Higher-derivative dynamics: stability, interactions and quantization

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Background

The Hamiltonian formulation for higher-derivative theories

$$S=\int dtL, \qquad L=L(\varphi,\dot{\varphi},\ddot{\varphi},\dots,\overset{(n)}{\varphi}), \qquad \overset{(n)}{\varphi}=\frac{d^n\varphi}{dt^n}$$

has been developed by Ostrogradski [Ostrogradski,1850].

$$H_O=\sum_{i=1}^{n}P_i\dot{Q}_i-L(\varphi,\dot{\varphi},\ddot{\varphi},\dots,\overset{(n)}{\varphi})\Big|_{\overset{(i)}{\varphi}=\overset{(i)}{\varphi}(Q,P)}$$

$$Q_i=\overset{(i-1)}{\varphi}, \qquad P_i=\sum_{j=i}^{n}\Big(-\frac{d}{dt}\Big)^{j-i}\frac{\partial L}{\partial\overset{(j)}{\varphi}}, \qquad i=1,\dots,n$$

For $n>1$, the Ostrogradski Hamiltonian is not bounded from below because it is linear in $P_i, i=1,\dots,n-1$.

The constrained systems may have bounded Hamiltonian with account of constraints. This is a very special case, the number of such models is small.

Instability problem and solution

When the Hamiltonian of the model is unbounded, the theory becomes unstable. The typical problems are

- Ghost states (quantum instability)
- Runaway/explosive behavior of solutions (classical instability)
  No selection rules for interaction vertices

Solution: To find the Hamiltonian formulation with a bounded Hamiltonian.

- If the Hamiltonian is Hermitian and bounded from below, the energy spectra is also bounded from below (quantum stability)
- If the level surfaces $H = E$ are bounded for all E's, the motion is bounded for all the initial data (classical stability)

Even if the bounded Hamiltonian is given from outside, it ensures the classical and quantum stability of the theory.

Model

Given the free Pais-Uhlenbeck oscillator equation of motion

$$0 = \frac{1}{\Omega} \prod_{i=1}^{n} \Big(\frac{d^2}{dt^2} + \omega_i^2 \Big) \varphi(t) \,, \tag{1}$$

where

$$0 < \omega_1 < \omega_2 < \ldots < \omega_n$$

and $\Omega > 0$ is a dimensional factor,

to find: a nonlinear deformation of equation (1) such that

- The nonlinear theory admits the Hamiltonian formulation
- The Hamiltonian of the nonlinear theory is bounded from below

Model

Given the free Pais-Uhlenbeck oscillator equation of motion

$$0 = \frac{1}{\Omega}\prod_{i=1}^{n}\Big(\frac{d^2}{dt^2} + \omega_i^2\Big)\varphi(t) + V(\varphi, \dot{\varphi}, \dots, \overset{(2n)}{\varphi}), \qquad (1)$$

where

$$0 < \omega_1 < \omega_2 < \dots < \omega_n$$

and $\Omega > 0$ is a dimensional factor,

to find: a nonlinear deformation of equation (1) such that

- The nonlinear theory admits the Hamiltonian formulation
- The Hamiltonian of the nonlinear theory is bounded from below

Remark: We do not require that the nonlinear term follows from least action principle

$$V(\varphi, \dot{\varphi}, \dots, \overset{(2n)}{\varphi}) \neq \frac{\delta}{\delta\varphi}\Big(\int dt L^{int}(\varphi, \dot{\varphi}, \dots, \overset{(n)}{\varphi})\Big).$$

Nonlinear oscillator of order 2n

Let us consider a model, whose dynamics is described by the equation

$$\frac{1}{\Omega}\prod_{i=1}^{n}\Big(\frac{d^2}{dt^2}+\omega_i^2\Big)\varphi+U'\Big(\sum_{i=1}^{n}\alpha_i\mathcal{P}_i\varphi\Big)=0\,,\qquad U'(\varphi)=\frac{\partial U(\varphi)}{\partial\varphi}\,,\quad (2)$$

where

$$\mathcal{P}_i=\prod_{j\neq i}\frac{1}{\omega_j^2-\omega_i^2}\Big(\frac{d^2}{dt^2}+\omega_j^2\Big)\,,\qquad V=U'\Big(\sum_{i=1}^{n}\alpha_i\mathcal{P}_i\varphi\Big)\,,$$

$U(\varphi)$ is some function of dynamical variable $\varphi(t)$ and $\alpha_i\neq 0$ are the parameters of the theory.

Outline:

- In the free limit ($U=0$), equation (2) describes the free Pais-Uhlenbleck oscillator of order $2n$.
- Equation (2) is not Lagrangian unless $U=0$ or $\alpha_i=1$.

$\varphi-$ and $\xi-$representations

There is one-to-one correspondence between the solutions of $2n$-th order equation

$$\frac{1}{\Omega}\prod_{i=1}^{n}\Big(\frac{d^2}{dt^2}+\omega_i^2\Big)\varphi+U'\Big(\sum_{i=1}^{n}\alpha_i\mathcal{P}_i\varphi\Big)=0\,,$$

and the system of n second-order differential equations

$$\frac{1}{\Omega}\Big[\prod_{j\neq i}\Big(\omega_j^2-\omega_i^2\Big)\Big]\Big(\frac{d^2}{dt^2}+\omega_i^2\Big)\xi_i+U'\Big(\sum_{i=1}^{n}\alpha_i\xi_i\Big)=0,\qquad i=1,\ldots,n\,.$$

This correspondence is established by the relations

$$\varphi=\sum_{i=1}^{n}\xi_i\,,\qquad \xi_i=\mathcal{P}_i\xi.$$

Bounded Hamiltonian

The $\xi-$representation system is Lagrangian. The Hamiltonian formulation reads

$$\dot{\xi}_i = \{\xi_i, H\}\,, \qquad \dot{\pi}_i = \{\pi_i, H\}\,,$$

where the Poisson bracket is canonical $\{\xi_i, \pi_j\} = \delta_{ij}$ and the Hamiltonian has the form

$$H = \frac{1}{2}\sum_{i=1}^{n}\Big(\frac{\pi_i^2}{\gamma_i} + \gamma_i\omega_i^2\xi_i^2\Big) + U\Big(\sum_{i=1}^{n}\alpha_i\xi_i\Big)\,, \qquad \gamma_i = -\frac{\alpha_i}{\Omega}\Big[\prod_{j\neq i}\Big(\omega_j^2 - \omega_i^2\Big)\Big] \tag{3}$$

In contrast to Ostragradski's Hamiltonian, the Hamiltonian (3) may be bounded from below if

$$(-1)^i\alpha_i > 0\,, \qquad U \geq 0\,.$$

The interactions with alternating sign of α's are not compatible with least action principle. The stable interactions of form (2) are not variational.

Further generalizations

The bounded Hamiltonians for more general class of theories may be found by the method of nonlinear factorization [arXiv:1407.8481]. It applies to the fourth-order theories with equations of motion of the form

$$\mathcal{P}\mathcal{Q}\varphi+U'(\alpha\mathcal{Q}\varphi-\beta\mathcal{P}\varphi)=0\,,\qquad U'(\alpha\mathcal{Q}\varphi-\beta\mathcal{P}\varphi)=\frac{\partial U(\varphi)}{\partial\varphi}\Big|_{\varphi=\alpha\mathcal{Q}\varphi-\beta\mathcal{P}\varphi}$$

where $U(\varphi)$ is a function of fields, $\alpha,\beta\neq 0$ are constants, $\mathcal{P}$, $\mathcal{Q}$ are self-adjoint second-order matrix linear differential operators subjected to the relation $\mathcal{P}+\mathcal{Q}=1$.

Each theory with factorizable structure is equivalent to the system of second-order Lagrangian equations

$$\mathcal{P}\xi+U'(\alpha\xi-\beta\eta)=0,\qquad \mathcal{Q}\xi+U'(\alpha\xi-\beta\eta)=0$$

$$\varphi=\xi+\eta,\qquad \xi=\mathcal{Q}\varphi\,,\qquad \eta=\mathcal{P}\varphi.$$

Summary and Perspectives

We have shown the following.

- The higher-derivative theories can admit bounded from below Hamiltonian even if the Ostrogradski Hamiltonian is unbounded.
- For the factorizable theories, the bounded Hamiltonian may be constructed by the method of nonlinear factorization.
- This method is well applied to the wide class of higher-derivative models, including the Pais-Uhlenbeck oscillator, higher-derivative scalar field, Podolsky electrodynamics.
- The non-variational interaction vertices are compatible with stability of nonlinear theory.

The open question:

- Is there any generalization of method of nonlinear factorization to more general class of theories?

And couple of slides instead conclusion.

Lagrangian counterpart

The energy is a Lagrangian counterpart of the Hamiltonian. It is the Noether current associated with invariance of action under the time translations.

When the energy is bounded form below, the theory is stable.

For non-Lagrangian equations, the correspondence between symmetries and conservation laws is established by the Lagrange anchor.

To prove stability, we should find the bounded from below conservation law and the Lagrange anchor that associates it with time translation.

And it is possible to do this for all factorizable theories...

Lagrange anchor for nonlinear oscillator

For the nonlinear oscillator, the bounded conservation law I reads

$$I=\frac{1}{2\Omega}\sum_{i=1}^{n}\Big[\alpha_i\prod_{j\neq i}\Big(\omega_j^2-\omega_i^2\Big)\Big(\Big(\frac{\mathcal{P}_i\varphi}{dt}\Big)^2+\omega_i^2(\mathcal{P}_i\varphi)^2\Big)\Big]+U\Big(\sum_{i=1}^{n}\alpha_i\mathcal{P}_i\varphi\Big)$$

and the Lagrange anchor V has the form

$$V=-\sum_{i=1}^{n}\frac{P_i}{\alpha_i}-W\Big[\frac{d^2U}{d\varphi^2}\Big(\sum_{i=1}^{n}\alpha_i\mathcal{P}_i\varphi\Big)\cdot\Big],$$

where

$$W=\Omega\sum_{i<j}\Big[\Big(\frac{\alpha_i}{\alpha_j}+\frac{\alpha_j}{\alpha_i}-2\Big)\prod_{k\neq i}\frac{1}{\omega_k^2-\omega_i^2}\prod_{l\neq j}\frac{1}{\omega_l^2-\omega_j^2}\prod_{s\neq i,s\neq j}\Big(\frac{d^2}{dt^2}+\omega_s^2\Big)\Big]$$

By construction, the Lagrange anchor V associates the conservation law I with time translation.