$S = \frac{1}{2} \int \mathrm{d}^d x\, e \left(\nabla^a \varphi \nabla_a \varphi + \frac{1}{4} \frac{d-2}{d-1} R \varphi^2 + \lambda \varphi^{2d/(d-2)} \right)$$

Imposing a Weyl gauge condition $\varphi = \frac{1}{2\kappa} \sqrt{\frac{d-1}{d-2}} = \mathrm{const}$
reduces the action to

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^d x\, e\, R - \frac{\Lambda}{\kappa^2} \int \mathrm{d}^d x\, e$$

Conformal isometries

Conformal Killing vector fields

A vector field $\xi = \xi^m \partial_m = \xi^a e_a$, with $e_a := e_a{}^m \partial_m$, is conformal Killing if there exist local Lorentz, $K^{bc}[\xi]$, and Weyl, $\sigma[\xi]$, parameters such that

$$(\delta_{\mathcal{K}} + \delta_\sigma)\nabla_a = \Big[\xi^b \nabla_b + \frac{1}{2} K^{bc}[\xi] M_{bc}, \nabla_a\Big] + \sigma[\xi]\nabla_a + (\nabla^b \sigma[\xi]) M_{ba} = 0$$

A short calculation gives

$$K^{bc}[\xi] = \frac{1}{2}\big(\nabla^b \xi^c - \nabla^c \xi^b\big) , \qquad \sigma[\xi] = \frac{1}{d} \nabla_b \xi^b$$

Conformal Killing equation

$$\nabla^a \xi^b + \nabla^b \xi^a = 2\eta^{ab} \sigma[\xi]$$

Equivalent spinor form in $d = 4$: $\qquad \nabla_{(\alpha}{}^{(\dot\alpha} \xi_{\beta)}{}^{\dot\beta)} = 0$

($\nabla_a \to \nabla_{\alpha\dot\alpha}$ and $\xi_a \to \xi_{\alpha\dot\alpha}$)

Equivalent spinor form in $d = 3$: $\qquad \nabla_{(\alpha\beta} \xi_{\gamma\delta)} = 0$

Conformal isometries

- Lie algebra of conformal Killing vector fields
- Conformally related spacetimes (∇_a, φ) and $(\widetilde{\nabla}_a, \widetilde{\varphi})$

$$\widetilde{\nabla}_a = \mathrm{e}^{\rho}\Big(\nabla_a + (\nabla^b \rho) M_{ba}\Big) , \qquad \widetilde{\varphi} = \mathrm{e}^{\frac{1}{2}(d-2)\rho}\varphi$$

have the same conformal Killing vector fields $\xi = \xi^a e_a = \tilde{\xi}^a \tilde{e}_a$.

The parameters $K^{cd}[\tilde{\xi}]$ and $\sigma[\tilde{\xi}]$ are related to $K^{cd}[\xi]$ and $\sigma[\xi]$ as follows:

$$\mathcal{K}[\tilde{\xi}] := \tilde{\xi}^b \widetilde{\nabla}_b + \frac{1}{2} K^{cd}[\tilde{\xi}] M_{cd} = \mathcal{K}[\xi] ,$$
$$\sigma[\tilde{\xi}] = \sigma[\xi] - \xi \rho$$

- Conformal field theories

Isometries

Killing vector fields

Let $\xi = \xi^a e_a$ be a conformal Killing vector,

$$(\delta_{\mathcal{K}} + \delta_\sigma)\nabla_a = \Big[\xi^b \nabla_b + \frac{1}{2} K^{bc}[\xi] M_{bc}, \nabla_a\Big] + \sigma[\xi]\nabla_a + (\nabla^b \sigma[\xi]) M_{ba} = 0~.$$

It is called Killing if it leaves the compensator invariant,

$$(\delta_{\mathcal{K}} + \delta_\sigma)\varphi = \xi\varphi + \frac{1}{2}(d-2)\sigma[\xi]\varphi = 0~.$$

These Killing equations are Weyl invariant in the following sense: Given a conformally related spacetime $(\widetilde{\nabla}_a, \widetilde{\varphi})$

$$\widetilde{\nabla}_a = \mathrm{e}^\rho \Big(\nabla_a + (\nabla^b \rho) M_{ba}\Big)~, \qquad \widetilde{\varphi} = \mathrm{e}^{\frac{1}{2}(d-2)\rho}\varphi~,$$

the above Killing equations have the same functional form when rewritten in terms of $(\widetilde{\nabla}_a, \widetilde{\varphi})$, in particular

$$\xi\widetilde{\varphi} + \frac{1}{2}(d-2)\sigma[\tilde{\xi}]\widetilde{\varphi} = 0~.$$

Isometries

Because of Weyl invariance, we can work with a conformally related spacetime such that

$$\varphi = 1$$

Then the Killing equations turn into

$$\Big[\xi^b \nabla_b + \frac{1}{2} K^{bc}[\xi] M_{bc}, \nabla_a\Big] = 0 \ , \qquad \sigma[\xi] = 0$$

Standard Killing equation

$$\nabla^a \xi^b + \nabla^b \xi^a = 0$$

- Lie algebra of Killing vector fields
- Rigid symmetric field theories in curved space

(Conformal) symmetries of curved superspace

Weyl-invariant approach to spacetime symmetries has a natural superspace extension in all cases when supergravity is formulated as conformal supergravity coupled to certain conformal compensator(s) Ξ

$z^M = (x^m, \theta^\mu)$	local coordinates of curved superspace
$\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha) = E_A + \Omega_A + \Phi_A$	superspace covariant derivatives
$E_A = E_A{}^M(z)\partial_M$	superspace vielbein
$\Omega_A = \frac{1}{2}\Omega_A{}^{bc}(z)M_{bc}$	superspace Lorentz connection
$\Phi = \Phi_A{}^I(z)T_I$	superspace R-symmetry connection

Supergravity gauge transformation

$$\delta_\mathcal{K}\mathcal{D}_A = [\mathcal{K}, \mathcal{D}_A] \ , \qquad \delta_\mathcal{K}\Xi = \mathcal{K}\Xi \ , \qquad \mathcal{K} := \xi^B\mathcal{D}_B + \frac{1}{2}K^{bc}M_{bc} + K^I T_I$$

Super-Weyl transformation

$$\delta_\sigma\mathcal{D}_a = \sigma\mathcal{D}_a + \cdots \ , \qquad \delta_\sigma\mathcal{D}_\alpha = \frac{1}{2}\sigma\mathcal{D}_\alpha + \cdots \ , \qquad \delta_\sigma\Xi = w_\Xi\sigma\Xi \ ,$$

with w_Ξ a non-zero super-Weyl weight

Conformal isometries of curved superspace

Let $\xi = \xi^B E_B$ be a real supervector field. It is called conformal Killing if

$$(\delta_{\mathcal{K}} + \delta_\sigma)\mathcal{D}_A = 0 ,$$

for some Lorentz $K^{bc}[\xi]$, R-symmetry $K^I[\xi]$ and super-Weyl $\sigma[\xi]$ parameters.

- All parameters $K^{bc}[\xi]$, $K^I[\xi]$ and $\sigma[\xi]$ are uniquely determined in terms of ξ^B.
- The spinor component ξ^β is uniquely determined in terms of ξ^b.
- The vector component obeys an equation that contains all the information about the conformal Killing supervector field.

$$d = 3 \qquad \mathcal{D}^I_{(\alpha}\xi_{\beta\gamma)} = 0$$

$$d = 4 \qquad \mathcal{D}^I_{(\alpha}\xi_{\beta)\dot\beta} = 0$$

Isometries of curved superspace

Let $\xi = \xi^B E_B$ be a conformal Killing supervector field,

$$(\delta_{\mathcal{K}[\xi]} + \delta_{\sigma[\xi]})\mathcal{D}_A = 0 \ , \qquad (\star)$$

for uniquely determined parameters $K^{bc}[\xi]$, $K^I[\xi]$ and $\sigma[\xi]$.
It is called Killling if the compensators are invariant,

$$(\delta_{\mathcal{K}[\xi]} + w_\Xi \sigma[\xi])\Xi = 0 \ . \qquad (\star\star)$$

The Killing equations $(\star)$ and $(\star\star)$ are super-Weyl invariant in the sense that they hold for all conformally related superspace geometries.

Using the compensators Ξ we can always construct a scalar superfield $\phi = \phi(\Xi)$, which is an algebraic function of Ξ, nowhere vanishing, and has a nonzero super-Weyl weight w_ϕ, $\delta_\sigma \phi = w_\phi \sigma \phi$.

$$(\delta_{\mathcal{K}[\xi]} + w_\Xi \sigma[\xi])\phi = 0 \ .$$

Super-Weyl invariance may be used to impose the gauge $\phi = 1$, and then

$$\sigma[\xi] = 0 \ .$$

(Conformal) supersymmetries of curved superspace

Of special interest are curved backgrounds which admit at least one (conformal) supersymmetry. Such a superspace must possess a conformal Killing supervector field ξ^A with the property

$$\xi^a| = 0 , \qquad \xi^\alpha| \neq 0$$

and describe a bosonic background with the property that all spinor components of the superspace torsion and curvature tensors

$$[\mathcal{D}_A, \mathcal{D}_B\} = T_{AB}{}^C \mathcal{D}_C + \frac{1}{2} R_{AB}{}^{cd} M_{cd} + R_{AB}{}^I T_I$$

have zero bar-projections,

$$\varepsilon(T_{\dots}{}^{\dots}) = 1 \rightarrow T_{\dots}{}^{\dots}| = 0 , \qquad \varepsilon(R_{\dots}{}^{\dots}) = 1 \rightarrow R_{\dots}{}^{\dots}| = 0 .$$

These conditions are supersymmetric.
At the component level, all spinor fields may be gauged away.

$d = 3$, $\mathcal{N} = 2$ supergravity

3D $\mathcal{N} = 2$ curved superspace, $\mathcal{M}^{3|4}$, parametrized by local coordinates
$z^M = (x^m, \theta^\mu, \bar\theta_\mu)$, $m = 0, 1, 2$ and $\mu = 1, 2$
Superspace structure group $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{U}(1)_R$
Superspace covariant derivatives

$$\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}^\alpha) = E_A + \Omega_A + \mathrm{i}\,\Phi_A \mathcal{J} \ .$$

Algebra of covariant derivatives

$$\begin{aligned} \{\mathcal{D}_\alpha, \mathcal{D}_\beta\} &= -4\bar{R}\mathcal{M}_{\alpha\beta} \ , \qquad \{\bar{\mathcal{D}}_\alpha, \bar{\mathcal{D}}_\beta\} = 4R\mathcal{M}_{\alpha\beta} \ , \\ \{\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta\} &= -2\mathrm{i}(\gamma^c)_{\alpha\beta}\mathcal{D}_c - 2\mathcal{C}_{\alpha\beta}\mathcal{J} - 4\mathrm{i}\varepsilon_{\alpha\beta}\mathcal{S}\mathcal{J} \\ &\quad +4\mathrm{i}\mathcal{S}\mathcal{M}_{\alpha\beta} - 2\varepsilon_{\alpha\beta}\mathcal{C}^{\gamma\delta}\mathcal{M}_{\gamma\delta} \ . \end{aligned}$$

$\mathcal{M}_{ab} = -\mathcal{M}_{ba} \longleftrightarrow \mathcal{M}_{\alpha\beta} = \mathcal{M}_{\beta\alpha}$ Lorentz generators
Dimension-1 torsion superfields: (i) real scalar $\mathcal{S}$; (ii) complex scalar R such that $\mathcal{J}R = -2R$; (iii) real vector $\mathcal{C}_a \longleftrightarrow \mathcal{C}_{\alpha\beta}$.
Bianchi Identities:

$$\bar{\mathcal{D}}_\alpha R = 0 \ , \qquad (\bar{\mathcal{D}}^2 - 4R)\mathcal{S} = 0 \quad \ldots$$

Conformal isometries

The conformal Killing supervector fields obey the equation

$$(\delta_{\mathcal{K}} + \delta_\sigma)\mathcal{D}_A = 0 \ ,$$

where

$$\delta_{\mathcal{K}}\mathcal{D}_A = [\mathcal{K}, \mathcal{D}_A] \ , \qquad \mathcal{K} = \xi^C \mathcal{D}_C + \frac{1}{2} K^{cd} \mathcal{M}_{cd} + \mathrm{i}\, \tau \mathcal{J}$$

and

$$\delta_\sigma \mathcal{D}_\alpha = \frac{1}{2}\sigma \mathcal{D}_\alpha + (\mathcal{D}^\gamma \sigma)\mathcal{M}_{\gamma\alpha} - (\mathcal{D}_\alpha \sigma)\mathcal{J} \ , \quad \dots$$

It suffices to require $(\delta_{\mathcal{K}} + \delta_\sigma)\mathcal{D}_\alpha = 0$, which implies

$$\xi^\alpha = -\frac{\mathrm{i}}{6}\bar{\mathcal{D}}_\beta \xi^{\beta\alpha} \ , \qquad K_{\alpha\beta} = \mathcal{D}_{(\alpha}\xi_{\beta)} - \bar{\mathcal{D}}_{(\alpha}\bar{\xi}_{\beta)} - 2\xi_{\alpha\beta}\mathcal{S}$$

$$\sigma = \frac{1}{2}\big(\mathcal{D}_\alpha \xi^\alpha + \bar{\mathcal{D}}^\alpha \bar{\xi}_\alpha\big) \ , \qquad \tau = -\frac{\mathrm{i}}{4}\big(\mathcal{D}_\alpha \xi^\alpha - \bar{\mathcal{D}}^\alpha \bar{\xi}_\alpha\big)$$

All parameters ξ^α, $K_{\alpha\beta}$, σ and τ are expressed in terms of ξ^a.

Conformal isometries

The remaining vector parameter ξ^a satisfies

$$\mathcal{D}_{(\alpha}\xi_{\beta\gamma)} = 0 \qquad (\star)$$

and its conjugate.
Implication: superfield analogue of the conformal Killing equation

$$\mathcal{D}_a\xi_b + \mathcal{D}_b\xi_a = \frac{2}{3}\eta_{ab}\mathcal{D}^c\xi_c \ .$$

Eq. $(\star)$ is fundamental in the sense that it implies $(\delta_{\mathcal{K}} + \delta_\sigma)\mathcal{D}_A \equiv 0$ provided the parameters ξ^α, $K_{\alpha\beta}$, σ and τ are defined as above.
The conformal Killing supervector field is a real supervector field

$$\xi = \xi^A E_A \ , \qquad \xi^A \equiv (\xi^a, \xi^\alpha, \bar{\xi}_\alpha) = \Big(\xi^a, -\frac{\mathrm{i}}{6}\bar{\mathcal{D}}_\beta\xi^{\beta\alpha}, -\frac{\mathrm{i}}{6}\mathcal{D}^\beta\xi_{\beta\alpha}\Big)$$

which obeys the master equation $(\star)$.
If ξ_1 and ξ_2 are two conformal Killing supervector fields, their Lie bracket $[\xi_1, \xi_2]$ is a conformal Killing supervector field.

Conformal isometries

Equation $(\delta_{\mathcal{K}} + \delta_\sigma)\mathcal{D}_\alpha = 0$ implies some additional results that have not been discussed above. Define

$$\Upsilon := (\xi^B, K^{\beta\gamma}, \tau)$$

It turns out that

- $\mathcal{D}_A \Upsilon$ is a linear combination of Υ, σ and $\mathcal{D}_C\sigma$;
- $\mathcal{D}_A\mathcal{D}_B\sigma$ can be represented as a linear combination of Υ, σ and $\mathcal{D}_C\sigma$.

The super Lie algebra of the conformal Killing vector fields on $\mathcal{M}^{3|4}$ is finite dimensional. The number of its even and odd generators cannot exceed those in the $\mathcal{N} = 2$ superconformal algebra $\mathfrak{osp}(2|4)$.

Charged conformal Killing spinors

Look for curved superspace backgrounds admitting at least one conformal supersymmetry. Such a superspace must possess a conformal Killing supervector field ξ^A with the property

$$\xi^a| = 0 \ , \qquad \epsilon^\alpha := \xi^\alpha| \neq 0 \ .$$

All other bosonic parameters are assumed to vanish, $\sigma| = \tau| = K_{\alpha\beta}| = 0$.
Bosonic superspace backgrounds without covariant fermionic fields:

$$\mathcal{D}_\alpha \mathcal{S}| = 0 \ , \qquad \mathcal{D}_\alpha R| = 0 \ , \qquad \mathcal{D}_\alpha \mathcal{C}_{\beta\gamma}| = 0 \ .$$

These conditions mean that the gravitini can completely be gauged away

$$\mathcal{D}_a| = \mathbf{D}_a := e_a + \frac{1}{2}\omega_a{}^{bc}\mathcal{M}_{bc} + \mathrm{i}b_a\mathcal{J} \equiv \mathfrak{D}_a + \mathrm{i}b_a\mathcal{J} \ , \qquad e_a := e_a{}^m\partial_m \ .$$

Introduce scalar and vector fields associated with the superfield torsion:

$$s := \mathcal{S}| \ , \qquad r := R| \ , \qquad c_a := \mathcal{C}_a| \ .$$

S-supersymmetry parameter: $\eta_\alpha := \mathcal{D}_\alpha\sigma|$.

Charged conformal Killing spinors

Q-supersymmetry parameter $\epsilon^\alpha := \xi^\alpha|$ obeys the equation

$$\mathbf{D}_a\epsilon^\alpha + \frac{\mathrm{i}}{2}(\tilde{\gamma}_a\bar{\eta})^\alpha + \mathrm{i}\varepsilon_{abc}\, c^b(\tilde{\gamma}^c\epsilon)^\alpha - s(\tilde{\gamma}_a\epsilon)^\alpha - \mathrm{i}r(\tilde{\gamma}_a\bar{\epsilon})^\alpha = 0~,$$

which is equivalent to

$$\Big(\mathbf{D}_{(\alpha\beta} - \mathrm{i}c_{(\alpha\beta}\Big)\epsilon_{\gamma)} = \Big(\mathfrak{D}_{(\alpha\beta} - \mathrm{i}(b+c)_{(\alpha\beta}\Big)\epsilon_{\gamma)} = 0~,$$

$$\bar{\eta}_\alpha = -\frac{2\mathrm{i}}{3}\Big((\gamma^a\mathbf{D}_a\epsilon)_\alpha + 2\mathrm{i}(\gamma^a\epsilon)_\alpha c_a + 3s\epsilon_\alpha + 3\mathrm{i}r\bar{\epsilon}_\alpha\Big)~.$$

This follows by bar-projecting the equation ($\boldsymbol{C}_{\alpha\beta\gamma} = -\mathrm{i}\mathcal{D}_{(\alpha}\mathcal{C}_{\beta\gamma)}$)

$$0 = \mathcal{D}_a\xi_\alpha + \frac{\mathrm{i}}{2}(\gamma_a)_\alpha{}^\beta\bar{\mathcal{D}}_\beta\sigma - \mathrm{i}\varepsilon_{abc}(\gamma^b)_\alpha{}^\beta\mathcal{C}^c\xi_\beta - (\gamma_a)_\alpha{}^\beta(\xi_\beta\mathcal{S} + \bar{\xi}_\beta R)$$
$$-\frac{1}{2}\varepsilon_{abc}\xi^b(\gamma^c)^{\beta\gamma}\Big(\mathrm{i}\bar{\boldsymbol{C}}_{\alpha\beta\gamma} - \frac{4\mathrm{i}}{3}\varepsilon_{\alpha(\beta}\bar{\mathcal{D}}_{\gamma)}\mathcal{S} - \frac{2}{3}\varepsilon_{\alpha(\beta}\mathcal{D}_{\gamma)}R\Big)~,$$

which is one of the implications of $(\delta_{\mathcal{K}} + \delta_\sigma)\mathcal{D}_a = 0$.

Supersymmetric backgrounds

Rigid supersymmetry transformations (in super-Weyl gauge $\phi = 1$) are characterised by

$$\sigma[\xi] = 0 \quad \Longrightarrow \quad \eta_\alpha = 0 ,$$

The conformal Killing spinor equation turns into

$$\mathbf{D}_a \epsilon^\alpha = -\mathrm{i}\varepsilon_{abc} c^b (\tilde{\gamma}^c \epsilon)^\alpha + s(\tilde{\gamma}_a \epsilon)^\alpha + \mathrm{i} r(\tilde{\gamma}_a \bar{\epsilon})^\alpha .$$

$$\mathbf{D}_a = e_a + \frac{1}{2}\omega_a{}^{bc}\mathcal{M}_{bc} + \mathrm{i} b_a \mathcal{J} = \mathfrak{D}_a + \mathrm{i} b_a \mathcal{J} .$$

$$[\mathbf{D}_a, \mathbf{D}_b] = \frac{1}{2}\mathcal{R}_{ab}{}^{cd}\mathcal{M}_{cd} + \mathrm{i}\mathcal{F}_{ab}\mathcal{J} = [\mathfrak{D}_a, \mathfrak{D}_b] + \mathrm{i}\mathcal{F}_{ab}\mathcal{J} .$$

Supersymmetric backgrounds with four supercharges

The existence of rigid supersymmetries imposes non-trivial restrictions on the background fields. In the case of four supercharges, these are

$$\mathfrak{D}_a s = 0 \ , \qquad \mathfrak{D}_a r = 2\mathrm{i} b_a r \ , \qquad \mathfrak{D}_a c_b = 2\varepsilon_{abc} c^c s \ ,$$
$$r\, s = 0 \ , \qquad r\, c_a = 0 \ .$$

c_a is a Killing vector field,

$$\mathfrak{D}_a c_b + \mathfrak{D}_b c_a = 0 \ .$$

The $\mathrm{U}(1)_R$ field strength proves to vanish, $\mathcal{F}_{ab} = 0$.
The Einstein tensor $\mathcal{G}_{ab} := \mathcal{R}^{ab} - \frac{1}{2}\eta^{ab}\mathcal{R}$ is

$$\mathcal{G}_{ab} = 4\Big[c_a c_b + \eta_{ab}\big(s^2 + \bar{r} r\big)\Big] \ .$$

For the Cotton tensor $\mathcal{W}_{ab} := \frac{1}{2}\varepsilon_{acd}\mathcal{W}^{cd}{}_b = \mathcal{W}_{ba}$, with $\mathcal{W}_{abc} = 2\mathfrak{D}_{[a}\mathcal{R}_{b]c} + \frac{1}{2}\eta_{c[a}\mathfrak{D}_{b]}\mathcal{R}$, we obtain

$$\mathcal{W}_{ab} = -24 s\Big[c_a c_b - \frac{1}{3}\eta_{ab} c^2\Big] \ .$$

Compensators

Type I supergravity: Chiral compensator

$$\bar{\mathcal{D}}_\alpha \Phi = 0 , \qquad \delta_\sigma \Phi = \frac{1}{2}\sigma\Phi , \qquad \mathcal{J}\Phi = -\frac{1}{2}\Phi .$$

The freedom to perform the super-Weyl and local $\mathrm{U}(1)_R$ transformations can be used to impose the gauge $\Phi = 1$.
Consistency conditions:

$$\mathcal{S} = 0 , \qquad \Phi_\alpha = 0 , \qquad \Phi_{\alpha\beta} = \mathcal{C}_{\alpha\beta} .$$

Supersymmetric backgrounds with four supercharges:

$$r\, c_a = 0 , \qquad \mathfrak{D}_a r = 0 , \qquad \mathfrak{D}_a c_b = 0 .$$

Such spacetimes are necessarily conformally flat,

$$\mathcal{W}_{ab} = 0 .$$

Solution with $c_a = 0$ corresponds to (1,1) AdS superspace.

Compensators

Type II supergravity: Real linear compensator

$$\bar{\mathcal{D}}^2\mathbb{G} = 0 , \qquad \delta_\sigma \mathbb{G} = \sigma \mathbb{G} .$$

Super-Weyl invariance allows us to choose the gauge $\mathbb{G} = 1$.
Consistency conditions:

$$R = \bar{R} = 0 .$$

All supersymmetric backgrounds with four supercharges:

$$\mathfrak{D}_a s = 0 , \qquad \mathfrak{D}_a c_b = 2\varepsilon_{abc} c^c s .$$

The Cotton tensor

$$\mathcal{W}_{ab} = -24 s\Big[c_a c_b - \frac{1}{3}\eta_{ab} c^d c_d\Big] = -6c\Big[\mathcal{R}_{ab} - \frac{1}{3}\eta_{ab}\mathcal{R}\Big]$$

Solution with $c_a = 0$ corresponds to (2,0) AdS superspace.

General feature of maximally supersymmetric backgrounds

For any background admitting four supercharges, if there exists a tensor superfield T such that its bar-projection vanishes, $T| = 0$, and this condition is supersymmetric, then the entire superfield is zero, $T = 0$.

Supersymmetric conditions

$$\mathcal{D}_\alpha \mathcal{S}| = 0 \ , \qquad \mathcal{D}_\alpha R| = 0 \ , \qquad \mathcal{D}_\alpha \mathcal{C}_{\beta\gamma}| = 0 \ .$$

imply

$$\mathcal{D}_\alpha \mathcal{S} = 0 \ , \qquad \mathcal{D}_\alpha R = 0 \ , \qquad \mathcal{D}_\alpha \mathcal{C}_{\beta\gamma} = 0 \ .$$

Maximally supersymmetric backgrounds in Type I SUGRA

Dimension-1 torsion superfields

$$\mathcal{S} = 0 \ , \qquad R\,\mathcal{C}_a = 0 \ , \qquad \mathcal{D}_A R = 0 \ , \qquad \mathcal{D}_A \mathcal{C}_b = 0$$

are covariantly constant.

Algebra of covariant derivatives

$$\begin{aligned}
\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} &= -4\bar{R}\mathcal{M}_{\alpha\beta} \ , \qquad \{\bar{\mathcal{D}}_\alpha, \bar{\mathcal{D}}_\beta\} = 4R\mathcal{M}_{\alpha\beta} \ , \\
\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta\} &= -2\mathrm{i}(\gamma^c)_{\alpha\beta}\mathcal{D}_c - 2\mathcal{C}_{\alpha\beta}\mathcal{J} - 2\varepsilon_{\alpha\beta}\mathcal{C}^{\gamma\delta}\mathcal{M}_{\gamma\delta} \ , \\
[\mathcal{D}_a, \mathcal{D}_\beta] &= \mathrm{i}\varepsilon_{abc}(\gamma^b)_\beta{}^\gamma\mathcal{C}^c\mathcal{D}_\gamma - \mathrm{i}(\gamma_a)_{\beta\gamma}\bar{R}\bar{\mathcal{D}}^\gamma \ , \\
[\mathcal{D}_a, \bar{\mathcal{D}}_\beta] &= -\mathrm{i}\varepsilon_{abc}(\gamma^b)_\beta{}^\gamma\mathcal{C}^c\bar{\mathcal{D}}_\gamma - \mathrm{i}(\gamma_a)_\beta{}^\gamma R\mathcal{D}_\gamma \ , \\
[\mathcal{D}_a, \mathcal{D}_b] &= 4\varepsilon_{abc}\Big(\delta^c{}_d\bar{R}R + \mathcal{C}^c\mathcal{C}_d\Big)\mathcal{M}^d \ .
\end{aligned}$$

4 different superalgebras.