Discrete evolution operator for q-deformed top and Faddeev's modular double

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Notion of Faddeev's modular double

Faddeev's example. Consider the standard Heisenberg algebra (*HA*) generated by operators $x\,,p$

$$[x\,,\,p] = i\ .$$

Introduce the algebra **T** (quantum torus or Weyl pair) with generators $U\,,V$

$$U\ =\ e^{i\alpha x}\ ,\quad V\ =\ e^{i\beta p}\ ,$$

(α, β are parameters) with commutation relations

$$U\,V\ =\ q\,V\,U\qquad q\ =\ e^{-i\,\alpha\beta}\ .$$

One can think that *HA* can be obtained from **T** by means of log - function. So we have got a question:
Is the algebra **T** of quantum torus (in above realization) is "equivalent" (representation theories are identical) to the Heisenberg algebra?
The answer is NO!

Notion of Faddeev's modular double

To demonstrate this we note that from HA one can construct another "dual" algebra $\widetilde{\mathbf{T}}$ of quantum torus

$$\tilde{U} = e^{i\tilde{\alpha}x} \,, \quad \tilde{V} = e^{i\tilde{\beta}p} \,.$$

$$\tilde{U}\,\tilde{V} = \tilde{q}\,\tilde{V}\,\tilde{U} \,, \quad \tilde{q} = e^{-i\,\tilde{\alpha}\tilde{\beta}} \,,$$

with another parameters $\tilde{\alpha}, \tilde{\beta}$. Then, if

$$\boxed{\tilde{\alpha} = \frac{2\pi}{\beta} \,, \quad \tilde{\beta} = -\frac{2\pi}{\alpha} \,,}$$

the generators U, V of $\mathbf{T}$ commute with $\tilde{U}\,\tilde{V}$ of $\widetilde{\mathbf{T}}$ and parameters q and $\tilde{q}$ are related by modular transformation

$$q = e^{-i\,\alpha\beta} = e^{i2\pi\tau} \;\to\; \tilde{q} = e^{-i\,\tilde{\alpha}\tilde{\beta}} = e^{-\frac{i2\pi}{\tau}} \quad (\tau \;\to\; \tilde{\tau} = -\frac{1}{\tau}) \,.$$

Thus, the dual algebra $\tilde{\mathbf{T}}$ centralizes the algebra $\mathbf{T}$ and vice versa.

The double of algebras **T** and $\tilde{\mathbf{T}}$ is called modular double.

The modular double of **T** and $\tilde{\mathbf{T}}$ is "equivalent" to HA!

The notion of the modular double was introduced by L.D.Faddeev in 1999.

We use this simple example of the modular double to explain what kind of discrete evolution will be considered in the case of quantum groups. Let x be a coordinate and p be a momentum of a free particle. The time evolution is defined by the evolution operator

$$\Theta(t) = \exp(\frac{i}{2}p^2 t) ,$$

and we have the standard formulas for free evolution

$$p \to \Theta(t) \cdot p \cdot \Theta(t)^{-1} = p , \quad x \to \Theta(t) \cdot x \cdot \Theta(t)^{-1} = x + p\,t .$$

From these formulas, for coordinates U, V of quantum torus **T**, we obtain the evolution

$$V \to \Theta(t) \cdot V \cdot \Theta(t)^{-1} = V \ ,$$

$$U \to \Theta(t) \cdot U \cdot \Theta(t)^{-1} = U\, e^{i\alpha tp}\, e^{\frac{i\alpha^2 t}{2}} \ .$$

Note that for special time interval $t = \frac{\beta}{\alpha} = -\frac{\tilde{\beta}}{\tilde{\alpha}}$ we obtain intrinsic discrete evolution on **T**

$$V \to \Theta \cdot V \cdot \Theta^{-1} = V \ , \qquad U \to \Theta \cdot U \cdot \Theta^{-1} = U\, V\, q^{-\frac{1}{2}} \ , \tag{1}$$

where we denote $\Theta = \Theta(\frac{\beta}{\alpha})$. Since in (1) the first relation is $[V, \Theta] = 0$ one can search the operator Θ as a function $\theta(V)$. The second relation in (1) gives the equation:

$$\theta(V) = q^{\frac{1}{2}}\, \theta(qV)\, V \ .$$

For $|q| < 1$, this equation can be solved in terms of the Jacobi theta-function

$$\theta(V) = \prod_{n=1}^{\infty}(1 + q^{n-1/2}V)\prod_{n=1}^{\infty}(1 + q^{n-1/2}V^{-1})\ .$$

The operator $\theta(V)$ describes the evolution of the coordinates V, U of the torus $\mathbf{T}$ for the finite time interval $t = \frac{\beta}{\alpha} = -\frac{\tilde{\beta}}{\tilde{\alpha}}$.
In view of the condition $|q| < 1$ the evolution operator $\theta(V)$ is called compact.
We stress that the operator $\theta(V)$ leaves the dual torus $\tilde{\mathbf{T}}$ in rest:

$$\tilde{V} \to \theta(V)\cdot\tilde{V}\cdot\theta(V)^{-1} = \tilde{V}\ ,$$
$$\tilde{U} \to \theta(V)\cdot\tilde{U}\cdot\theta(V)^{-1} = \tilde{U}\ ,$$

The free motion evolution operator for finite time interval $t = \frac{\alpha}{\beta}$:

$$\Theta = \exp(\frac{i}{2}p^2\, t)\Big|_{t=\frac{\alpha}{\beta}} \;,$$

should be proportional to the "compact" evolution operator $\theta(V, q)$:

$$\Theta = \exp(\tfrac{i}{2}p^2\, t)\Big|_{t=\frac{\alpha}{\beta}} \sim C(\tilde{V}, \tilde{U}) \cdot \theta(V, q) \;,$$

where the "constant" $C(\tilde{V}, \tilde{U})$ should commute with U, V.

In the same way as before for **T** one can consider the discrete time evolution of the coordinates $\tilde{V}, \tilde{U}$ of the dual torus $\tilde{\mathbf{T}}$. We note that

$$\alpha, \beta \;\to\; \tilde{\alpha}, \tilde{\beta} \;\Rightarrow\; V, U \;\to\; \tilde{V}, \tilde{U}$$

Thus the discrete evolution operator $\tilde{\Theta}$ for $\tilde{V}, \tilde{U}$ is defined by Θ with substitution $\alpha, \beta \rightarrow \tilde{\alpha}, \tilde{\beta}$. Recall that $\frac{\tilde{\beta}}{\tilde{\alpha}} = -\frac{\beta}{\alpha} = -t$ and it means that

$$\tilde{\Theta} = \Theta|_{\alpha=\tilde{\alpha};\beta=\tilde{\beta}} = \exp(\frac{i}{2}p^2 t)\Big|_{t=-\frac{\alpha}{\beta}} = \Theta^{-1} ,$$

$$\tilde{V} \rightarrow \Theta^{-1} \cdot \tilde{V} \cdot \Theta = \tilde{V} , \quad \tilde{U} \rightarrow \Theta^{-1} \cdot \tilde{U} \cdot \Theta = \tilde{U}\, \tilde{V}\, \tilde{q}^{-\frac{1}{2}} .$$

We again look for the solution $\Theta^{-1} \sim \theta(\tilde{V}, \tilde{q})$ which is given as before

$$\Theta^{-1} \sim \theta(\tilde{V}, \tilde{q}) = \prod_{n=1}^{\infty}(1 + \tilde{q}^{n-1/2}\tilde{V}) \prod_{n=1}^{\infty}(1 + \tilde{q}^{n-1/2}\tilde{V}^{-1}) ,$$

and which is ”compact” (for $|\tilde{q}| < 1$) evolution operator for dual quantum torus $\widetilde{T}$. Finally the combination of both results gives the answer for complete discrete time evolution operator $\Theta(\frac{\beta}{\alpha})$ in the form of well known identity for theta-functions

$$\exp\left(\frac{i}{2}p^2\,\frac{\beta}{\alpha}\right) \sim \frac{\Theta(V, q)}{\Theta(\tilde{V}, \tilde{q})} .$$

The important remark is that the operator

$$\exp\left(\frac{i}{2}p^2\,\frac{\beta}{\alpha}\right) \sim \frac{\Theta(V,q)}{\Theta(\tilde{V},\tilde{q})}\,.$$

is well defined for any values of q and $\tilde{q}$!!!

Below we obtain the similar formulas in the context of a discrete evolution of $SL_q(N)$- quantum top considered by Faddeev and Alekseev.

We will consider as the analog of Weyl pair $\{U,V\}$ (quantum torus) the "Heisenberg double" of the RTT algebra and the $RLRL$ - or reflection equation algebra.

1. R-matrices

Let V be a finite dimensional $\mathbb{C}$ - linear space. For any operator $X \in \mathrm{End}(V \otimes V)$ and integers $i > 0$, $j > 0$ we denote

$$X_{i\,i+1} := I^{\otimes(i-1)} \otimes X \otimes I^{\otimes(j-1)} \in \mathrm{End}(V^{\otimes(i+j)}),$$

where $I \in \mathrm{Aut}(V)$ is the identity operator.

Def 1. An operator $\hat{R} \in \mathrm{Aut}(V \otimes V)$ is called an R-*matrix* if

$$\boxed{\hat{R}_{12}\,\hat{R}_{23}\,\hat{R}_{12} \;=\; \hat{R}_{23}\,\hat{R}_{12}\,\hat{R}_{23} \in \mathrm{Aut}(V \otimes V \otimes V)\,.}$$

Def 2. An R-matrix $\hat{R}$ is called <u>a Hecke type R-matrix</u> if

$$\boxed{(\hat{R} - q\,\mathbf{1})(\hat{R} + q^{-1}\,\mathbf{1}) = 0\,,} \quad (\mathbf{1} = I \otimes I)\,.$$

1. R-matrices

Consider the set of antisymmetrizers $\mathcal{A}^{(k)}(q)$ which can be defined by recurrent relations: $\mathcal{A}^{(1)} = 1$,

$$\mathcal{A}^{(k+1)} = \frac{[k]_q}{[k+1]_q} \mathcal{A}^{(k)} \left(\frac{q^k}{[k]_q} - \hat{R}_k \right) \mathcal{A}^{(k)} \in \mathrm{End}(V^{\otimes (k+1)}) \, .$$

Def 3. A Hecke type R-matrix $\hat{R}$ for q – generic is called $GL_q(n)$ type R-matrix if it satisfies

$$1.) \;\; \mathcal{A}^{(n+1)} = 0 \;\; \Leftrightarrow \;\; \mathcal{A}^{(n)} \Big(\frac{q^n}{[n]_q} I \; - \; \hat{R}_n \Big) \mathcal{A}^{(n)} = 0 \, , \quad 2.) \;\; \mathrm{rk}(\mathcal{A}^{(n)}) = 1 \, .$$

An example – the standard Drinfeld-Jimbo's $GL_q(n)$ type R-matrix:

$$\hat{R}^\circ \; = \; \sum_{i,j=1}^{n} q^{\delta_{ij}} E_{ij} \otimes E_{ji} \; + \; (q - q^{-1}) \sum_{i<j} E_{ii} \otimes E_{jj} \, ,$$

where $(E_{ij})_{kl} := \delta_{ik}\delta_{jl}$ are $(n \times n)$ matrix units.

Def 4. $\hat{R}$ is called *skew invertible* if $\exists\, \Psi \in End(V^{\otimes 2})$ such that

$$\hat{R}^{i_1 m_2}_{j_1 k_2}\, \Psi^{k_2 i_3}_{m_2 j_3} = \Psi^{i_1 m_2}_{j_1 k_2}\, \hat{R}^{k_2 i_3}_{m_2 j_3} = \delta^{i_1}_{j_3}\, \delta^{i_3}_{j_1} .$$

With any skew invertible $\hat{R}$ we associate matrix $D \in \mathrm{End}(V)$:

$$D_1 = \mathrm{Tr}_{(2)} \Psi_{12} \, ,$$

where $\mathrm{Tr}_{(i)}$ – trace in i-th space. Then, we define a quantum trace (q-traces) for any quantum matrix Y

$$Y \mapsto \mathrm{Tr}_{_D}(Y) := \mathrm{Tr}(D\, Y)\, ,$$

which possesses many remarkable properties, e.g.,

$$\mathrm{Tr}_{_{D\,(2)}}(\hat{R}^{\varepsilon}_{12}\, Y_1\, \hat{R}^{-\varepsilon}_{12}) = I_1\, \mathrm{Tr}_{_D}(Y) \;\; (\varepsilon = \pm 1)\, ,$$

$$\mathrm{Tr}_{_{D(1,\dots,k)}} \left(\left[\hat{R}_{i\, i+1},\, Y_{(1\dots k)} \right] \right) = 0 \;\; (\forall\, 1 < i < k\, ,\; \forall Y_{(1\dots k)})\, .$$

3. *RTT* and Reflection equation (RE) algebras

Quantized functions over matrix group (RTT algebra) (L.Faddeev,N.Reshetikhin,L.Takhtajan (1989)).

Let $\hat{R}$ be a skew invertible R-matrix. Consider an associative unital algebra generated by matrix components $\|T^i_j\|^{\dim V}_{i,j=1}$ which satisfy

$$\boxed{\hat{R}_{12}\, T_1\, T_2 = T_1\, T_2\, \hat{R}_{12}\ .}$$

The extension of this algebra by a set of components $\|(T^{-1})^i_j\|^{\dim V}_{i,j=1}$:

$$\sum_k T^i_k\,(T^{-1})^k_j \;=\; \sum_k (T^{-1})^i_k\, T^k_j \;=\; \delta^i_j\, 1\ ,$$

is a Hopf algebra with coproduct, counit and antipode mappings:

$$\Delta(T^i_j) = \sum_k T^i_k \otimes T^k_j\ , \qquad \epsilon(T^i_j) = \delta^i_j\ , \qquad S(T^i_j) \;=\; (T^{-1})^i_j\ .$$

This algebra is called an *RTT algebra* and denoted by $\mathcal{F}[\hat{R}]$.

Def 5. Let $\hat{R}$ be a skew invertible R-matrix. An associative unital algebra $\mathcal{L}[\hat{R}]$ with generators $\|L_j^i\|_{i,j=1}^{\dim V}$ satisfying relations

$$L_1\,\hat{R}_{12}\,L_1\,\hat{R}_{12} \;=\; \hat{R}_{12}\,L_1\,\hat{R}_{12}\,L_1\;,$$

is called a *reflection equation (RE) algebra*.

Consider REA $\mathcal{L}[\hat{R}]$ for Hecke type $\hat{R}$ and introduce elements ($a_0 = 1$)

$$a_i = \mathrm{Tr}_{D(1,\dots,i)}\left(\mathcal{A}^{(i)}L_{\overline{1}}\dots L_{\overline{i}}\right),\quad p_i = \mathrm{Tr}_D(L^i)\quad (i \geq 1)$$

where $L_{\overline{1}} := L_1\,,\quad L_{\overline{k+1}} := \hat{R}_k\,L_{\overline{k}}\,\hat{R}_k^{-1}$. Elements p_i and a_i are central in REA $\mathcal{L}[\hat{R}]$ and called *power sums* and *elementary symmetric functions*, respectively.

Proposition 1. Quantum Newton relations and q - Cayley-Hamilton identity hold for REA $\mathcal{L}[\hat{R}]$

$$k_q\, a_k + (-1)^k \sum_{j=0}^{k-1} (-q)^j a_j\, p_{k-j} = 0 \quad \forall\, 1 \le k \le n\,,$$
$$\sum_{j=0}^{n} (-q)^j\, a_j\, L^{n-j} \;=\; 0\,.$$

Proposition 2. *The set of* elementary symmetric functions $\{a_j,\, j = 1, ..., n\}$ *generate the whole center in REA* $\mathcal{L}[\hat{R}_{GL_q(n)}]$.

Def 5. A spectral extension of REA $\mathcal{L}[\hat{R}]$ for $GL_q(n)$ type $\hat{R}$-matrix is the extension of $\mathcal{L}[\hat{R}]$ by a set of invertible central elements μ_α $(\alpha = 1, \ldots, n)$ such that

$$[\mu_\alpha, L^i_j] = 0$$

and

$$a_i = \sum_{1 \le j_1 < \cdots < j_i \le n} \mu_{j_1} \mu_{j_2} \ldots \mu_{j_i} \quad \forall\, i = 1, \ldots, n.$$

It means that the Cayley-Hamilton identity can be written in factorized form

$$\sum_{j=0}^{n} (-q)^j \, a_j \, L^{n-j} \; = \prod_{\alpha=1}^{n} \big(L - q\mu_\alpha I\big) = 0\,.$$

We need projectors

$$P^\alpha = \prod_{\beta \ne \alpha} \frac{(q^{-1}L - \mu_\beta)}{(\mu_\alpha - \mu_\beta)}$$

Def 6. A *Heisenberg double (HD) algebra* of the *RTT* and RE algebras is an associative unital algebra generated by elements $T^i_j \in \mathcal{F}[\hat{R}]$ and $L^i_j \in \mathcal{L}[\hat{R}]$ subject to commutation relations

$$\hat{R}_{12}\, T_1\, T_2 = T_1\, T_2\, \hat{R}_{12}\ .$$

$$L_1\, \hat{R}_{12}\, L_1\, \hat{R}_{12} \ = \ \hat{R}_{12}\, L_1\, \hat{R}_{12}\, L_1\ ,$$

$$\gamma^2\, T_1\, L_2 \ = \ \hat{R}_{12}\, L_1\, \hat{R}_{12}\, T_1\,, \quad (\gamma \in \{\mathbf{C}\backslash 0\})\,.$$

Parameter γ will be fixed below.

4. Heisenberg double of *RTT* and RE algebras

In the limit

$$T^i_j \to T^i_j\ ,\ \ L^i_j \to \delta^i_j + h\,\ell^i_j + \dots\ ,\ \ R^{ij}_{km} \to \delta^i_m \delta^j_k + h\,\delta^i_k \delta^j_m + \dots\ .$$

we obtain from HD algebra structure relations the following Poisson brackets

$$\{T^i_j,\, T^k_m\} = 0\ , \quad \{\ell^i_j,\, \ell^k_m\} = 2(\delta^i_m \ell^k_j - \delta^k_j\, \ell^i_m),$$

$$\{\ell^i_j,\, T^k_m\} = \delta^k_j\, T^i_m.$$

Thus, the HD algebra is a quantization of the Poisson structure on $T^*(GL(n))$.

HD algebra is interpreted as quantum group cotangent bundle, where *RTT* algebra is a base and RE algebra is a bundle.

For the spectral extension of HD we have additional commutators of T^i_k and L^i_k with spectral elements $\{\mu_\alpha\}$

$$[\mu_\alpha, L^i_k] = 0,$$

$$\mu_\alpha \cdot T^i_k - \gamma^2\ T^i_k \cdot \mu_\alpha = (1 - q^2)\ \mu_\alpha \cdot (P^\alpha \cdot T)^i_k\ .$$

The last commutation relation can be rewritten equivalently

$$\gamma^2\ (P^\beta \cdot T)^i_k \cdot \mu_\alpha = q^{2\delta_{\alpha\beta}}\ \mu_\alpha \cdot (P^\beta \cdot T)^i_k\ .$$

5. Discrete time evolution on quantum group cotangent bundle

Consider sequence of automorphisms on the HD $(\mathcal{F} \sharp \mathcal{L})[\hat{R}]$

$$\{T,\, L\} \stackrel{\theta^k}{\longrightarrow} \{T(k),\, L(k)\}\,, \;\; \forall\, k = 0, 1, 2, \dots\,,$$

$$\hat{R}_{12}\, T_1(k)\, T_2(k) \;=\; T_1(k)\, T_2(k) \hat{R}_{12}$$

$$\hat{R}_{12}\, L_1(k) \hat{R}_{12}\, L_1(k) \;=\; L_1(k)\, \hat{R}_{12}\, L_1(k) \hat{R}_{12}\,,$$

$$\gamma^2\, T_1(k)\, L_2(k) \;=\; \hat{R}_{12}\, L_1(k)\, \hat{R}_{12}\, T_1(k)\,.$$

Here k is a discrete time. For any $\hat{R}$-matrix these automorphisms can be realized as (Faddeev–Alekseev discrete time evolution for the quantum top)

$$T(k) = L^k \cdot T\;, \quad L(k) = L\,.$$

One can compare with: $U^{(k)} = (q^{\frac{1}{2}} V)^k \cdot U\;, \quad V^{(k)} = V$.

5. Discrete time evolution for $SL_q(n)$ case

Consider the case when RTT algebra is $SL_q(n)$ quantum group. In this case we require

$$\det_q(T) = \mathrm{Tr}_{(1,\dots,n)}\left(\mathcal{A}^{(n)} T_1 T_2 \cdots T_n\right) = 1 \, .$$

Discrete time evolution must conserve this relation, i.e., we have $\det_q(L^k T) = 1 \; (\forall k > 0)$. This leads to the conditions

$$a_n = \mathrm{Tr}_{D(1,\dots,n)}\left(\mathcal{A}^{(n)} L_{\overline{1}} L_{\overline{2}} \cdots L_{\overline{n}}\right) = q^{-1} \, , \quad \gamma^n = q \, .$$

We will investigate the discrete evolution for HD of $SL_q(N)$ type. The key point is that $\exists$ the special evolution operator Θ:

$$T(k+1) = L\,T(k) = \Theta\,T(k)\,\Theta^{-1} \, , \quad L(k+1) = \Theta\,L(k)\,\Theta^{-1} = L \, .$$

For the case of "ribbon Hopf algebra" the Faddeev-Alekseev evolution is given by $\Theta \equiv$ ribbon element.

Thus, we have for the first shift $k = 1$:

$$L\,T = \Theta\,T\,\Theta^{-1}\,, \quad L = \Theta\,L\,\Theta^{-1}\,, \tag{2}$$

and we assume $\Theta = \Theta(\mu_1, \dots, \mu_n)$, where $\prod_{\alpha=1}^{n} \mu_\alpha = q^{-1}$.

For the HD with $\hat{R}$-matrix of the $SL_q(n)$-type the evolution operator $\Theta(\mu_\alpha)$ is a solution of first eq. in (2) which is written as

$$\Theta\big(\nabla^\alpha(\mu_\beta)\big) = q^{-1}\mu_\alpha^{-1}\,\Theta(\mu_\beta) \quad \forall\,\alpha = 1, \dots, n\,, \tag{3}$$

where ∇^α are finite shift operators $\nabla^\alpha(\mu_\beta) := q^{2X_{\alpha\beta}}\,\mu_\beta$ and the matrix X is a Gram matrix

$$X_{\alpha\beta} = \langle \vec{e}^{\,*}_\alpha, \vec{e}^{\,*}_\beta \rangle = \delta_{\alpha\beta} - \frac{1}{n} \quad (\alpha, \beta = 1, \dots, n)\,,$$

for the set of vectors: $\vec{e}^{\,*}_\alpha = \frac{1}{n}\,(\underbrace{-1, \dots, -1}_{(\alpha-1)\ \mathit{times}}, n-1, -1, \dots, -1\,)$.

As a result we obtain (special solution):

Proposition. *In case $|q| < 1$ a solution is expressed via multidimensional theta-function*

$$\Theta^{(1)}(\mu_\alpha) = \theta(\vec{p}, \Omega) = \sum_{\vec{k} \in \mathbf{Z}^{n-1}} \exp\left\{\pi \mathrm{i}\,(\vec{k}, \Omega\,\vec{k}) + 2\pi \mathrm{i}\,(\vec{k}\,, \vec{p})\right\},$$

where τ is a modular parameter, Ω is $(n-1) \times (n-1)$ matrix of periods

$$q = \exp(2\pi \mathrm{i}\,\tau), \quad q^{1/n}\mu_\alpha = \exp(2\pi \mathrm{i}\, p_\alpha), \ \ \textstyle\sum_{\alpha=1}^{n} p_\alpha = 0,$$

$$\Omega_{\alpha\beta} = \tfrac{2\tau}{n} A^*_{\alpha\beta} = 2\tau\,(\delta_{\alpha\beta} - \tfrac{1}{n})\,,$$

Expression $\Theta^{(1)}(\mu_\alpha)$ converges either if $|q| < 1$, or if $q^m = 1$ (the series is truncated).

The $(n-1) \times (n-1)$ matrix $A^*_{\alpha\beta}$ is a Gram matrix of a lattice A^*_{n-1} dual to the root lattice $A_{n-1} = sl(n)$, since we have $A^{*\,-1}_{\alpha\beta} = A_{\alpha\beta} = (\delta_{\alpha\beta} + 1)$ and $A_{\alpha\beta} = (e\alpha, e_\beta)$, where vectors $e_\alpha = (\underbrace{0, \ldots, 0}_{(\alpha-1)\ \text{times}}, 1, 0, \ldots, 0, -1)$

form the basis in the root space of $sl(n)$.

7. "Noncompact" solution for the evolution operator Θ

Proposition. *In case $|q| \geq 1$ one can find another solution:*

$$\Theta^{(2)}(p_\alpha) := \exp\Big(-\frac{\pi \mathrm{i}}{2\tau}\sum_{\beta=1}^{n} p_\beta^2\Big),$$

of the evolution equations.

Written in the independent variables $\vec{p} = \{p_1, \ldots, p_{n-1}\}$ it reads

$$\Theta^{(2)}(\vec{p}) = \exp\Big(-\frac{\pi \mathrm{i}}{\tau}\sum_{1\leq\alpha\leq\beta\leq n-1} p_\alpha p_\beta\Big) = \exp\Big\{-\pi \mathrm{i}\,(\vec{p},\,\Omega^{-1}\vec{p})\Big\},$$

where the inverse matrix of periods is

$$\Omega^{-1}_{\alpha\beta} = \frac{1}{2\tau}\,(\delta_{\alpha\beta} + 1) = \frac{1}{2\tau}\,A_{\alpha\beta}\,,$$

and $A_{\alpha\beta} = \langle e_\alpha, e_\beta\rangle$ is the Gram matrix for the root lattice A_{n-1}. Note that the logarithmic change of variables: $\log(\mu_\alpha)/(2\pi i) = p_\alpha - \tau/n$ which was rather superficial in case of $\Theta^{(1)}$, is inevitable for the derivation of $\Theta^{(2)}$.

Finally, we comment on relation between the two evolution operators $\Theta^{(1)} = \theta(\vec{p}, \Omega)$ and $\Theta^{(2)} = \exp\Big\{-\pi \mathrm{i}\,(\vec{p}, \Omega^{-1}\vec{p})\Big\}$. The relation is based on the identity for multidimensional theta functions

$$\theta(\Omega^{-1}\vec{p}, -\Omega^{-1}) = \Big(\det(\Omega/\mathrm{i})\Big)^{\frac{1}{2}} \exp\Big\{\pi \mathrm{i}(\vec{p}, \Omega^{-1}\vec{p})\Big\}\, \theta(\vec{p}, \Omega)\,.$$

With our particular matrix of periods Ω we find

$$\Theta^{(2)}(\vec{p}) = \frac{1}{\sqrt{n}}\Big(\frac{2\tau}{\mathrm{i}}\Big)^{\frac{n-1}{2}} \frac{\theta(\vec{p}, \Omega)}{\theta(\Omega^{-1}\vec{p}, -\Omega^{-1})}\,.$$

Note that theta function $\theta(\Omega^{-1}\vec{p}, -\Omega^{-1})$ (in the denominator) commutes with the elements of HD (defined by $SL_q(n)$ $\hat{R}$-matrix) and can be thought as an evolution operator on a ‘modular dual’ quantum cotangent bundle associated to dual $\hat{R}$-matrix of $SL_{\tilde{q}}(n)$ type.

8. Example

In the $SL_q(2)$ case the evolution operator $\Theta^{(1)}$ becomes the Jacobi theta function (L.D. Faddeev (1995)):

$$\Theta^{(1)}(\mu_1) = \sum_{k\in\mathbb{Z}} q^{\frac{1}{2}k(k+1)}\mu_1^k = \sum_{k\in\mathbb{Z}} \exp(\pi \mathrm{i}\, k^2\tau + 2\pi\mathrm{i}\, kz_1) = \theta_3(z_1; q)\,,$$

where $q = \exp(2\pi\mathrm{i}\,\tau)$, $\mu_1 = \exp(2\pi\mathrm{i}\, z_1)q^{-1/2}$. A multiplicative form for Θ is

$$\frac{1}{\eta(q)}\Theta^{(1)}(\mu_1) = \prod_{n=1}^{\infty}(1+q^n\mu_1)(1+q^{n-1}/\mu_1) = \prod_{n=1}^{\infty}(1+q^n\sigma_1+q^{2n-1})\,,$$

where $\eta(q) = \prod_{n=1}^{\infty}(1-q^n)$. For dual evolution operator we have

$$\widetilde{\Theta}^{(1)}(\mu_1) = \sum_{k\in\mathbb{Z}} \exp(-\frac{\pi\mathrm{i}}{\tau}k^2 + \frac{2\pi\mathrm{i}}{\tau}kz_1) = \sum_{k\in\mathbb{Z}} \tilde{q}^{\frac{1}{2}k(k+1)}\tilde{\mu}_1^k\,,$$

where $\tilde{q} = \exp(-\frac{2\pi\mathrm{i}}{\tau})$, $\tilde{\mu}_1 = \exp(\frac{2\pi\mathrm{i}}{\tau}z_1)\tilde{q}^{-1/2}$.

Summary

- What is a dual $\overline{HD}$ for the standard HD of $SL_q(n)$ type ($\overline{HD}$ centralizes HD and vice versa)?
- Explicit expressions for evolution operator Θ in the case of B, C, D quantum groups. In these cases Gram matrices A and their dual $A^* = (A)^{-1}$ are such that B and C type evolution operators are dual to each other.
- 3D analogue of RE (tetrahedron RE) were proposed in A.P.Isaev and P.P.Kulish, Mod. Phys. Lett. **A12** (1997) 427 (hep-th/9702013). The analog of 3D RTT algebra is also known: $R_{123}T_1T_2T_3 = T_3T_2T_1R_{123}$. What kind of cross-commutation relations are needed to describe discrete evolution in 3D case?