Dubna
sept. 14

NEW IDENTITIES FOR JACOBI ELLIPTIC FUNCTIONS & DISCRETE NONLINEAR EQUATIONS

AVINASH KHARE

IISER PUNE (INDIA)

khare@iiserpune.ac.in

Linear Eqs. $\Rightarrow$ superposition principle

$$\psi_1, \psi_2 \Rightarrow a_1\psi_1 + a_2\psi_2$$

Nonlinear eqn $\Rightarrow$ No superposition!

$$-i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\vec{\nabla}^2\psi + V\psi + g|\psi|^2\psi$$

$$\psi_1, \psi_2 \not\Rightarrow a_1\psi_1 + a_2\psi_2$$

why? $a_1^2 a_2 |\psi_1|^2 \psi_2 + a_2^2 a_1 |\psi_2|^2 \psi_1$

<u>Our claim</u>

A kind of linear superposition does work!

If periodic sols. are in terms of Jacobi elliptic functions

Not a serious restriction!

KdV, mKdV, NLS, SG, $\lambda\phi^4$...

<u>Why does it work?</u>

Jacobi Elliptic Functions satisfy highly unusual identities

Bill Reinhardt (Univ. of Washington, Seattle)

Handbook of Mathematical Functions (Abromowitz & Stegun)

http://dlmf.nist.gov/22 22.9, 22.7

Mathematica

Kim Rasmussen (Los Alamos Lab.)

Identities are (almost) solutions of discrete Nonlinear Eqs.!

Exact sols. of Saturable discrete Nonlinear Schrödinger Equation!

Another Form of Linear Superposition

$cn, dn \Rightarrow dn \pm cn$; $dn^2 \Rightarrow dn^2 \pm cn\, dn$

PLAN

JEF: Brief Introduction

Why we thought (LS for Nonlinear eqs.)

$\lambda \phi^4$ in 1+1

Structure of Identities

Saturable DNLS

Open Problems

Ref. U. Sukhatme & AK, JMP 43 (2002) 3798

US, AK & A. Lakshminarayan, JMP 44 (2003) 1822;

Pramana 62 (2004) 1201

Jacobi Elliptic Functions

$sn(x,m)$ $cn(x,m)$ $dn(x,m)$

$0 \le m \le 1$

m = 0

$\sin x$ $\cos x$ 1

m = 1

$\tanh x$ $\operatorname{sech} x$ $\operatorname{sech} x$

Identities

$$sn^2(x,m) + cn^2(x,m) = 1 = dn^2(x,m) + m\, sn^2(x,m)$$

Derivative $\frac{d}{dx} f(x)$

$cn(x,m)\,dn(x,m)$ $-sn(x,m)\,dn(x,m)$ $-m\,sn(x,m)\,cn(x,$

$x \to -x$

$sn(-x,m) = -sn(x,m)$ $cn(-x,m) = cn(x,m)$

$dn(-x,m) = dn(x,m)$

Doubly Periodic

$4K(m),\ 2iK(1-m)$ $4K(m),\ 4iK(1-m)$ $2K(m),\ 4iK(1-m)$

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}, \quad K(0) = \frac{\pi}{2}, \quad K(1) = \infty$$

Pole structure

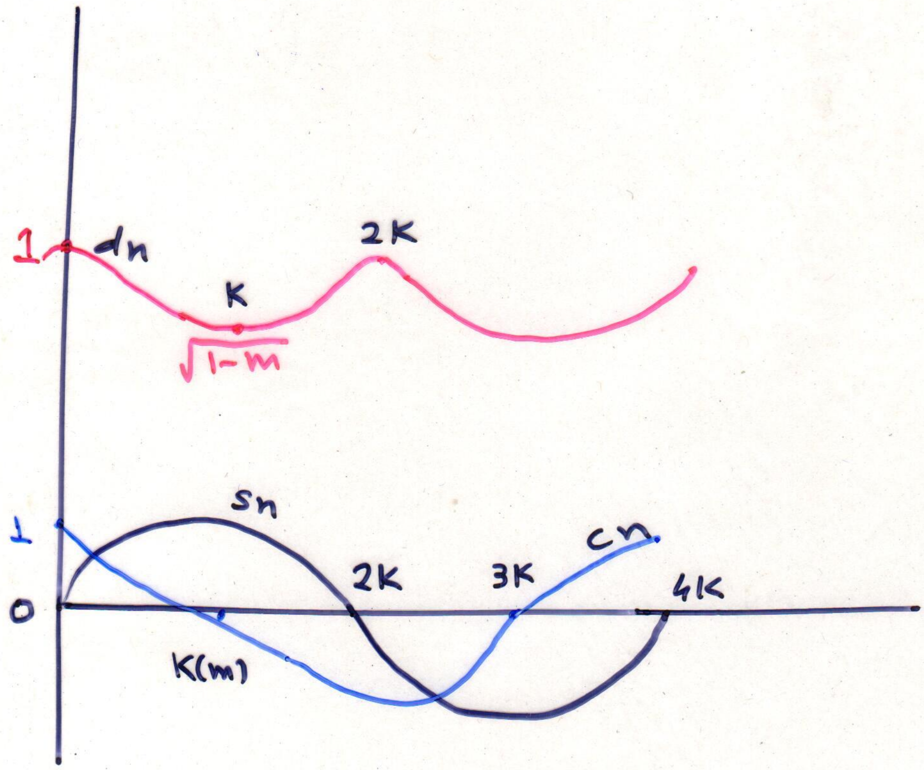
Only poles at $x = iK(1-m)$, $x = 2K(m) + iK(1-m)$ with equal & opposite residue

Addition Theorem

$$\mathrm{sn}(a+b) = \frac{\mathrm{sn}a\,\mathrm{cn}b\,\mathrm{dn}b + \mathrm{sn}b\,\mathrm{cn}a\,\mathrm{dn}a}{1 - m\,\mathrm{sn}^2a\,\mathrm{sn}^2b}$$

$$\mathrm{cn}(a+b) = \frac{\mathrm{cn}a\,\mathrm{cn}b - \mathrm{sn}b\,\mathrm{dn}b\,\mathrm{sn}a\,\mathrm{dn}a}{1 - m\,\mathrm{sn}^2a\,\mathrm{sn}^2b}$$

$$\mathrm{dn}(a+b) = \frac{\mathrm{dn}a\,\mathrm{dn}b - m\,\mathrm{sn}b\,\mathrm{cn}b\,\mathrm{sn}a\,\mathrm{cn}a}{1 - m\,\mathrm{sn}^2a\,\mathrm{sn}^2b}$$



$m \approx 0.5$

Pendulam

Why did we think about LS for Nonlinear Eqs.?

Susy QM for Periodic Potentials (1999)

$\downarrow$

band structure

Kronig-Penney Model $\quad \delta(x)$

∞ no. of bands & ∞ no. of band gaps.

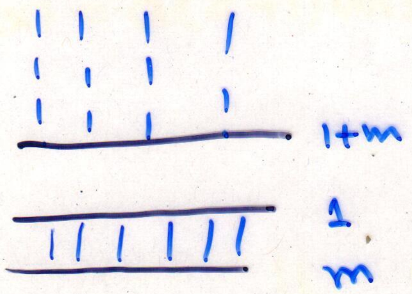
Lame Potentials ... Unusual band structure

$$V(x) = n(n+1)\, m\, \mathrm{sn}^2(x,m) \;; \quad n = 1, 2, 3, \ldots \quad 0 < m < 1$$

finite (n) no. of band gaps!

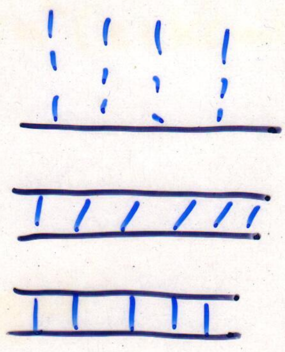
$n=1$

$V = 2m\,\mathrm{sn}^2(x,m)$



$n=2$

$V = 6m\,\mathrm{sn}^2(x,m)$



Dispersion Relation: Supersymmetry in Q.M.

Cooper, Khare & Sukhatme

Remarkable Observation

KdV eq. ··· Integrable Eq.

$$\frac{\partial u(x,t)}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

1 soliton sol.

$$u(x,t) = -2\,\mathrm{sech}^2(x-4t) \qquad \mathrm{sech}\,x = \frac{1}{\cosh x}$$

$\downarrow$

reflectionless potential in QM

$R(k) = 0$

Periodic 1 soliton sol.

$$u(x,t) = 2m\,\mathrm{sn}^2(x-bt, m)$$

$\downarrow$

one band gap!

Associated Lamé Eq. (1999)

$$V(x) = a(a+1)m\,\mathrm{sn}^2(x,m) + b(b+1)m\,\mathrm{sn}^2(x+K(m),m)$$

$\downarrow a=b=1$

$$V = 2m\,\mathrm{sn}^2(x,m) + 2m\,\mathrm{sn}^2(x+K(m),m)$$

$\Downarrow$ has only 1 band gap

Q.: Is it a sol. of KdV eq.?. $\boxed{\text{YES}}$

Curiosity

Is

$$V = 2m\left[sn^2(x,m) + sn^2\left[x + \frac{2K(m)}{3}, m\right] + sn^2\left[x + \frac{4K(m)}{3}, m\right]\right]$$

also a sol. of KdV eq.?

$\boxed{\text{YES}}$ $\therefore$ 1 band gap!!

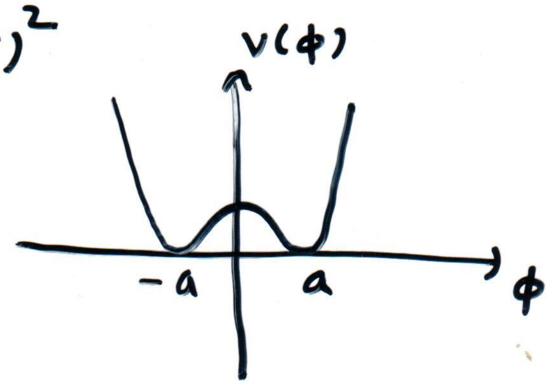
* $sn(x,m)$ must satisfy nontrivial identity!

Not known Before!

Illustration $\lambda\phi^4$ in 1+1

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{\lambda}{4}(\phi^2-a^2)^2$$

$$\phi_{xx} - \phi_{tt} = \lambda\phi(\phi^2-a^2)$$



Static kink sol.

$$\phi(x) = \pm\, a \tanh\left[\sqrt{\frac{\lambda}{2}}\, a(x+x_0)\right]$$

Periodic kink sol.

$$\phi(x) = \pm\sqrt{\frac{2m}{1+m}}\, a\, \mathrm{sn}\left[\sqrt{\frac{\lambda}{1+m}}\, a x + x_0, m\right]$$

Claim

$$\phi(x) = \pm\sqrt{2m}\, a\alpha \sum_{i=1}^{p} s_i, \quad p \text{ odd} \quad \ldots \text{also a sol.}$$

$$s_i \equiv \mathrm{sn}\left[y + \frac{4(i-1)K(m)}{p}, m\right], \quad y = \sqrt{\lambda}\, a\alpha x$$

Note: each term in the sum is exact sol. with $\alpha = \frac{1}{\sqrt{1+m}}$

Proof $p = 3$

$$\phi(x) = \pm\sqrt{2m}\, a\alpha\,[s_1 + s_2 + s_3]$$

$$s_1 \equiv \mathrm{sn}\, y, \quad s_2 \equiv \mathrm{sn}\left(y + \frac{4K(m)}{3}\right), \quad s_3 \equiv \mathrm{sn}\left(y + \frac{8K(m)}{3}\right)$$

$$\phi_{xx} = \lambda\phi(\phi^2 - a^2)$$

$$\phi = s_1 + s_2 + s_3 \qquad \alpha = \frac{1}{\sqrt{1+m}}$$

$$\phi^3 = (s_1+s_2+s_3)^3 = s_1^3 + s_2^3 + s_3^3$$

$$+3\left[\underbrace{s_1^2(s_2+s_3) + s_2^2(s_3+s_1) + s_3^2(s_1+s_2)}_{= A(s_1+s_2+s_3)}\right] + \underbrace{6 s_1 s_2 s_3}_{= B(s_1+s_2+s_3)}$$

$$\alpha = \frac{1}{\sqrt{1+m+6mA+12mB}} \overset{p=3}{=} \frac{1}{\sqrt{1+m+\frac{6m}{1-q^2} - 6(1-q^2)}}$$

$$q \equiv dn\left[\frac{2K(m)}{3}, m\right]$$

Q. Are these new sols. of $\lambda\phi^4$?

$\downarrow$

No! $\Rightarrow$ Landen Transformation!

$$sn(x, \tilde{m}) = \beta \sum_{j=1}^{p} sn\left[\alpha_1 x + \frac{4(j-1)K(m)}{p}, m\right], \quad p \text{ odd}$$

Lawden: Elliptic Functions & applications

Abromowitz-Stegen: only $p=2$ Landen!

hundreds of such identitis

Rank 2 identitis

$$\sum_{j=1}^{p} d_j\, d_{j+1} = \frac{A}{2}\,, \quad d_j \equiv dn\left[x + \frac{2(j-1)K(m)}{p}, m\right]$$

$p = 3$

$$dn\,x\, dn\left[x + \frac{2K(m)}{3}\right] + dn\left[x + \frac{2K(m)}{3}\right] dn\left[x + \frac{4K(m)}{3}\right]$$

$$+ dn\left[x + \frac{4K(m)}{3}\right] dn\,x$$

$p = 2, 3, 4 \ldots$ explicit proof

$p \geqslant 5 \ldots$ numerical to 6-7 decimal places using Maple

Arul Lakshminarayan [PRL, Ahmadabad IIT, Chennai]

Proof by Poisson summation formula (Liouville theorem) for arbitrary p!

Also proof using local identitis!

$$\sum_{j=1}^{p} d_j^2(x,m)\left[d_{j+1} + d_{j-1}\right] = A \sum_{i=1}^{p} d_i$$

$$d_j \equiv dn\left[x + \frac{2(j-1)K(m)}{p}, m\right]$$

Liouville Theorem

An elliptic function $f(z)$ with no poles in a cell, is merely a constant.

$$\sum_{j=1}^{p} d_j d_{j+1} = \frac{A}{2}, \quad d_j \equiv \mathrm{dn}\left[x + \frac{2(j-1)K(m)}{p}, m\right].$$

has terms like

$$\mathrm{dn}\, x \left[\mathrm{dn}\left(x + \frac{2K(m)}{p}\right) + \mathrm{dn}\left(x - \frac{2K(m)}{p}\right)\right]$$

pole at $x = iK(1-m)$

has zero at $x = iK(1-m)$

$$\frac{A}{2} = \frac{p}{2K(m)} \int_0^{2K(m)} \mathrm{dn}[x, m]\, \mathrm{dn}\left[x + \frac{2K(m)}{p}, m\right] dx$$

$$= p\left[\mathrm{dn}\left(\frac{2K(m)}{p}\right) - \frac{\mathrm{cn}(2K/p)}{\mathrm{sn}(2K/p)} Z\left(\frac{2K}{p}\right)\right]$$

Local Identities of Arbitrary Rank

Simpler Proof

$$\mathrm{dn}^2 x\,[\mathrm{dn}(x+a)+\mathrm{dn}(x-a)] = A\,\mathrm{dn}\,x + B[\mathrm{dn}(x+a)+\mathrm{dn}(x-a)]$$

$$A = 2\,\frac{\mathrm{dn}\,a}{\mathrm{sn}^2 a} \quad ; \quad B = -\frac{\mathrm{cn}^2 a}{\mathrm{sn}^2 a}$$

Multiply by $\mathrm{dn}^2 x$

$$\mathrm{dn}^4 x\,[\mathrm{dn}(x+a)+\mathrm{dn}(x-a)] = A\,\mathrm{dn}^3 x$$

$$+ B\Big[A\,\mathrm{dn}\,x + B\big(\mathrm{dn}(x+a)+\mathrm{dn}(x-a)\big)\Big]$$

$\vdots$

$$\mathrm{dn}^{2n} x\,[\mathrm{dn}(x+a)+\mathrm{dn}(x-a)] = B^n\,[\mathrm{dn}(x+a)+\mathrm{dn}(x-a)]$$

$$+ A\sum_{j=1}^{n} B^{j-1}\,(\mathrm{dn}\,x)^{2(n-j)+1} \qquad ; \quad n = 1, 2, \cdots$$

Add p such identities with $x \to x+a, x+2a, \cdots$ and choose $a = 2K/p$

$$\sum_{j=1}^{p} d_j^{2n}\,[d_{j+1}+d_{j-1}] = 2B^n \sum_{j=1}^{p} d_j + A\sum_{j=1}^{p}\sum_{k=1}^{n} B^{k-1}\, d_j^{2(n-k)+1}$$

on multiplying by $1, \omega, \cdots, \omega^{p-1}$, $\omega \equiv e^{2i\pi/s}$

$p = 0 \pmod{s}$

$$\sum_{j=1}^{p} \omega^{j-1}\, d_j^{2n}\,(d_{j+1}+d_{j-1}) = 2B^n \cos\left(\frac{2\pi}{s}\right) \sum_{j=1}^{p} \omega^{j-1} d_j$$

$$+ A\sum_{j=1}^{p}\sum_{k=1}^{n} \omega^{j-1}\, B^{k-1}\, d_j^{2(n-k)+1}$$

Structure of Identities

If l.h.s. $= f(z)$, satisfies one of 4 possibilities

$$f(z+2K) = \pm f(z), \quad f(z+2iK') = \pm f(z)$$

dn	dn^2	sn	cn
$+-$	$++$	$-+$	$--$
I	II	III	IV

$\therefore$ r.h.s. will always be of the <u>same</u> type.

$\therefore$ Constant can only appear in identity of type II.

<u>Identity of type I</u>

r.h.s. is linear combination of dn's and its various derivatives.

$\downarrow$
$dn^{2k+1}(x, m)$

<u>Note</u>

$Z(z) = E(z) - z\frac{E}{K}$, $\frac{dZ(z)}{dz} = dn^2(z) - \frac{E}{K}$

$\downarrow$

almost elliptic!

$$Z(z+2iK') = Z(z) - \frac{i\pi}{K}$$

$dn^2 x[dn(x+a) \pm dn(x-a)]$ ..... $dn x$, $dn(x \pm a)$ & their derivatives!

Some of these identities are already known!

$$\sum_{j=1}^{p} d_j\, d_{j+1} = pA/2$$

Poristic polygons of Poncelet

$$d_1 d_2 d_3 = B(d_1 + d_2 + d_3)$$

$$d_1 \equiv \operatorname{dn} x, \quad d_2 \equiv \operatorname{dn}\left(x + \frac{2K(m)}{3}\right), \quad d_3 \equiv \operatorname{dn}\left(x + \frac{4K(m)}{3}\right)$$

$$d_j \equiv \operatorname{dn}\left[x + \frac{2(j-1)K(m)}{p}, m\right]$$

Ref: G. H. Halpern : Traité des Fonctions Elliptiques (1859)

$$B = \left[\frac{1}{\operatorname{sn}^2\left(\frac{2K(m)}{3}, m\right)} - 1\right]$$

Identitis For Ratios of Jacobi Θ functions

$$\mathrm{sn}(x,m) = \frac{1}{m^{1/4}} \frac{\theta_1(z,\tau)}{\theta_4(z,\tau)}, \qquad z = \frac{\pi x}{2K(m)}$$

$$\mathrm{cn}(x,m) = \left(\frac{1-m}{m}\right)^{1/4} \frac{\theta_2(z,\tau)}{\theta_4(z,\tau)} \qquad \tau = \frac{iK'(m)}{K(m)}$$

$$\mathrm{dn}(x,m) = (1-m)^{1/4} \frac{\theta_3(z,\tau)}{\theta_4(z,\tau)}$$

Shift	Elliptic fns.	Θ fns.
	$\frac{2K}{p}$	$\frac{\pi}{p}$
	$\frac{2iK'}{p}$	$\frac{\tau\pi}{p}$
	$\frac{2(K+iK')}{p}$	$(1+\tau)\frac{\pi}{p}$

$$\prod_{j=1}^{p} d_j = \prod_{n=1}^{\frac{p-1}{2}} \mathrm{cs}^2\left(\frac{2Kn}{p}\right) \sum_{j=1}^{p} d_j \quad , \quad p \cdots \text{odd}$$

$$\downarrow$$

$$\prod_{j=1}^{p} \frac{\theta_3(z+(j-1)\pi/p)}{\theta_4(z+(j-1)\pi/p)} = \left[\prod_{n=1}^{\frac{p-1}{2}} \frac{\theta_2^2(2nK/p)}{\theta_1^2(2nK/p)}\right] \sum_{j=1}^{p} \frac{\theta_3(z+(j-1)p)}{\theta_4(z+(j-1)p)}$$

Identities for Weierstrass $\wp$-function

$$\wp'^2(u) = 4\wp^3(u) - g_2\wp(u) - g_3.$$

$$\equiv 4(\wp(u)-e_1)(\wp(u)-e_2)(\wp(u)-e_3)$$

$$\mathrm{dn}^2[\sqrt{e_1-e_3}\,u] = \frac{\wp(u)-e_2}{\wp(u)-e_3},$$

$$\boxed{\begin{array}{l} e_1 > e_2 > e_3 \text{ if} \\ g_2^3 - 27g_3^2 > 0 \end{array}}$$

$$\sum_{j=1}^{p} d_j^2\, d_{j+1}^2 = A\sum_{j=1}^{p} d_j^2 + B$$

$$d_j \equiv \mathrm{dn}\left[x + \frac{2(j-1)K(m)}{p}, m\right]$$

$$\sum_{j=1}^{p} \wp\left(u + \frac{2(j-1)}{p}\omega_1\right)\wp\left(u + \frac{2j\omega_1}{p}\right)$$

$$= B + pAe_1 - pe_1^2 - (A-2e_1)\sum_{j=1}^{p}\wp\left(u + \frac{2(j-1)\omega_1}{p}\right)$$

$$\wp \equiv \wp(u,\omega_1,\omega_3), \quad \omega_1 = \frac{K(m)}{\sqrt{e_1-e_3}}, \quad \omega_3 = \frac{iK'(m)}{\sqrt{e_1-e_2}}$$

Application to Discrete Nonlinear Eqs.

What are discrete models?

$$\frac{d^2\phi}{dx^2} \longrightarrow \frac{\phi_{n+1} + \phi_{n-1} - 2\phi_n}{h^2}$$

differential eq. $\longrightarrow$ difference eq.

NLS

$$i\,\frac{\partial u(x,t)}{\partial t} + \frac{\partial^2 u}{\partial x^2} \pm 2|u|^2 u = 0$$

$+ \ldots$ bright soliton

DNLS

$$i\,\frac{\partial u_n}{\partial t} + \frac{u_{n+1} + u_{n-1} - 2u_n(t)}{h^2} + f(u_{n+1}, u_n, u_{n-1}) =$$

$\downarrow$ continuum lim

$\pm 2|u|^2 u$

Normal choice

$$f = 2|u_n|^2 u_n$$

Optical waveguides, B.E. condensation, propagation of e.m. waves in glass fibers

"Integrable choice" (Ablowitz - Ladik)

replace $2|u_n|^2 u_n$ with

$$\downarrow$$

$$|u_n|^2 (u_{n+1} + u_{n-1})$$

Saturable DNLS

replace $2|u_n|^2 u_n$ with

$$\downarrow$$

$$\frac{2|u_n|^2 u_n}{1+\mu |u_n|^2}$$

A popular model in quantum optis.
Useful in the context of
"Optical Pulse propagation in various doped fibers"

Another solution

$dn \rightleftarrows cn$ } Sols. for finite lattice.

Sol. for infinite lattice

$m \to 1$, $N_p \to \infty$

$$u_n = \frac{\sinh\beta}{\sqrt{\mu}} \operatorname{sech}[\beta(n+\delta_1)]$$

where

$$\frac{2\mu}{g} = \operatorname{sech}\beta$$

Stability of Solutions

$N_p > 4$ stable solutions

H & $P = \sum |\psi_n|^2$ conserved.

By Now have examined several discrete nonlinear eq. (relevant in Physics problems).

$$i\frac{\partial\psi_n}{\partial t}+\psi_{n+1}+\psi_{n-1}-2\psi_n+\frac{g|\psi_n|^2\psi_n}{1+\mu|\psi_n|^2}=0$$

JPA 38 (2005) 807

$\boxed{\text{Sept. 26/27, 2004}}$

$$\psi_n = e^{-i(\omega t+\delta)} u_n$$

$$(\omega-2)u_n+[(\omega-2)\mu+g]u_n^3+(1+\mu u_n^2)(u_{n+1}+u_{n-1})=$$

$$\mu u_n^2[u_{n+1}+u_{n-1}]=-(u_{n+1}+u_{n-1})+(2-\omega)u_n$$

$$-\underbrace{[(\omega-2)\mu+g]}_{=0}u_n^3$$

Local Identity

$$dn^2x[dn(x+a)+dn(x-a)]=-\frac{cn^2a}{sn^2a}[dn(x+a)+dn(x-a)]$$

$$+2\frac{dn\,a}{sn^2a}dn(x,m)$$

$\therefore$ Exact Solution is

$$u_n=\frac{1}{\sqrt{\mu}}\frac{sn(\beta,m)}{cn(\beta,m)}dn[\beta(n+\delta_1),m]$$

with

$$g=(2-\omega)\mu\,;\quad \frac{2\mu}{g}=\frac{cn^2(\beta,m)}{dn(\beta,m)},$$

$$N_p\beta=2K(m)$$

Another Form of Linear Superposition

A. Saxena & AK : PLA **377** (2013) 2761
JMP **55** (2014) 032701

Claims

① If $cn(x,m)$ & $dn(x,m)$ sol.

Then so is $dn(x,m) \pm \sqrt{m}\, cn(x,m)$

② If $dn^2(x,m)$ sol.

Then so is $dn^2(x,m) \pm \sqrt{m}\, dn(x,m)\, cn(x,m)$

(i) $u_n = A\, dn[\beta(n+\delta_1), m]$

$$A^2 = \frac{sn^2(\beta,m)}{cn^2(\beta,m)}, \quad \frac{2\mu}{g} = \frac{cn^2\beta}{dn\beta}$$

(ii) $u_n = A\sqrt{m}\, cn[\beta(n+\delta_1), m]$

$$A^2 = \frac{sn^2(\beta,m)}{dn^2(\beta,m)}, \quad \frac{2\mu}{g} = \frac{dn^2\beta}{cn\beta}$$

(iii) $u_n = \frac{A}{2} dn[\beta(n+\delta_1),m] + \frac{B}{2}\sqrt{m}\, cn[\beta(n+\delta_1),m]$

$$B = \pm A, \quad A^2 = \frac{4\, sn^2(\beta,m)}{[cn(\beta,m) + dn(\beta,m)]^2},$$

$$\frac{2\mu}{g} = \frac{2}{cn\beta + dn\beta}$$

NLS

$$i u_t + u_{xx} + g|u|^2 u = 0$$

(i) $u = A\,\mathrm{dn}[\beta(x - vt + \delta_1), m]\, e^{-i(\omega t - kx + \delta)}$

$gA^2 = 2\beta^2, \; v = 2k, \quad \omega = k^2 - (2-m)\beta^2$

(ii) $u = A\sqrt{m}\,\mathrm{cn}[\beta(x - vt + \delta_1)]\, e^{-i(\quad)}$

$gA^2 = 2\beta^2, \; v = 2k, \quad \omega = k^2 - (2m-1)\beta^2$

(iii) $u = \left[\frac{A}{2}\mathrm{dn} + \frac{B}{2}\sqrt{m}\,\mathrm{cn}\right] e^{-i(\quad)}$

$B = \pm A, \; gA^2 = 2\beta^2, \; v = 2k, \quad \omega = k^2 - \frac{1+m}{2}\beta^2$

KdV

$$u_t + u_{xxx} + g u u_x = 0$$

(i) $u = A\,\mathrm{dn}^2[\beta(x - vt + \delta_1), m]$

$gA = 12\beta^2, \quad v = 4(2-m)\beta^2$

(ii) $u = \frac{A}{2}\mathrm{dn}^2[\quad] + \frac{B}{2}\sqrt{m}\,\mathrm{dn}[\quad]\,\mathrm{cn}[\quad]$

$B = \pm A, \; gA = 12\beta^2, \quad v = (5-m)\beta^2$

Note: cn dn not a sol. of KdV eq!

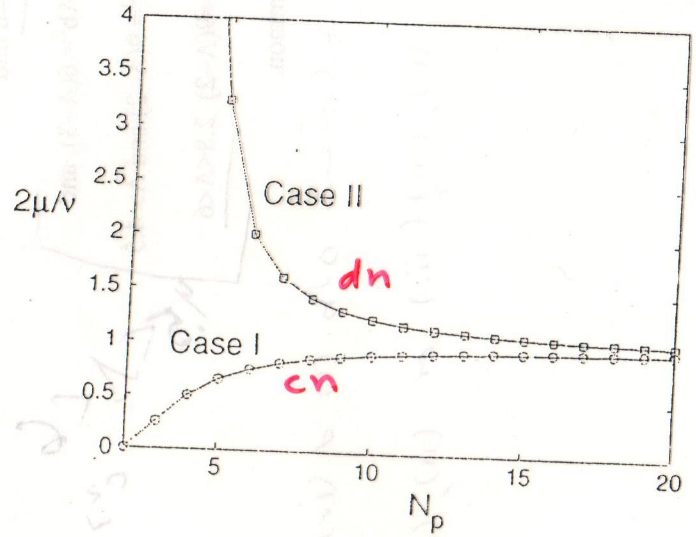


Figure 2. Illustration of parameter values μ, ν and N_p for which the exact solutions are allowed. Case I: $2\mu/\nu$ between 0 and $\cos^2\frac{\pi}{N_p}$ and $N_p \geqslant 3$. Case II: $2\mu/\nu$ between 0 and $1/\cos^2\frac{2\pi}{N_p}$ and $N_p \geqslant 3$ except for $N_p = 4$.

dn

cn

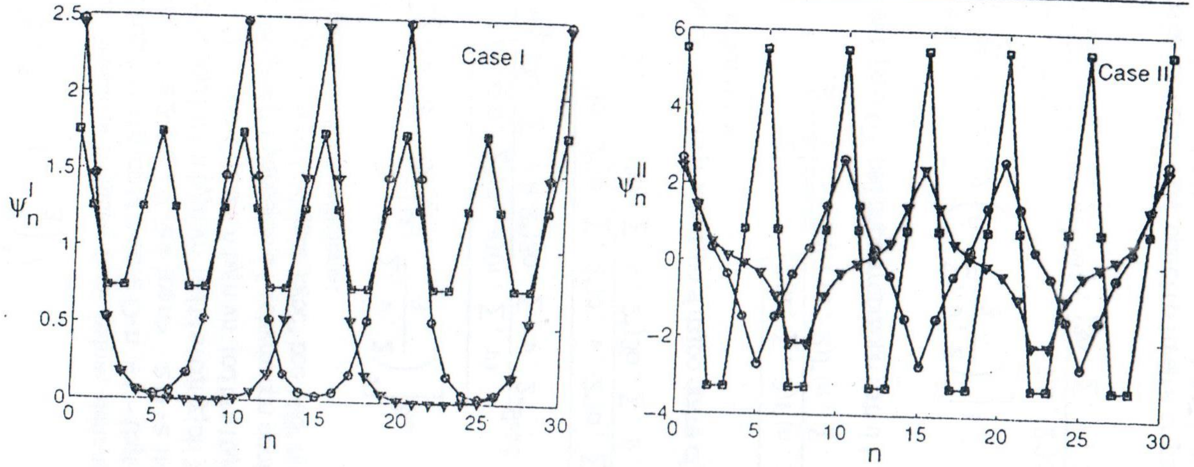


Figure 1. Illustration of the exact solutions of two types. $\nu = 1$, $\mu = 0.3$, $\omega = -1.33$ and $c = t = \delta = 0$. $N_p = 5$ (squares), $N_p = 10$ (circles) and $N_p = 15$ (triangles). Lines are guides to the eye.

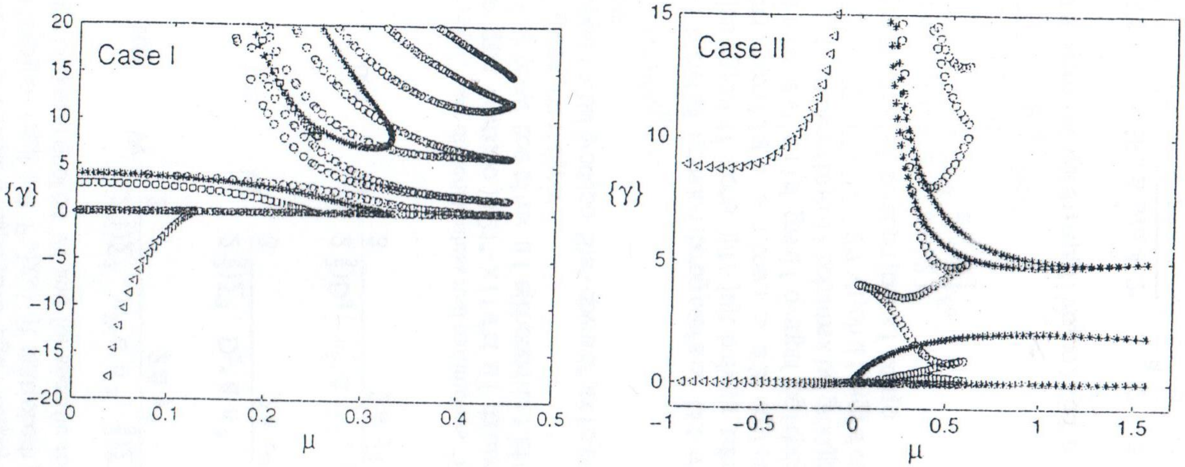


Figure 3. Illustration of the stability of the exact solutions. Shown is the eigenvalue spectrum $\{\gamma\}$ for the matrix product AB, $\nu = 1$. Case I (left panel) and $N_p = 3$ (triangles), $N_p = 4$ (squares), $N_p = 5$ (stars), and $N_p = 10$ (circles). Case II (right panel) $N_p = 3$ (triangles), $N_p = 5$ (stars) and $N_p = 10$ (circles).

Sol. I not stable for $N_p = 3$