The structure of invariants in conformal mechanics

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Motivation

- Quantum Calogero model describes the set of N identical particles on a circle interacting pairwise with inverse square potential.
- CM is a rare example of integrable many-body problem.
- CM is maximally superintegrable (has N Liouville plus N-1 additional integrals of motion
- Extensions: for trigonometric potentials, for particles with spins, for supersymmetric systems, and for other Lie algebras
  Hamiltonian with oscillator confining potential:

$$H_\omega = \sum_{i=1}^{N} \left(\frac{p_i^2}{2} + \frac{\omega^2 q_i^2}{2} \right) + \sum_{i<j} \frac{g^2}{(q_i - q_j)^2} \qquad (1)$$

Dynamical $SL(2,\mathbb{R})$ symmetry

Generators of conformal symmetry:

$$H_0 = H_\omega|_{\omega=0}, \qquad D = \sum_{i=1}^{N} p_i q_i, \qquad K = \frac{1}{2}\sum_{i=1}^{N} q_i^2,$$

form algebra

$$\{H_0, D\} = 2H_0, \qquad \{K, D\} = -2K, \qquad \{H_0, K\} = D.$$

Casimir $\mathcal{I} = 4H_0K - D^2$ is the angular part of the Calogero model commutes with $H_\omega = H_0 + \omega^2 K$ at any ω.
For any function f on phase space $F \to \hat{f} = \{f, .\}$ then

$$\hat{H}_0 = \sum_i (p_i \frac{\partial}{ddq_i} - \frac{\partial V}{ddq_i}\frac{\partial}{\partial p_i}), \qquad \hat{K} = -\sum p_i \frac{\partial}{\partial q_i}, \qquad \hat{D} = \sum_i q_i \frac{\partial}{\partial q_i - p_i \frac{\partial}{\partial p_i}}.$$

$$H_0 = S^+, \qquad {}_K = S^-, \qquad S^z = -\frac{1}{2}D.$$

Representation

Casimir $\hat{\mathbf{S}}^2 = \sum_{\alpha=1}^{3} \hat{S}_\alpha \hat{S}^\alpha$ is a second-order operator, while the vector field $\hat{\mathcal{I}} = \{\mathcal{I}, .\} = 8\sum_{\alpha=1}^{3} S_\alpha \hat{S}^\alpha$ generated by $\hat{\mathbf{S}}^2$ is a first-order one.

The descendants $I_{s,k} = (\hat{S}_-)^k I_s, \qquad k = 0, 1, 2, \ldots$ form the basic states of the spin-s representation

If s is positive half-integer and $I_{s,2s+1} = 0$, $I_{s,k}$ is $2s$-dimensional nonunitary multiplet of integrals of motion.

If $I_{s,2s+1} \neq 0$, $I_{s,k}$ is indecomposable (not fully reducible).

Suppose I_{s_1}, I_{s_2} are integrals of motion (highest weight vectors: $\dot{I}_s = S^+ I_s = 0$, then:

$$I^{(s_1,s_2)}_{s_1+s_2-k} = \sum_{j=0}^{k} (-1)^j C_j^k \frac{\Gamma(2s_1 - k + j + 1)\Gamma(2s_2 - j + 1)}{\Gamma(2s_1 - k + 1)\Gamma(2s_2 - k + 1)} I_{s_1,k-j} I_{s_2,j} \qquad k = 0, 1, 2, \ldots.$$

Quantum integrals of motion

$\{p_i, q_j\} = \delta_{ij} \qquad \rightarrow \qquad \frac{i}{\hbar}[p_i, q_j] = \delta_{ij}.$

The operator ordering affects only dilatation
$D = \frac{1}{2}\sum_{i=1}^{N}(p_i q_i + q_i p_i) = \sum_{i=1}^{N}(p_i q_i + i\hbar\ N/2.$

The hermitian generators obey the quantum commutation relations
$[H_0, D] = -2i\hbar\ H_0, \qquad [K, D] = 2i\hbar\ K, \qquad [H_0, K] = -i\hbar\ D,.$

The quantum physical observables must be hermitian: it suffices to symmetrize
$I_{s_1,k_1} I_{s_2,k_2} \ \rightarrow\ \frac{1}{2}(I_{s_1,k_1} I_{s_2,k_2} + I_{s_2,k_2} I_{s_1,k_1}).$

$$I^{(s_1,s_2)}_{s_1+s_2-k} = \sum_{j=0}^{k}(-1)^j C_j^k \frac{\Gamma(2s_1-k+j+1)\Gamma(2s_2-j+1)}{\Gamma(2s_1-k+1)\Gamma(2s_2-k+1)} \frac{1}{2}(I_{s_1,k_1} I_{s_2,k_2} + I_{s_2,k_2} I_{s_1,k_1}).$$

The simplest quantum integrals beyond the Liouville ones corresponds to
$k = 1,\ s_1 = 1$ and $I_1 = S_+$: $I_s^{(1,s)} = \frac{i}{2\hbar}[\mathcal{I}, I_s]$.

Angular part in general conformal mechanics

Angular $u = (\theta_\alpha, p_{\theta_\alpha})$
Radial $r^2 = \sum_i g_i^2, \qquad rp_r = \sum_i p_i q_i$
The conformal generators S_α and $\hat{S}_\alpha$ are:

Dual generators

Introduce dual set of $s\ell(2\mathbb{R})$-generators:

$$\hat{S}^R_+ = -p_r r^2 \frac{\partial}{\partial r}, \qquad \hat{S}^R - = \frac{1}{r}\frac{\partial}{\partial p_r}, \qquad \hat{S}^R_z = \frac{1}{2}(r\frac{\partial}{\partial r} + p_r\frac{\partial}{\partial p_r})$$

the highest weight condition $\hat{S}_+ I_s = 0$ $\hat{S}_z I_s = sI_s$ reads $\hat{\mathcal{I}} I_s = 2(\hat{S}^R_+ - \mathcal{I}\hat{S}^R_-)I_s$.
At zero angular part these relate $\hat{S}_a|_{\mathcal{I}=0} = R\hat{S}^R_a R$ by inversion:

$$R\ :\quad r \to \frac{1}{r}, \qquad p_r \to p_r, \qquad u \to u.$$

Convenient to replace $(\frac{1}{r}, p_r)$ to

$$z = \frac{1}{\sqrt{2}}(p_r - \frac{i\sqrt{\mathcal{I}}}{r}), \qquad \bar{z} = \frac{1}{\sqrt{2}}(p_r + \frac{i\sqrt{\mathcal{I}}}{r}).$$

Dual spin operators take the form:

$$\hat{S}^R_z = \frac{1}{2}(\bar{z}\frac{\partial}{\partial z} + z\frac{\partial}{\partial \bar{z}}), \quad \hat{S}^R_+ + \mathcal{I}\hat{S}^R_- = i\sqrt{\mathcal{I}}(z\frac{\partial}{\partial \bar{z}} + \bar{z}\frac{\partial}{\partial z}), \quad \hat{S}^R_+ - \mathcal{I}\hat{S}^R_- = i\sqrt{\mathcal{I}}(\bar{z}\frac{\partial}{\partial \bar{z}} + z\frac{\partial}{\partial z}).$$

Diagonalization of the hamiltonian

Addition of a harmonic confining potential is deformation compatible with conformal symmetry

$$H_\omega = S_+ - \omega^2 S_-, \qquad \hat{H}_\omega = \hat{S}_+ - \omega^2 \hat{S}_-.$$

It is easy to see that the operator

$$U = U(\omega) = (i\omega)^{-\hat{S}_z} e^{\frac{\pi}{2}\hat{S}_y}$$

links the Hamiltonian with harmonic potential to the diagonal conformal generator,

$$U \;\; : \quad -2i\omega S_z \;\rightarrow\; H_\omega, \qquad \hat{H}_\omega = -2i\omega U \hat{S}_z U^{-1}.$$

Note also:

$$U \;\; : \quad S_x \;\rightarrow\; S_z, \qquad 2iS_y \;\rightarrow \omega^{-1} S_+ + \omega S_-.$$

So defining

$$\tilde{I}_{s,\ell} = (i\omega)^s \sum_{k=0}^{2s} I_{s,k} (U^s)_{k\ell},$$

And one deduces:

$$\hat{H}_\omega \tilde{I}_{s,\ell} = -2i(s-\ell)\omega \tilde{I}_{s,\ell}.$$

Rational Calogero model with harmonic potential

can be obtained by $SU(N)$ reduction

$$H_\omega = \frac{1}{2}\mathrm{tr}(P^2) + \frac{\omega^2}{2}\mathrm{tr}(Q^2), \qquad H_0 = \frac{1}{2}\mathrm{tr}(P^2).$$

here

$$\{P_{ij}, Q_{i'j'}\} = \delta_{ij'}\delta_{ji'}.$$

Hamiltonians H_ω and H_0 describe N^2-dimensional oscillator and free particle in $\mathbb{R}^{N^2}$.

Upon gauge fixing

$$[P, Q] = -ig(1 - e \otimes e), \qquad e = (1, 1, \dots, 1)$$

by $SU(N)$ rotations one can turn Q_{ij} to diagonal form: $Q_{ij} = q_i\delta_{ij}$, then one obtains:

$$P_{ij} = p_i\delta_{ij} + (1 - \delta_{ij})\frac{ig}{q_i - q_j}.$$

Generators of conformal algebra are:

$$S_z = \frac{1}{2}\mathrm{tr}(PQ), \qquad S_- = -\frac{1}{2}\mathrm{tr}(Q^2), \qquad S_+ = \frac{1}{2}\mathrm{tr}(P^2).$$

Liouville integrals and their descendants are

$$I_s = \mathrm{tr}(P^{2s}), \qquad I_{s,\ell} = \frac{(2s)!}{(2s-\ell)!}\mathrm{tr}(P^{2s-\ell}Q^{\ell})_{sym}$$

here $s \leq N/2$, e index sym means symmetrization over all orderings of P, Q matrices. Symmetrized traces can be computed by means of the generating function

$$\mathrm{tr}((P+vQ)^{2s}) = \sum_{\ell=0}^{2s} C_\ell^{2s}\mathrm{tr}(P^{2s-\ell}Q^{\ell})_{sym}v^{\ell} = \sum_{\ell=0}^{2s}\frac{v^{\ell}}{\ell!}I_{s,\ell}.$$

The composite integrals are:

$$I^{(s_1,s_2)}_{s_1+s_2-k} = \frac{(2s_1)!(2s_2)!}{(2s_1-k)!(2s_2-k)!}\sum_{\ell=0}^{k}(-1)^{\ell}C_\ell^k\mathrm{tr}(P^{2s_1-k+\ell}Q^{k-\ell})_{sym}\mathrm{tr}(P^{2s_2-\ell}Q^{\ell})_{sym}.$$

Introduce:

$$A^{\pm} = \frac{1}{\sqrt{2\omega}}P \pm i\sqrt{\frac{\omega}{2}}Q,$$

then

$$H = \frac{\omega}{2}\mathrm{tr}(A^{+}A), \qquad \{A^{-}_{ij}, A^{+}_{i'j'}\} = i\delta_{ii'}\delta_{jj'}.$$

Time dependence

The matrix variables $A^{\pm}$ oscillate in time with frequency ω:

$$\dot{A}^{\pm} = \{A^{\pm}, H_\omega\} = \pm i\omega A^{\pm}, \qquad \mapsto \qquad A^{\pm}(t) = e^{\pm i\omega(t-t_0)} A^{\pm}(t_0).$$

The transformation U is:

$$UP = e^{-\frac{\pi}{4}} A^{-}, \qquad UQ = e^{-\frac{\pi}{4}} A^{+}.$$

The trace of any product of $A^{\pm}$ matrices

$$\mathrm{tr}(A^{\sigma_1} \dots A^{\sigma_1}), \qquad \sigma_i \in \{+, -\}.$$

Liouville integrals:

$$\tilde{I}_n = \mathrm{tr}((A^{+}A^{-})^n).$$

The first integral corresponds to the central $U(1)$ part

$$\tilde{I}_1 = \omega^{-1} H_\omega = \omega^{-1} H_0 + \omega K.$$

Additional integrals

Using definition of $A^{\pm}$ one obtains:

$$\mathrm{tr}(A^+A^+) = \omega^{-1}H_0 - \omega K - iD, \qquad \mathrm{tr}(A^-A^-) = \omega^{-1}H_0 - \omega K + iD.$$

The Casimir is:

$$\mathcal{I} = (\mathrm{tr}(A^+A^-))^2 - \mathrm{tr}(A^+A^+)\mathrm{tr}(A^-A^-).$$

is conserved:

$$\dot{\mathcal{I}} = \{H_\omega, \mathcal{I}\},$$

but does not commute with Liouville integrals: $J_n = \{\mathcal{I}, \tilde{I}_n\} =$

$$= 2in[\mathrm{tr}((A^-A^-)^n)\mathrm{tr}((A^+A^+)(A^+A^-)^{n-1}) - \mathrm{tr}((A^+A^+)^n)\mathrm{tr}((A^-A^-)(A^+A^-)^{n-1})].$$

Since $J_1 = 0$, one finds $2N-1$ integrals

$$(\tilde{I}_1, \ldots, \tilde{I}_N, J_2, \ldots, J_n),$$

which form a complete set of integrals for H_ω.

Functional independence of integrals

Consider free-particle limit $g \to 0$ one obtains:

$$A^{\pm} = diag(a_1, \ldots, a_N) + \mathcal{O}(g), \qquad a_j = \frac{p_j}{\sqrt{2\omega}} - iq_j\sqrt{\frac{\omega}{2}} = \sqrt{\rho_j}e^{-\frac{i}{2}\varphi_j}.$$

Then integrals take the form:

$$I_n = \sum_{j=1}^{N} \rho_j^n, \qquad J_n = 4n\sum_{i,j=1}^{N} \rho_i\rho_j^{n-1}\sin(\varphi_i - \varphi_j).$$

$$\frac{\partial(I_1,\ldots,I_N,J_2,\ldots,J_N)}{\partial(\rho_1,\ldots,\rho_N,\varphi_2,\ldots,\varphi_N)} = \begin{pmatrix} \frac{\partial(I_1,\ldots,I_N)}{\partial(\rho_1,\ldots,\rho_N)} & \frac{\partial(I_1,\ldots,I_N,)}{\partial(\varphi_2,\ldots,\varphi_N)} \\ \frac{\partial(J_2,\ldots,J_N)}{\partial(\rho_1,\ldots,\rho_N)} & \frac{\partial(J_2,\ldots,J_N)}{\partial(\varphi_2,\ldots,\varphi_N)} \end{pmatrix} =$$

$$= \left|\frac{\partial(I_1,\ldots,I_N)}{\partial(\rho_1,\ldots,\rho_N)}\right| \left|\frac{\partial(J_2,\ldots,J_N)}{\partial(\varphi_2,\ldots,\varphi_N)}\right|$$

due to:

$$\frac{\partial I_n}{\partial\varphi_k} = 0, \qquad \frac{\partial I_n}{\partial\rho_k} = n\rho_k^{n-1}, \qquad \frac{\partial J_n}{\partial\varphi_k} = \sum_i(\rho_k\rho_i^n - \rho_i\rho_k^n)\cos(\varphi_k - \varphi_i)$$

The first term is proportional to the Vandermonde determinant

$$\left|\frac{\partial(I_1,\ldots,I_N)}{\partial(\rho_1,\ldots,\rho_N)}\right| = N!\prod_{1\le i<j\le N}(\rho_j - \rho_i),$$

then one has:

$$\frac{\partial J_n}{\partial \varphi_k} = \sum_{i=1}^{N} \rho_i^{n-1} B_{ik}, \qquad B_{ik} = b_{ik} - \delta_{ik} \sum_{l=1}^{N} b_{lk}, \qquad b_{ik} = \rho_i \rho_k \cos(\varphi_i - \varphi_k).$$

One has $\sum_{i=1}^{N} B_{ij} = 0$ so adding first row and column to obtain $N \times N$ matrix:

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ \rho_1 & \partial J_2/\partial \varphi_2 & \dots & \partial J_2/\partial \varphi_N \\ \vdots & \vdots & \ddots & \vdots \\ \rho_1^{N-1} & \partial J_N/\partial \varphi_2 & \dots & \partial J_N/\partial \varphi_N \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 1 & \dots & 1 \\ \rho_1 & \rho_2 & \dots & \rho_N \\ \vdots & \vdots & \ddots & \vdots \\ \rho_1^{N-1} & \rho_2^{N-1} & \dots & \rho_N^{N-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & B_{12} & \dots & B_{1N} \\ 0 & B_{22} & \dots & B_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & B_{N2} & \dots & B_{NN} \end{pmatrix}$$

Extracting the diagonal matrix $B_{ik} = \tilde{B}_{ik} \rho_k$ one obtains:

$$\left| \frac{\partial(J_2, \dots, J_N)}{\partial(\varphi_2, \dots, \varphi_N)} \right| = \prod_{i=2}^{N} \rho_i \prod_{1 \le i < j \le N} (\rho_j - \rho_i) \ M_{11},$$

In the simplest case of equal phases $\varphi_i = \varphi$,

$$M_{11} = \det(\tilde{B}_{ij} - \rho \delta_{ij}) = \rho^{N-1} \rho_1, \qquad \rho = \sum_{i=1}^{N} \rho_i.$$