CURIOUS SYMMETRY OF CHIRAL FERMIONS

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''Chiral fermions as classical massless spinning particles,'' arXiv:1406.0718 [hep-th].

September 12, 2014

Stephanov & Yin PRL 2012 (semi)classical model derived from Weyl Hamiltonian $H = \boldsymbol{\sigma} \cdot \boldsymbol{p} \rightsquigarrow$ spin-1/2 system with positive helicity and energy, described by phase-space action $S =$

$$\int \left((\boldsymbol{p} + e\boldsymbol{A}) \cdot \frac{d\boldsymbol{x}}{dt} - (|\boldsymbol{p}| + e\phi) - \boxed{\boldsymbol{a} \cdot \frac{d\boldsymbol{p}}{dt}} \right) dt. \qquad (1)$$

$\boldsymbol{a}(\boldsymbol{p})$ "momentum-dependent vector potential" for "Berry monopole" in $\boldsymbol{p}$-space

$$\boldsymbol{\nabla}_{\boldsymbol{p}} \times \boldsymbol{a} = \boldsymbol{\Theta} \equiv \frac{\widehat{\boldsymbol{p}}}{2|\boldsymbol{p}|^2} \qquad (2)$$

where $\widehat{\boldsymbol{p}} = \boldsymbol{p}/|\boldsymbol{p}|$, $\boldsymbol{A}(\boldsymbol{x}, t)$ & $\phi(\boldsymbol{x}, t)$ "ordinary" vector & scalar potentials.

Fundamental assumption: Spin "enslaved"

$$\boxed{\boldsymbol{s} = s\,\widehat{\boldsymbol{p}}} \qquad s = \tfrac{1}{2}. \qquad (3)$$

Lack of manifest Lorentz symmetry .

Chen-Son-Stephanov-Yee-Yin, arXiv:1404.5963
Manuel, Torres-Rincon, arXiv:1404.6409 :
Lorentz algebra acts as

$$\begin{cases} \delta\boldsymbol{p} &= \boldsymbol{\omega}\times\boldsymbol{p} + |\boldsymbol{p}|\,\boldsymbol{\beta}\,, \\ \delta\boldsymbol{x} &= \boldsymbol{\omega}\times\boldsymbol{x} + \boldsymbol{\beta}\times(s\dfrac{\boldsymbol{p}}{|\boldsymbol{p}|^2}) - \boldsymbol{\beta}\cdot\boldsymbol{x}\,\dfrac{\boldsymbol{p}}{|\boldsymbol{p}|}\,. \end{cases} \tag{4}$$

N.B. : *not* a space-time action !
"dynamical" symmetry !

Duval, PAH arXiv: 1406.0718 :

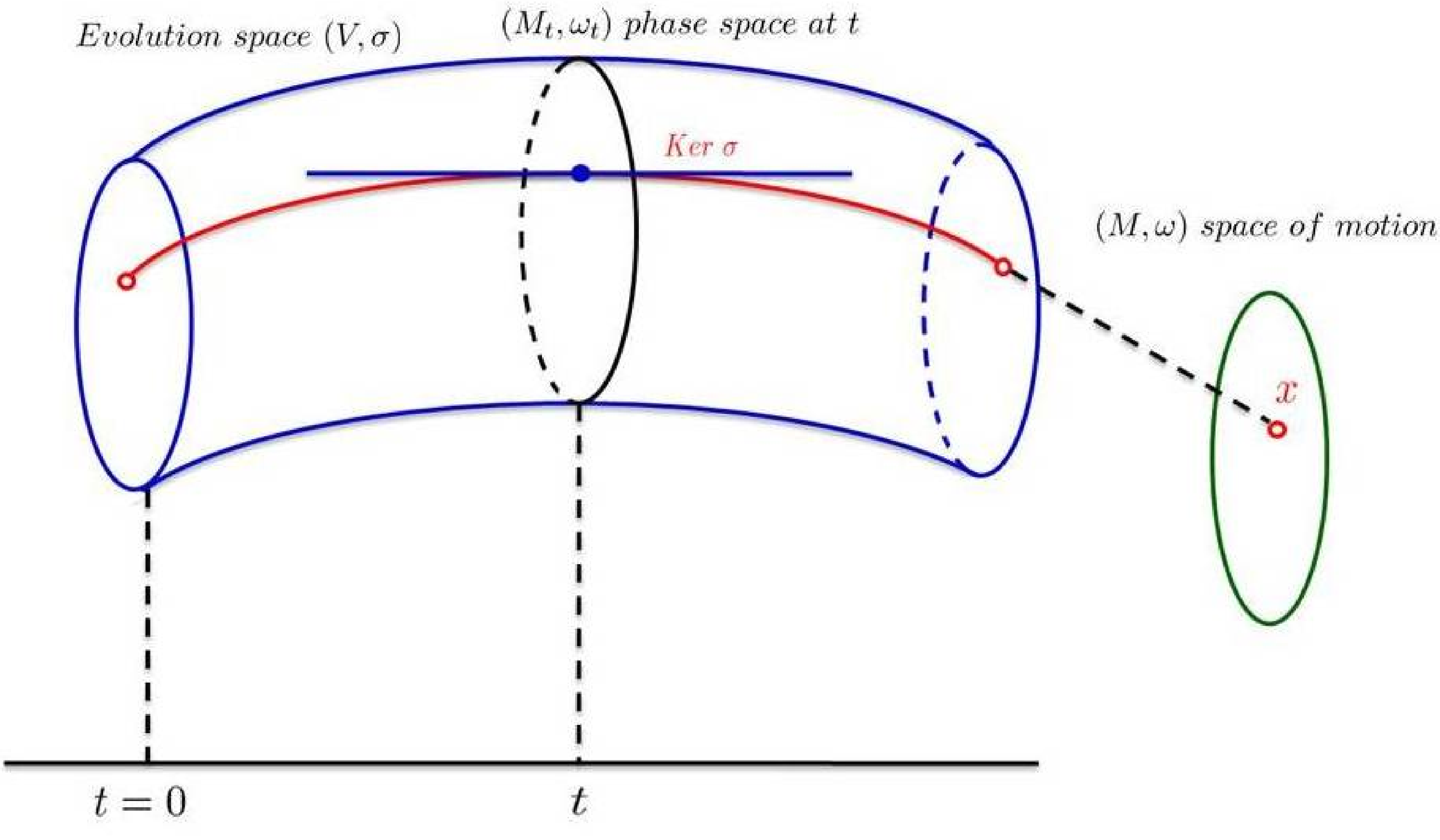
1. *chiral system embedded into Souriau's massless spinning particle with Poincaré symmetry;*

2. *Spin enslavement inconsistent* with *natural Lorentz symmetry*

3. *twisted (4) but no natural Lorentz symmetry for chiral system (1).*

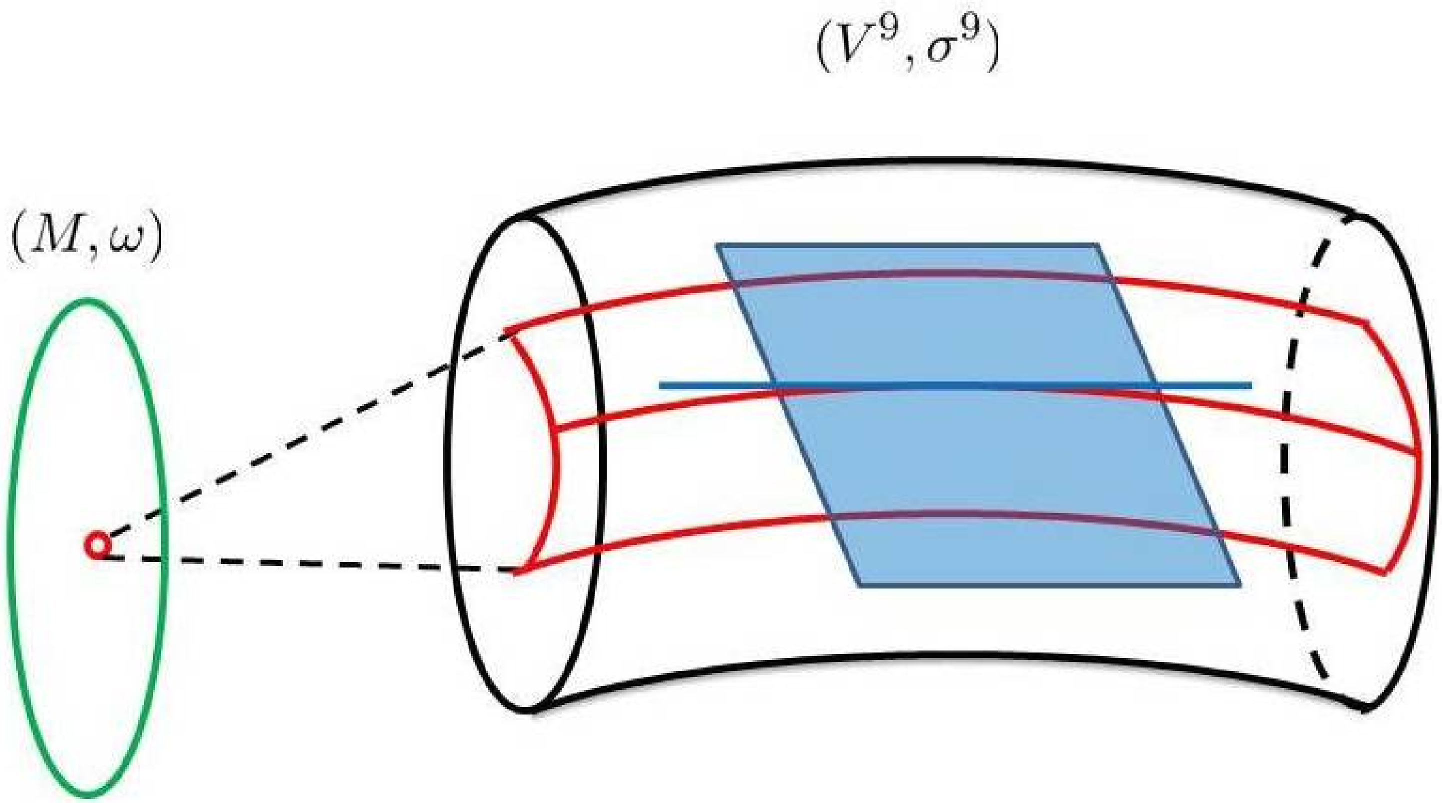
Souriau's mechanics

Dynamics determined by closed two-form σ of constant rank defined on "evolution space" V. Motions [curves or surfaces] $\equiv$ "characteristic leaves", tangent to kernel of two-form σ.



Can be viewed as generalized variational calculus.

Space of motions (M, ω) symplectic manifold, obtained from (V, σ) by factoring out "characteristic submanifolds".



phase space [at time t] is section of evolution space for fixed value t.

Symplectic description of chiral model

Variation of chiral action (1) yields eqns of motion for position $\boldsymbol{x}$ and momentum $\boldsymbol{p} \neq 0$ in 3-space,

$$\begin{cases} \left(1 + e\boldsymbol{\Theta}\cdot\boldsymbol{B}\right)\dfrac{d\boldsymbol{x}}{dt} &= \widehat{\boldsymbol{p}} + \boxed{e\,\boldsymbol{E}\times\boldsymbol{\Theta} + (\boldsymbol{\Theta}\cdot\widehat{\boldsymbol{p}})\,e\,\boldsymbol{B}} \\ \left(1 + e\boldsymbol{\Theta}\cdot\boldsymbol{B}\right)\dfrac{d\boldsymbol{p}}{dt} &= e\,\boldsymbol{E} + e\,\widehat{\boldsymbol{p}}\times\boldsymbol{B} + e^2(\boldsymbol{E}\cdot\boldsymbol{B})\,\boldsymbol{\Theta}, \end{cases} \tag{5}$$

$\boldsymbol{E}$, $\boldsymbol{B}$ electric/magnetic field. “anomalous velocity” $\rightsquigarrow$ “transverse shifts” or “side jumps” in spin-Hall-type effects.

Described within Souriau’s framework. Evolution space

$$V^7 = T(\mathbb{R}^3\setminus\{0\})\times\mathbb{R}, \tag{6}$$

endowed with two-form,

$$\omega = \omega_0 + \frac{e}{2}\epsilon_{ijk}B^i\,dx^j\wedge dx^k, \tag{7}$$

$$\omega_0 = dp_i\wedge dx^i - \frac{s}{2|\boldsymbol{p}|^3}\epsilon^{ijk}\,p_i\,dp_j\wedge dp_k, \tag{8}$$

$$h = |\boldsymbol{p}| + e\phi, \tag{9}$$

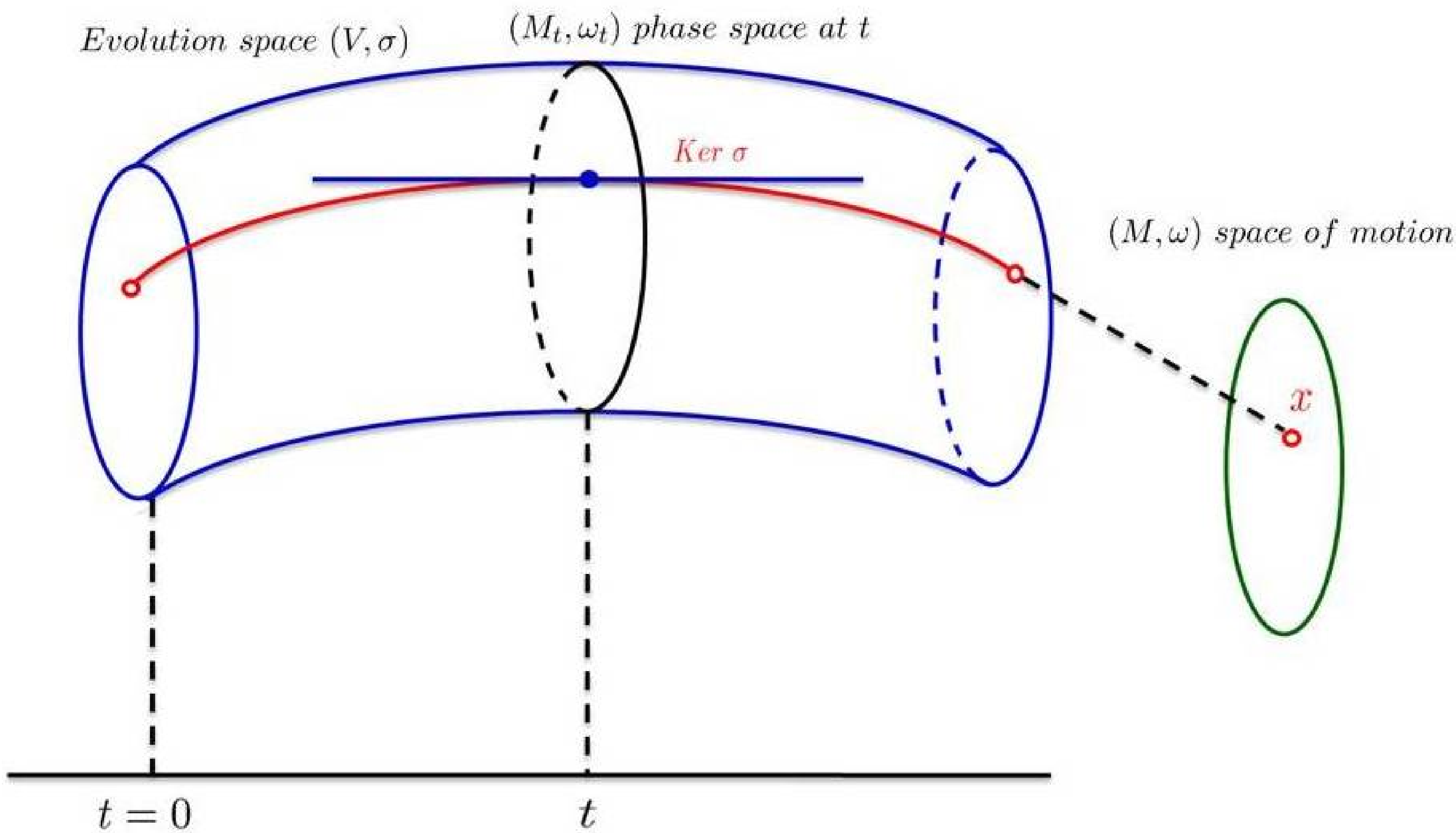
($s = 1/2$). $\omega \Rightarrow \sigma$ closed since

$$\nabla_{\boldsymbol{x}} \cdot \boldsymbol{B} = 0, \quad \nabla_{\boldsymbol{p}} \cdot \boldsymbol{\Theta} = 0 \tag{10}$$

see (2). Where

$$\det(\omega_{\alpha\beta}) = (1 + e\,\boldsymbol{\Theta} \cdot \boldsymbol{B})^2 \neq 0, \tag{11}$$

kernel of σ 1-dimensional; curve $\big(\boldsymbol{x}(\tau), \boldsymbol{p}(\tau), t(\tau)\big)$ tangent to it iff eqns of motion (5) satisfied.



Souriau's Massless Spinning Particles

Souriau 1969 *Free relativistic massless spinning particle* described by 9-dim evolution space V^9. Start with three four-vectors R, I, J in Minkowski space $\mathbb{R}^{3,1}$ with signature $(-,-,-,+)$.

$$V^9 = \left\{ R, I, J \in \mathbb{R}^{3,1} \,\middle|\, I_\mu I^\mu = J_\mu J^\mu = 0, I_\mu J^\mu = -1 \right\} \tag{12}$$

I future-directed. Lightlike vectors I, J generat null 2-plane. R represents spacetime event.

Equivalent but more convenient description uses the spin tensor. Renaming $P = I$ (will be interpreted as linear momentum),

$$S_{\mu\nu} = -s\, \epsilon_{\mu\nu\rho\sigma}\, P^\rho J^\sigma. \tag{13}$$

Satisfies $\frac{1}{2} S_{\mu\nu}\, S^{\mu\nu} = s^2$, where $s \neq 0$ scalar spin (also called helicity). Pauli-Lubanski condition $S_{\mu\nu} P^\nu = 0$ satisfied.

Identifying $S = (S_{\mu\nu})$ with element of Lorentz algebra $\mathfrak{o}(3,1)$, evolutions space also presented as

$$\left\{ R, P, S \,\middle|\, P_\mu P^\mu = 0, S_{\mu\nu} P^\nu = 0, \tfrac{1}{2} S_{\mu\nu} S^{\mu\nu} = s^2 \right\} \tag{14}$$

V^9 endowed with closed two-form

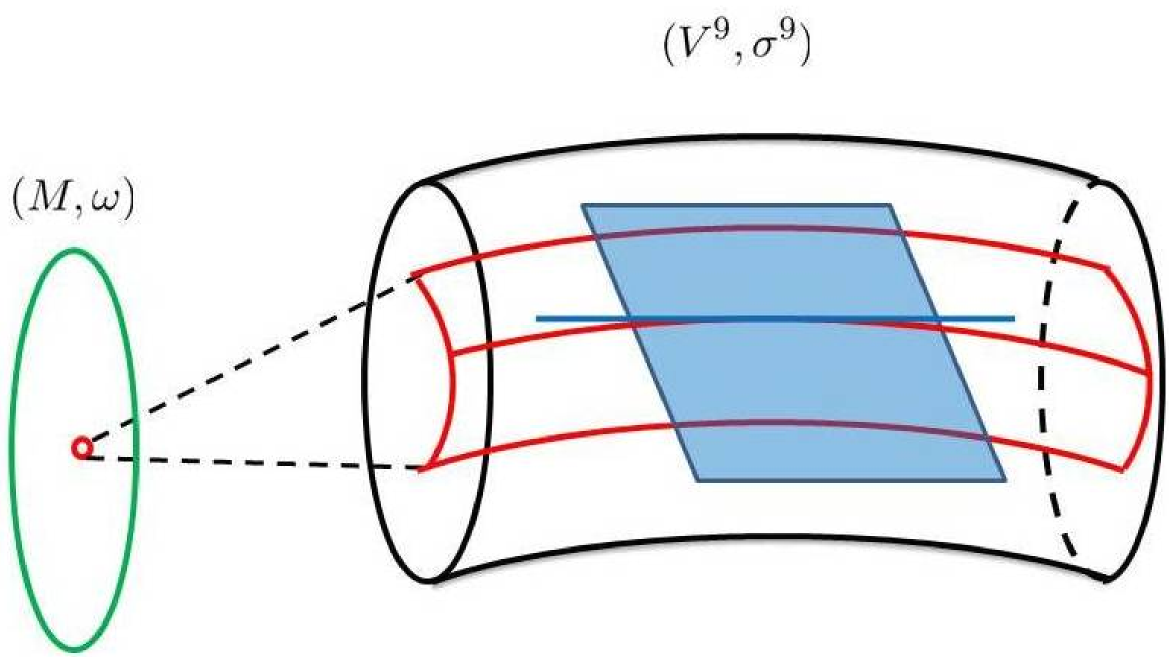
$$\sigma = -dP_\mu \wedge dR^\mu - \frac{1}{2s^2} dS^\mu{}_\lambda \wedge S^\lambda{}_\rho \, dS^\rho{}_\mu \,. \qquad (15)$$

Dynamics given by foliation whose leaves are tangent to $\ker\sigma$; "world-sheet" [or world-line] obtained by projecting leaf to Minkowski space, yielding corresponding spacetime track.

Calculating kernel using constraints which define evolution space $\Rightarrow$ curve $(R(\tau), P(\tau), S(\tau))$ tangent to $\ker\sigma$ iff

$$\begin{cases} P_\mu \dot{R}^\mu &= 0, \\ \dot{P}^\mu &= 0, \\ \dot{S}^{\mu\nu} &= P^\mu \dot{R}^\nu - P^\nu \dot{R}^\mu, \end{cases} \qquad (16)$$

where "dot" $\equiv d/d\tau$. Spacetime velocity $\dot{R}$ orthogonal to momentum P.



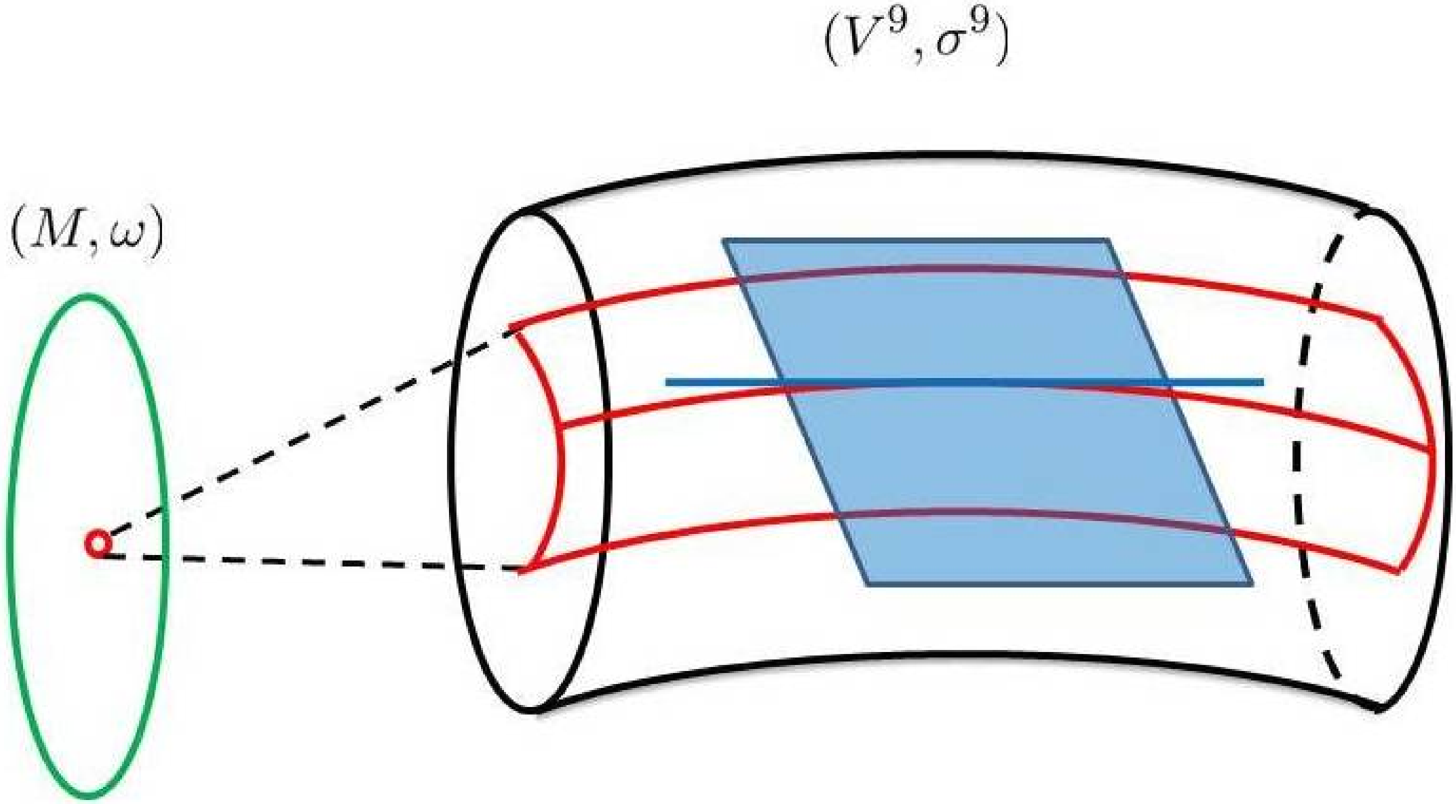
Wigner translations

(16) integrated using spacetime vectors Z orthogonal to P, $P_\mu Z^\mu = 0$,

$$\begin{cases} R^\mu & \to & R^\mu + Z^\mu, \\ P^\mu & \to & P^\mu, \\ S^{\mu\nu} & \to & S^{\mu\nu} + (P^\mu Z^\nu - P^\nu Z^\mu). \end{cases} \tag{17}$$

Any point in leaf reached by choosing suitable $Z \Rightarrow$ at each point kernel is 3-*dim*, projects to spacetime as affine subspace of $\mathbb{R}^{3,1}$, spanned by all vectors orthogonal to linear momentum $P \Rightarrow$ "motions" of free massless spinning particle take place on 3-dimensional "plane-wave" tangent to light-cone at each spacetime event R: particle is *not localized* in spacetime. Wigner 39, Penrose 67, Souriau 69.

All curves which lie in a leaf (left invariant by a "Z-shift" in (17)) should be considered as same motion. Each (3-dimensional) leaf defines therefore "motion" of particle. Space of motions is collection $M^6 = V^9/\ker\sigma$ of leaves. Explained below: spin is responsible for spacetime delocalization of massless particles.



Free massless spinning particle has 9-dim evolution space V^9. Dynamics defined by two-form σ. $\ker \sigma$ *3-dimensional, whose points reached by a "Z-shift" ("Wigner translation"). Characteristic leaves project into Minkowski as 3-planes orthogonal to momentum : massless spinning particle can not be localized.*

Put $R = (\mathbf{r}, t)$ where $\mathbf{r}$ and t are position & time in chosen Lorentz frame. Null-vectors $P = (\boldsymbol{p}, |\boldsymbol{p}|)$ and $J = (\mathbf{q}, -|\mathbf{q}|)$, where $\boldsymbol{p}$ and $\mathbf{q}$ two 3-vectors which satisfy $\boldsymbol{p} \cdot \mathbf{q} + |\boldsymbol{p}|\,|\mathbf{q}| = 1$.

$$S_{ij} = \epsilon_{ijk}\, s^k, \qquad \boldsymbol{s} = s\big(\boldsymbol{p}|\mathbf{q}| + \mathbf{q}|\boldsymbol{p}|\big), \quad (18)$$

$$S_{j4} = s\big(\boldsymbol{p} \times \mathbf{q}\big)_j = \big(\widehat{\boldsymbol{p}} \times \boldsymbol{s}\big)_j. \quad (19)$$

Important observation

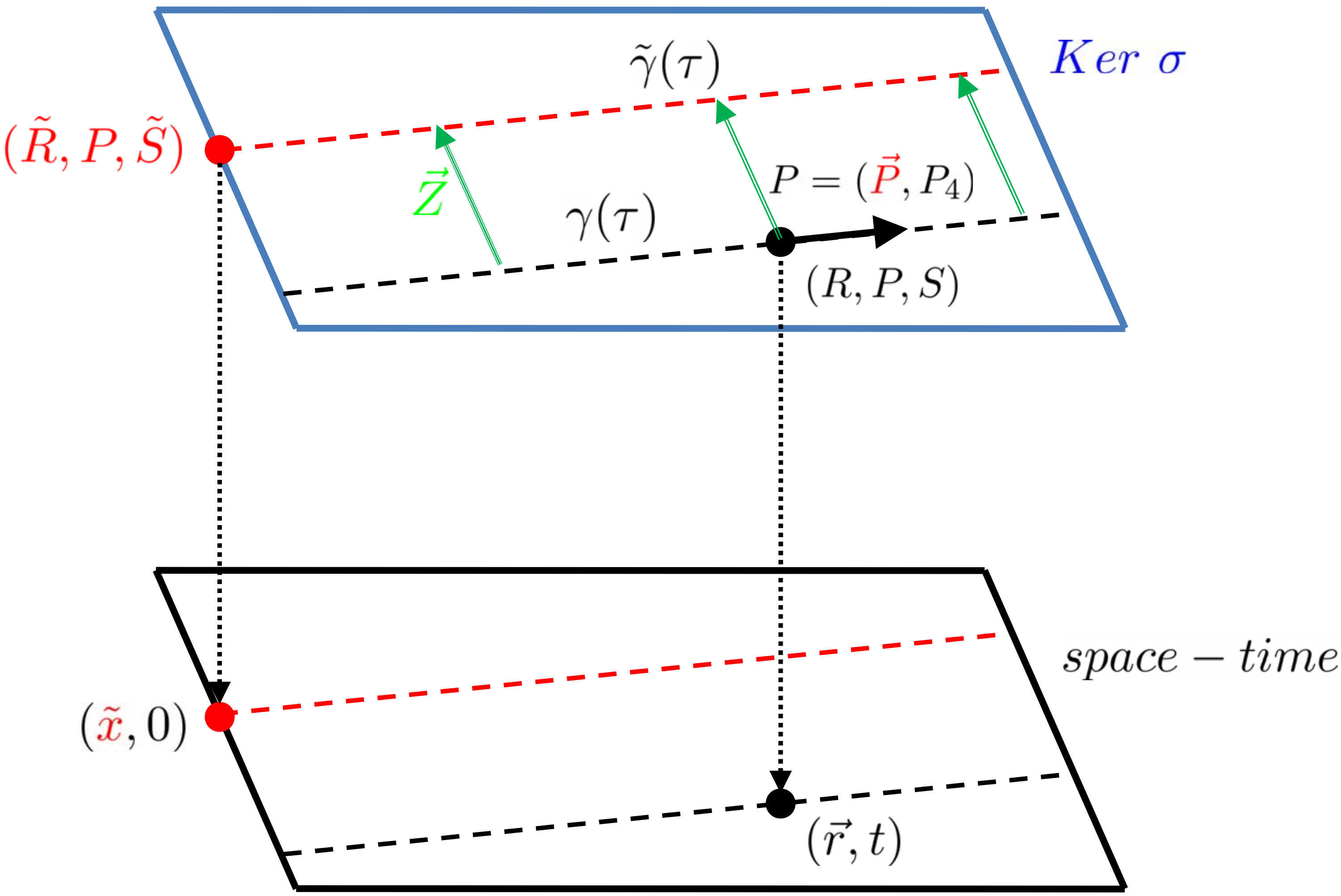
$$\widehat{\boldsymbol{p}} \cdot \boldsymbol{s} = s \tag{20}$$

in general. s *not* length of $\boldsymbol{s}$ but *projection* onto $\widehat{\boldsymbol{p}}$, which is a constant.

Wigner transl. used to "enslave" spin

$$\boxed{\boldsymbol{s} = s\widehat{\boldsymbol{p}}} \tag{21}$$

SPIN eliminated as indept degree of freedom.



The spin of a motion tangent to (3d) kernel of σ can be "enslaved" by suitable "Z-shift". Chosing point on shifted curve with vanishing time coordinate provides us with coordinate $\tilde{\boldsymbol{x}}$ of "motion". The displaced motions sweep a 3-plane under all Z-shifts.

Two-form σ descends to space of motions M^6 as symplectic two-form

$$\omega = d\tilde{p}_i \wedge d\tilde{x}^i - \frac{s}{2|\tilde{\boldsymbol{p}}|^3}\, \epsilon^{ijk}\, \tilde{p}_i\, d\tilde{p}_j \wedge d\tilde{p}_k\,. \qquad (22)$$

Poincaré symmetry

Poincaré algebra spanned by (Λ, Γ) where $\Lambda = (\Lambda_{\mu\nu})$ belongs to Lorentz algebra and $\Gamma = (\Gamma^\mu)$ is translation in Minkowski space. Acts on V^9:

$$\begin{cases} \delta R^\mu = \Lambda^\mu_\nu R^\nu + \Gamma^\mu, \\ \delta P^\mu = \Lambda^\mu_\nu P^\nu, \\ \delta S_{\mu\nu} = \Lambda^\mu_\rho S^{\rho\nu} - \Lambda^\nu_\rho S^{\rho\mu}. \end{cases} \tag{23}$$

Leaves two-form (15) invariant $\rightsquigarrow$ symmetry. Action descends to space of motions (M^6, ω). Noetherian conserved quantities

$$P^\mu = I^\mu, \qquad M^{\mu\nu} = R^\mu P^\nu - R^\nu P^\mu + S^{\mu\nu} \tag{24}$$

conserved linear and angular momentum.

In (3+1)-decomposition: parametrize Poincaré algebra by $\Lambda_{ij} = \epsilon_{ijk}\,\omega^k$, $\Lambda_{i4} = \beta^i$, $\Gamma = (\boldsymbol{\gamma}, \varepsilon)$, $\boldsymbol{\omega}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^3, \varepsilon \in \mathbb{R}$ (rotations, boosts and space-resp. time-translations). Inf. Poincaré-action on V^9 given by

$$\begin{cases} \delta \mathbf{r} &=& \boldsymbol{\omega} \times \mathbf{r} + \boldsymbol{\beta} t + \boldsymbol{\gamma}\,, \\ \delta t &=& \boldsymbol{\beta} \cdot \mathbf{r} + \varepsilon\,, \\ \delta \boldsymbol{p} &=& \boldsymbol{\omega} \times \boldsymbol{p} + \boldsymbol{\beta}\,|\boldsymbol{p}|\,, \\ \delta \boldsymbol{s} &=& \boldsymbol{\omega} \times \boldsymbol{s} - \boldsymbol{\beta} \times (\widehat{\boldsymbol{p}} \times \boldsymbol{s})\,, \end{cases} \tag{25}$$

projects as natural action on Minkowski space.

To write down explicit form of Poincaré momenta, present matrix $M = (M_{\mu\nu})$ as $M_{ij} = \epsilon_{ijk}\,\ell^k$, $M_{j4} = g^j$ with $\boldsymbol{\ell}$, $\boldsymbol{g}$ 3-vectors.

$$\begin{cases} \boldsymbol{\ell} &= \mathbf{r} \times \boldsymbol{p} + \boldsymbol{s}, \\ \boldsymbol{g} &= |\boldsymbol{p}|\,\mathbf{r} - \boldsymbol{p}t + \widehat{\boldsymbol{p}} \times \boldsymbol{s}. \end{cases} \tag{26}$$

Then

$$\boxed{\tilde{\boldsymbol{x}} = \frac{\boldsymbol{g}}{|\boldsymbol{p}|} = \mathbf{r} - \widehat{\boldsymbol{p}}\,t + \frac{\widehat{\boldsymbol{p}}}{|\boldsymbol{p}|} \times \boldsymbol{s}} \tag{27}$$

itself conserved.

Full Poincaré algebra acts on space of motions as

$$\begin{cases} \delta\tilde{\boldsymbol{p}} &= \boldsymbol{\omega} \times \tilde{\boldsymbol{p}} + |\tilde{\boldsymbol{p}}|\,\boldsymbol{\beta}\,, \\ \delta\tilde{\boldsymbol{x}} &= \boldsymbol{\omega} \times \tilde{\boldsymbol{x}} + \boldsymbol{\beta} \times (s\frac{\tilde{\boldsymbol{p}}}{|\tilde{\boldsymbol{p}}|^2}) - \boldsymbol{\beta} \cdot \tilde{\boldsymbol{x}}\,\frac{\tilde{\boldsymbol{p}}}{|\tilde{\boldsymbol{p}}|} \\ &\qquad + \boldsymbol{\gamma} - \varepsilon\,\frac{\tilde{\boldsymbol{p}}}{|\tilde{\boldsymbol{p}}|}\,. \end{cases} \tag{28}$$

10-parameter vectorfield (28) leaves free symplectic structure invariant, i.e., generates family of symmetries ⇝ Noether thm ⇝ 10 constants of the motion,

$$\left\{\begin{array}{lll}\boldsymbol{\ell} & = & \tilde{\boldsymbol{x}}\times\tilde{\boldsymbol{p}}+s\widehat{\boldsymbol{p}} \qquad \text{angular momentum}\\ \boldsymbol{g} & = & |\tilde{\boldsymbol{p}}|\,\tilde{\boldsymbol{x}} \qquad \text{boost momentum}\\ \boldsymbol{p} & = & \tilde{\boldsymbol{p}} \qquad \text{linear momentum}\\ \mathcal{E} & = & |\tilde{\boldsymbol{p}}| \qquad \text{energy}\end{array}\right. \tag{29}$$

Conservation follows also directly from free eqns of motions. NB: two terms in angular momentum $\boldsymbol{\ell}$ separately conserved.

Poisson brackets of calculated using (22),

$$\begin{array}{lll}\{\ell_i,\ell_j\}=-\epsilon_{ij}^k\,\ell_k, & \{\ell_i,g_j\}=-\epsilon_{ij}^k\,g_k, & \{\ell_i,p_j\}=-\epsilon_{ij}^k\,p_k,\\ \{\ell_i,\mathcal{E}\}=0, & \{g_i,g_j\}=\epsilon_{ij}^k\,\ell_k, & \{g_i,p_j\}=-\mathcal{E}\,\delta_{ij},\\ \{g_i,\mathcal{E}\}=-p_i, & \{p_i,p_j\}=0, & \{p_i,\mathcal{E}\}=0,\end{array} \tag{30}$$

Poincaré algebra. Casimir invariants

$$m^2=-\boldsymbol{p}^2+\mathcal{E}^2=0\,,\qquad \boldsymbol{\ell}\cdot\widehat{\boldsymbol{p}}=s\,, \tag{31}$$

zero-mass and spin-s representation.

Twisted Poincaré symmetry of free chiral model

Poincaré symmetry for free chiral system ?

In free case $\boldsymbol{E} = \boldsymbol{B} = 0$ motions determined explicitly: $\boldsymbol{\Theta}$-term drops out from (5),

$$\begin{cases} \Big(1 + e\boldsymbol{\Theta}\cdot\boldsymbol{B}\Big)\dfrac{d\boldsymbol{x}}{dt} &= \widehat{\boldsymbol{p}} + e\,\boldsymbol{E}\times\boldsymbol{\Theta} + (\boldsymbol{\Theta}\cdot\widehat{\boldsymbol{p}})\,e\,\boldsymbol{B}, \\ \Big(1 + e\boldsymbol{\Theta}\cdot\boldsymbol{B}\Big)\dfrac{d\boldsymbol{p}}{dt} &= e\,\boldsymbol{E} + e\,\widehat{\boldsymbol{p}}\times\boldsymbol{B} + e^2(\boldsymbol{E}\cdot\boldsymbol{B})\,\boldsymbol{\Theta}, \end{cases}$$

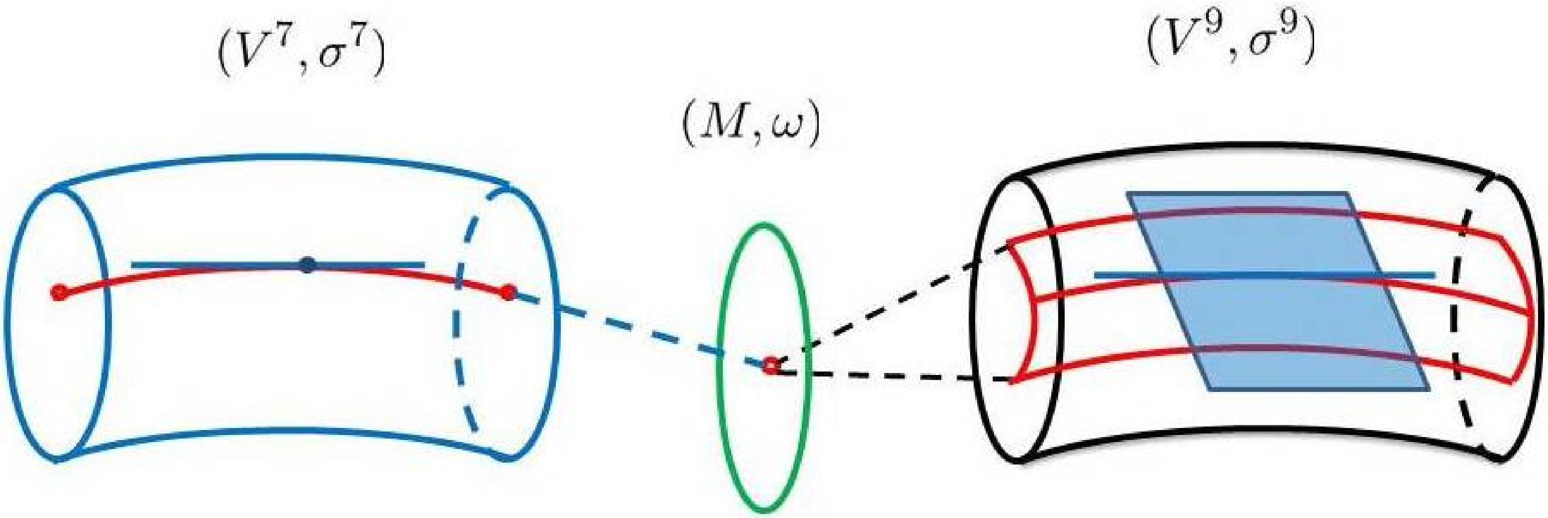
integrated as:

$$\boldsymbol{x}(t) = \tilde{\boldsymbol{x}} + \widehat{\boldsymbol{p}}\,t, \qquad \boldsymbol{p}(t) = \tilde{\boldsymbol{p}}, \qquad (32)$$

with $\tilde{\boldsymbol{x}}$, $\tilde{\boldsymbol{p}}$ const. Space of motions $M^6 = V^7/\ker\sigma$ described by $\tilde{\boldsymbol{x}} = \boldsymbol{x}(t) - \widehat{\boldsymbol{p}}\,t$ and $\tilde{\boldsymbol{p}}$. Two-form ω is (22) :

$$\omega \;=\; \omega_0 = dp_i\wedge dx^i - \frac{s}{2|\boldsymbol{p}|^3}\epsilon^{ijk}\,p_i\,dp_j\wedge dp_k.$$

Free chiral model has same space of motions as massless spinning particle.



Chiral and massless spinning systems admit same space of motions.

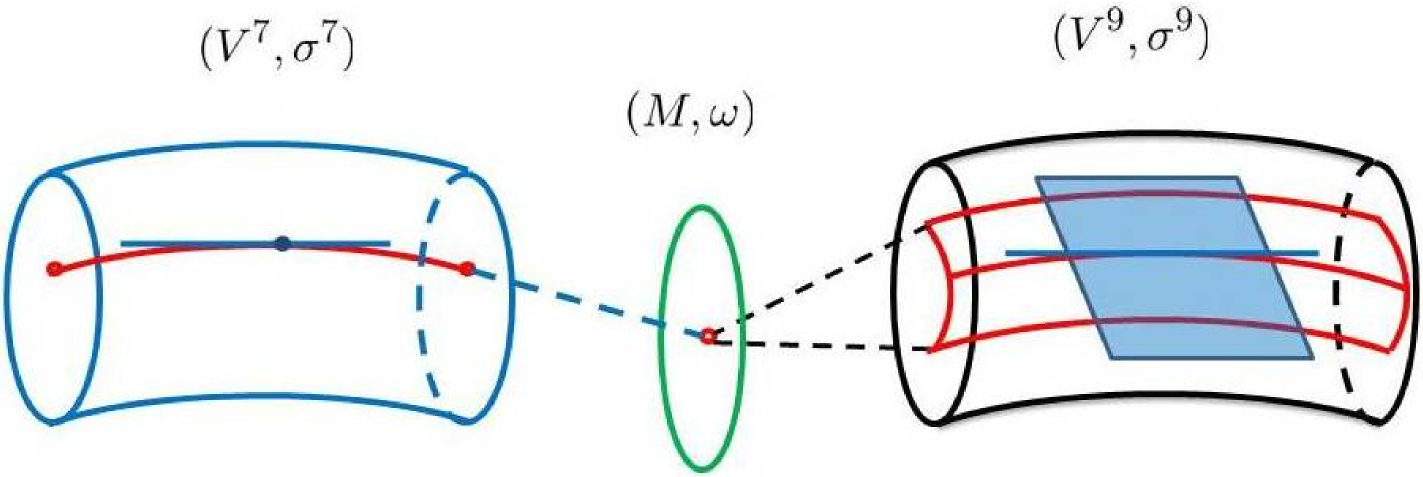
Strategy: *import" natural Poincaré symmetry of Souriau model to chiral system through their common space of motions* (M, ω).

From identity of space of motions coordinates conclude that position of chiral particle and of massless Poincaré model are *same*, $\boldsymbol{x} = \mathbf{r}$. In terms of coordinates $(\boldsymbol{x}, \boldsymbol{p}, t)$ on chiral evolution space V^7, Poincaré action (28) becomes "twisted",

$$\begin{cases} \delta \boldsymbol{x} &= \boldsymbol{\omega} \times \boldsymbol{x} + \boldsymbol{\beta} \times \dfrac{\widehat{\boldsymbol{p}}}{2|\boldsymbol{p}|} + \boldsymbol{\beta}\, t + \boldsymbol{\gamma}, \\ \delta \boldsymbol{p} &= \boldsymbol{\omega} \times \boldsymbol{p} + |\boldsymbol{p}|\, \boldsymbol{\beta}, \\ \delta t &= \boldsymbol{\beta} \cdot \boldsymbol{x} + \varepsilon. \end{cases} \tag{33}$$

Vector fields generate same (Poincaré) algebra as in (28). Eqn (33) confirms and extends recently proposed action of Lorentz subalgebra Chen, Son, Stephanov ... Conserved Lorentz quantities,

$$\begin{cases} \boldsymbol{\ell} &= \boldsymbol{x} \times \boldsymbol{p} + \frac{1}{2}\widehat{\boldsymbol{p}}\,, \\ \boldsymbol{g} &= |\boldsymbol{p}|\,\boldsymbol{x} - \boldsymbol{p}t\,. \end{cases} \tag{34}$$



Identity of spaces of motion allows to "export" natural Poincaré symmetry of latter to become "twisted" ("dynamical") symmetry of former.

N.B. action (33) is *not* usual, natural one on ordinary spacetime. In fact, it is *not* action on spacetime at all, since it also involves momentum variable $\boldsymbol{p}$; is sort of "dynamical symmetry". $\boldsymbol{x}$ should *not* be viewed as position variable, because *does not transform* under a boost as positions should.

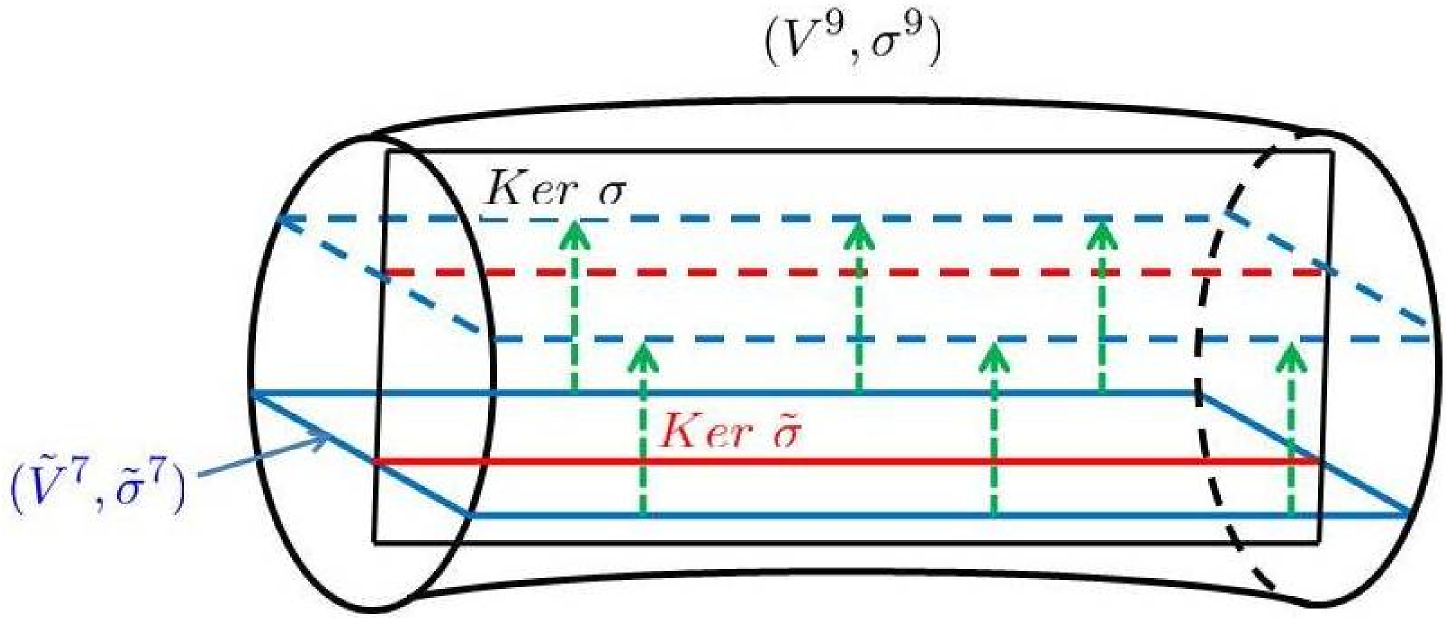
Embedding chiral system into massless spinning model

Further insight gained by *embedding evolution space of chiral system*, V^7, into that, V^9, of massless spinning particle by "enslaving spin". Condition

$$S_{j4} = (\hat{\boldsymbol{p}} \times \boldsymbol{s})_j = 0 \qquad (35)$$

defines submanifold $\widetilde{V}^7$ of V^9 identified with V^7; restriction of free two-form of V^9 to $\widetilde{V}^7$ is (7)–(8).

Dynamics consistent with embedding. Motions of chiral system lie within leaves of 3-dimensional foliation of V^9 and remain therefore motions also for extended system.



Chiral evolution space can be embedded, $V^7 \to \widetilde{V}^7$, *into that,* $\mathbf{V}^9$, *of massless spinning particle by enslaving spin by requiring* $\boldsymbol{p} \times \boldsymbol{s} = 0$. *Constraint is* *not* *Lorentz-invariant : Lorentz boost* $\boldsymbol{\beta}$ *does not leave* $\boldsymbol{p} \times \boldsymbol{s}$ *invariant.*

Lorentz symmetry inherited from embedding ?

no : consider chiral motion $\mathbf{r} - \widehat{\boldsymbol{p}}t$ embedded into V^9 s.t. $\boldsymbol{s} = s\,\widehat{\boldsymbol{p}}$ and boost. Although spin is boost-invariant,

$$\delta(\boldsymbol{p} \times \boldsymbol{s}) = s\,\boldsymbol{\beta} \times \boldsymbol{p} \neq 0 \qquad (36)$$

in general, $\Rightarrow$ *spin and momentum do not remain parallel* : embedding chiral system through spin enslavement is *not* boost invariant.

While full Poincaré group acts on V^9, it is only its "Aristotle subgroup" generated by space rotations and by space- and time translations which leaves $\widetilde{V}^7$ invariant : *natural boosts are broken* by spin enslavement.

Lorentz symmetry restored by "unchaining spin". Boosting a trajectory in V^9 according to (25),

$$\begin{aligned} \delta\Big(\mathbf{r}-\widehat{\boldsymbol{p}}t\Big) &= -\widehat{\boldsymbol{p}}\Big(\boldsymbol{\beta}\cdot(\mathbf{r}-\widehat{\boldsymbol{p}}t)\Big), \\ \delta\Big(\frac{\widehat{\boldsymbol{p}}}{|\boldsymbol{p}|}\times\boldsymbol{s}\Big) &= \boldsymbol{\beta}\times\Big(s\frac{\boldsymbol{p}}{|\boldsymbol{p}|^2}\Big)-\frac{\boldsymbol{p}}{|\boldsymbol{p}|^2}\Big(\boldsymbol{\beta}\cdot(\widehat{\boldsymbol{p}}\times\boldsymbol{s})\Big). \end{aligned}$$

Terms combine to yield action (28) on space of motions,

$$\delta\tilde{\boldsymbol{x}}=\delta\left(\mathbf{r}-\widehat{\boldsymbol{p}}\,t+\frac{\widehat{\boldsymbol{p}}}{|\boldsymbol{p}|}\times\boldsymbol{s}\right)=\boldsymbol{\beta}\times\Big(s\frac{\widehat{\boldsymbol{p}}}{|\boldsymbol{p}|}\Big)-\widehat{\boldsymbol{p}}\,(\boldsymbol{\beta}\cdot\tilde{\boldsymbol{x}}). \tag{37}$$

Coupling to electromagnetic field

Conventional "minimal coupling" rule : 4-momentum shifted by 4-potential,

$$p_\mu \to p_\mu - eA_\mu. \tag{38}$$

not what is proposed in (1): rule (38) used for 4-momentum $(\boldsymbol{p}, h)$, momentum in "Berry term" $\Theta(\boldsymbol{p})$ *not* shifted.

"half-way-rule" in (1) is instead consistent with Souriau's prescription, who works with same evolution space as for free particle, but adds electromagnetic field strength F to free two-form (15) ,

$$\sigma \to \sigma + eF\,, \tag{39}$$

where e is electric charge. σ closed, $d\sigma = 0$, because F is closed 2-form of Minkowski space.

Rules (38) and (39) equivalent in spinless case only. Why should (39) be chosen ? Argument in its favor comes form experience in plane, where yielded insight into Hall-type phenomena cf. Duval et al Phys. Lett. **B 479**, 284 (2000).

Minimal coupling of massless spinning model

Souriau's prescription (39) applied to massless spinning model yields, on evolution space V^9, closed two-form

$$\sigma = -dP_\mu \wedge dR^\mu - \frac{1}{2s^2}\, dS^\mu{}_\lambda \wedge S^\lambda{}_\rho \, dS^\rho{}_\mu + \frac{e}{2} F_{\mu\nu}\, dR^\mu \wedge dR^\nu . \tag{40}$$

Lengthy calculation $\Rightarrow$ eqns of motions (16) change into

$$\left\{ \begin{array}{rcl} \dot{R}^\mu & = & P^\mu + \dfrac{S^{\mu\nu}F_{\nu\rho}P^\rho}{\frac{1}{2}S\cdot F}, \\ \dot{P}^\mu & = & -eF^\mu_\nu \dot{R}^\nu, \\ \dot{S}^{\mu\nu} & = & P^\mu \dot{R}^\nu - P^\nu \dot{R}^\mu \end{array} \right. \tag{41}$$

assuming

$$S \cdot F \equiv S_{\alpha\beta}F^{\alpha\beta} \neq 0. \tag{42}$$

Dim of $\ker\sigma$ drops $3 \to 1$: spin-field coupling breaks "Z-shift" invariance $\Rightarrow$
spin degree can not be eliminated, we are left with $9 - 1 = 8$-*dim.* *space of motions* (phase space locally).

In terms of $(3+1)$-decomposition : assuming

$$(a) \quad \frac{1}{2}S \cdot F \equiv \frac{1}{2}S_{\alpha\beta}F^{\alpha\beta} = \boldsymbol{s} \cdot (\boldsymbol{B} - \widehat{\boldsymbol{p}} \times \boldsymbol{E}) \neq 0,$$

$$(b) \quad \widehat{\boldsymbol{p}} \cdot \boldsymbol{B} \neq 0,$$

$$\begin{cases} \dfrac{d\mathbf{r}}{dt} = \dfrac{\boldsymbol{B} - \widehat{\boldsymbol{p}} \times \boldsymbol{E}}{\widehat{\boldsymbol{p}} \cdot \boldsymbol{B}}, \\[2ex] \dfrac{d\boldsymbol{p}}{dt} = e\Big(\boldsymbol{E} + \dfrac{d\mathbf{r}}{dt} \times \boldsymbol{B}\Big) = e\,\dfrac{\boldsymbol{E} \cdot \boldsymbol{B}}{\widehat{\boldsymbol{p}} \cdot \boldsymbol{B}}\,\widehat{\boldsymbol{p}}, \\[2ex] \dfrac{d\boldsymbol{s}}{dt} = \boldsymbol{p} \times \dfrac{d\mathbf{r}}{dt} = \dfrac{\boldsymbol{p} \times \boldsymbol{B}}{\widehat{\boldsymbol{p}} \cdot \boldsymbol{B}} - \dfrac{\boldsymbol{p} \times (\widehat{\boldsymbol{p}} \times \boldsymbol{E})}{\widehat{\boldsymbol{p}} \cdot \boldsymbol{B}}. \end{cases} \quad (43)$$

Unusual features :

- $\widehat{\boldsymbol{p}}$ one would have expected on r.h.s. of velocity relation cancels out;

- electric charge drops out;

- Dynamics of the momentum decouples from spin as long as latter does not vanish; scalar spin $s \neq 0$ disappears also from all equations.

Eqs (43) $\Rightarrow$ $d\widehat{\boldsymbol{p}}/dt = 0$ $\Rightarrow$ direction of $\boldsymbol{p}$ unchanged during motion.

Put, for example, into crossed constant electric & magnetic fields (like in Hall effect),

$$\boldsymbol{E} = E\,\widehat{\boldsymbol{x}}, \quad \boldsymbol{B} = B\,\widehat{\boldsymbol{z}}$$

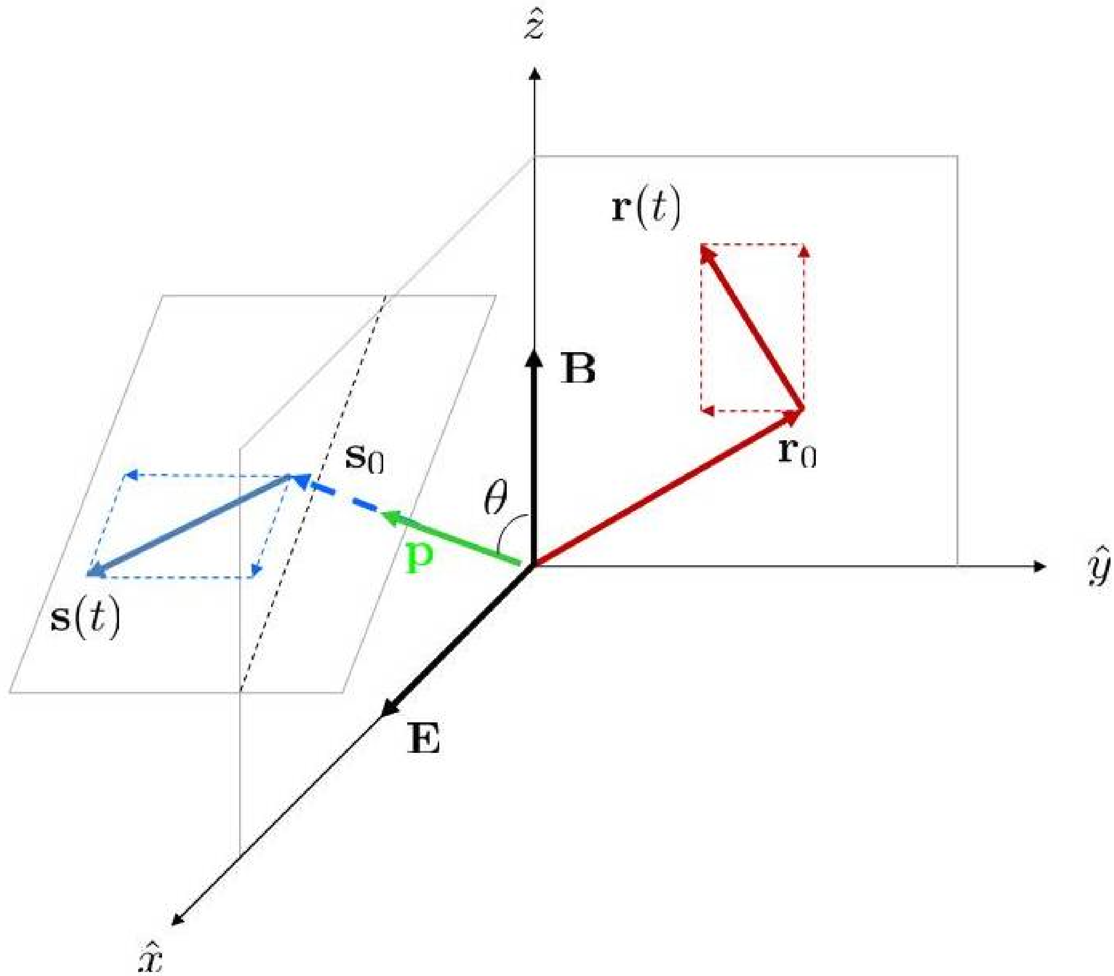
$\boldsymbol{E} \cdot \boldsymbol{B} = 0 \Rightarrow \boldsymbol{p}$ itself const. of motion, $\Rightarrow$ so is angle θ between $\boldsymbol{B}$ and $\boldsymbol{p}$ ($\theta \neq \pi/2$ for $\boldsymbol{p} \cdot \boldsymbol{B} \neq 0$).

Assume for simplicity that initial momentum lies in x-z plane. Then eqns of motion solved by

$$\begin{cases} \mathbf{r}(t) &= \Big((\cos\theta)^{-1}\,\widehat{\boldsymbol{z}} - \dfrac{E}{B}\,\widehat{\boldsymbol{y}}\Big)t + \mathbf{r}_0, \\ \boldsymbol{p}(t) &= \boldsymbol{p}_0, \\ \boldsymbol{s}(t) &= |\boldsymbol{p}|\Big(-\tan\theta\,\widehat{\boldsymbol{y}} + \dfrac{E}{B}\big(\cos\theta\,\widehat{\boldsymbol{x}} - \sin\theta\,\widehat{\boldsymbol{z}}\big)\Big)\,t + \boldsymbol{s}_0. \end{cases} \tag{44}$$

Thus, in addition to const.-speed vertical motion, particle also drifts perpendicularly to electric field with *Hall velocity* E/B.

Spin follows curious motion perpendicularly to $\widehat{\boldsymbol{p}} \Rightarrow$ projection on $\widehat{\boldsymbol{p}}$ remains const, $\boldsymbol{s}(t) \cdot \widehat{\boldsymbol{p}} = \boldsymbol{s}_0 \cdot \widehat{\boldsymbol{p}}$. Spin decoupled, but can *not* consistently "enslaved" : $\boldsymbol{s}$ and $\widehat{\boldsymbol{p}}$ do *not* remain parallel.



Motion in constant Hall-type electric and magnetic fields described by eqn (44). Initial position, $\mathbf{r}_0$, chosen in y-z plane, initial momentum and spin chosen parallel and in x-z plane. Spatial motion, $\mathbf{r}(t)$, is combination of constant-velocity Hall drift perp. to $\boldsymbol{E}$ and $\boldsymbol{B}$, combined with const-velocity vertical drift. Momentum, $\boldsymbol{p}$, conserved, but spin, $\boldsymbol{s}(t)$, has curious, Hall-type motion in plane perp to momentum.

Velocity superluminal ($|d\mathbf{r}/dt| > 1$), diverges as $\theta \to \pi/2$; for $\widehat{\boldsymbol{p}} \cdot \boldsymbol{B} = 0$ instantaneous motions with infinite velocity, parallel to z axis.

Anomalous coupling

Minimal model curious, but not completely satisfactory. Generalize. Clue : allow "mass-square" to depend on coupling of spin to e.m. field,

$$P_\mu P^\mu = -\frac{eg}{2} S \cdot F, \tag{45}$$

where used shorthand $S \cdot F \equiv S_{\alpha\beta} F^{\alpha\beta}$. Real constant g : gyromagnetic ratio.

$$P^\mu = I^\mu + \frac{eg}{4} (S \cdot F) J^\mu, \tag{46}$$

where I, J still as in (12). Allows to implement (45). Pauli-Lubanski condition

$$S_{\mu\nu} P^\nu = 0 \tag{47}$$

automatically satisfied.

Novel evolution space

$$\left\{P, R, S \,\middle|\, P_\mu P^\mu = -\frac{eg}{2} S\cdot F,\ S_{\mu\nu}P^\nu = 0,\ \tfrac{1}{2}S_{\mu\nu}S^{\mu\nu} = s^2\right\}, \tag{48}$$

endowed with closed two-form,

$$\sigma = -dP_\mu \wedge dR^\mu - \frac{1}{2s^2} dS^\mu_\lambda \wedge S^\lambda_\rho\, dS^\rho_\mu + \tfrac{1}{2}eF_{\mu\nu}\, dR^\mu \wedge dR^\nu. \tag{49}$$

formally same as (40) up to different mass-shell constraint.

New eqns of motion from kernel using constraints which define $\widetilde{V}^9$.

$$\begin{aligned}
\dot{R}^\mu &= P^\mu - \frac{1}{(g+1)}\frac{1}{S_{\alpha\beta}F^{\alpha\beta}}\Big[(g-2)\, S^{\mu\nu}F_{\nu\rho}P^\rho - g\, S^{\mu\nu}\partial_\nu F_{\rho\sigma}\, S^{\rho\sigma}\Big],\\
\dot{P}^\mu &= -eF^\mu_{\ \nu}\,\dot{R}^\nu - \frac{eg}{4}\,\partial^\mu F_{\rho\sigma}\, S^{\rho\sigma},\\
\dot{S}^{\mu\nu} &= P^\mu \dot{R}^\nu - P^\nu \dot{R}^\mu + \frac{eg}{2}\left[S^\mu_{\ \rho}\, F^{\rho\nu} - S^\nu_{\ \rho}\, F^{\rho\mu}\right].
\end{aligned} \tag{50}$$

Zero-rest-mass counterparts of Bargmann-Michel-Telegdi eqns for massless particle. Reduce to (40) for $g=0$. In "normal" case $g=2$ $\sim$ Dirac eqn anomalous velocity canceled, but new contribution involving derivative of em field.

$(3+1)$-decomposition:

$$R = (\mathbf{r}, t), \quad P = (\boldsymbol{p}, \mathcal{E}), \quad S_{j4} = \left(\frac{\boldsymbol{p}}{\mathcal{E}} \times \boldsymbol{s}\right)_j \qquad (51)$$

where spin tensor still defined as in (13), but new dispersion relation,

$$\mathcal{E} = \sqrt{|\boldsymbol{p}|^2 - \frac{eg}{2} S \cdot F}\,. \qquad (52)$$

Decomposing em field into electric and magnetic components,

$$\tfrac{1}{2} S \cdot F = \boldsymbol{s} \cdot \left(\boldsymbol{B} - \frac{\boldsymbol{p}}{\mathcal{E}} \times \boldsymbol{E}\right). \qquad (53)$$

Tedious calculation yields $(3+1)$-form of eqns of motion (50).

N.B. For weak pure magnetic field, $\boldsymbol{s} = \frac{1}{2}\widehat{\boldsymbol{p}}$

$$\mathcal{E} \approx |\boldsymbol{p}| - \frac{eg}{4} S \cdot F = |\boldsymbol{p}| - e\,\frac{\widehat{\boldsymbol{p}} \cdot \boldsymbol{B}}{2|\boldsymbol{p}|}\,, \qquad (54)$$

also proposed recently by Chen et al, Manuel et al.

Consider $g = 2$ + fields constant; field-derivative terms & anomalous velocity drop out. Simplifies to one reminiscent of a massive relativistic particle,

$$(g{=}2)\quad \begin{cases} \mathcal{E}\,\dfrac{d\mathbf{r}}{dt} &= \boldsymbol{p}, \\ \dfrac{d\boldsymbol{p}}{dt} &= e\left(\boldsymbol{E} + \dfrac{d\mathbf{r}}{dt}\times\boldsymbol{B}\right), \\ \dfrac{d\boldsymbol{s}}{dt} &= e\left(\left(\dfrac{\boldsymbol{p}}{\mathcal{E}^2}\times\boldsymbol{s}\right)\times\boldsymbol{E} + \dfrac{\boldsymbol{s}}{\mathcal{E}}\times\boldsymbol{B}\right) \end{cases} \tag{55}$$

assuming that $\mathcal{E} \neq 0$ [sort of "effective mass"] is real. [NB: $\boldsymbol{p} \neq 0$ implies that $\mathcal{E}$ can not vanish].

In pure magnetic field momentum and spin satisfy eqns of identical form,

$$\frac{d\boldsymbol{p}}{dt} = \frac{e}{\mathcal{E}}\boldsymbol{p}\times\boldsymbol{B}\,, \qquad \frac{d\boldsymbol{s}}{dt} = \frac{e}{\mathcal{E}}\boldsymbol{s}\times\boldsymbol{B}\,. \tag{56}$$

Multiplying by $\boldsymbol{p}$, $\boldsymbol{s}$ and by $\boldsymbol{B}$, resp. $\Rightarrow$

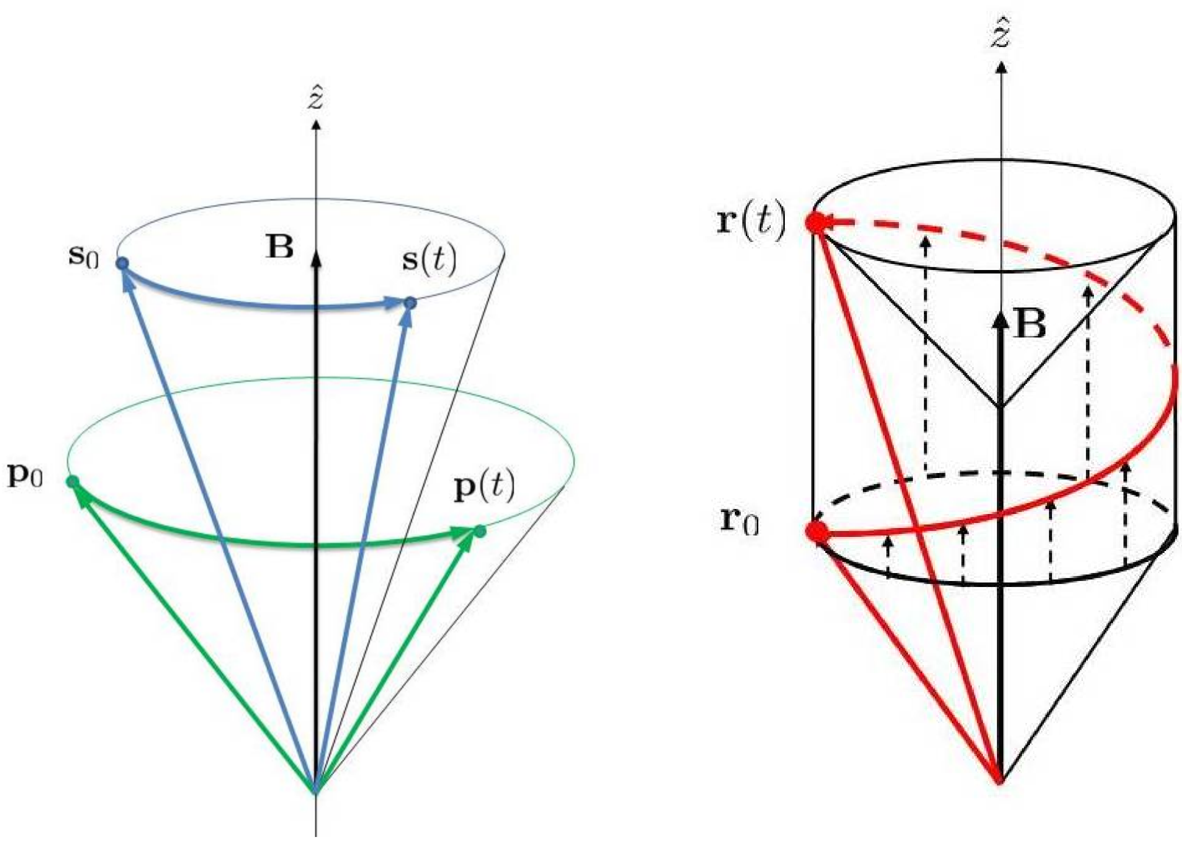
$$\begin{aligned} |\boldsymbol{p}| = \text{const} \neq 0, &\quad \boldsymbol{p}\cdot\boldsymbol{B} = \text{const}, \\ |\boldsymbol{s}| = \text{const} \neq 0, &\quad \boldsymbol{s}\cdot\boldsymbol{B} = \text{const}, \\ p_z \;=\; \text{const}, &\qquad s_z = \text{const}, \\ \mathcal{E} \;=\; \sqrt{|\boldsymbol{p}|^2 - e\boldsymbol{s}\cdot\boldsymbol{B}} &= \text{const}\,. \end{aligned}$$

Both momentum and spin vectors precess around direction of magnetic field, $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ with common angular velocity $\omega = -eB/\mathcal{E}$,

$$\boldsymbol{p}(t) = (p_0 e^{-i(eB/\mathcal{E})t}, p_z), \tag{57}$$
$$\boldsymbol{s}(t) = (s_0 e^{-i(eB/\mathcal{E})t}, s_z), \tag{58}$$

where $p_0 = p_x + ip_y$, $s_0 = s_x + is_y$. Therefore

$$\boxed{\mathbf{r}(t) = \left(\frac{ip_0}{eB} e^{-i(eB/\mathcal{E})t}, \frac{p_z}{\mathcal{E}} t\right) + \mathbf{r}_0 \,.} \tag{59}$$



Motion in pure const magnetic field $\boldsymbol{B}$. Both momentum $\boldsymbol{p}(t)$ and spin $\boldsymbol{s}(t)$ precess around $\boldsymbol{B}$ direction. Position $\mathbf{r}(t)$ spirals on cylinder around $\boldsymbol{B}$-axis.