Conformal Newton–Hooke symmetry of Pais–Uhlenbeck oscillator

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Motivations

- Current studies of the nonrelativistic version of the AdS/CFT correspondence stimulate a renewed interest in non–relativistic conformal algebras. In this context the so called l–conformal Galilei algebra, where l is a (half)integer parameter, and its Newton-Hooke counterpart play the central role. Dynamical realizations of these algebras for $l > 1$ are poorly understood.

- Recently, there has been an upsurge of interest in $d = 1$ conformal mechanics ($N = 4$ and $D(2,1|\alpha)$ supersymmetric extensions, a link to superparticles propagating near extreme black hole horizons, isospin degrees of freedom, angular sector, solutions to the Witten-Dijkgraaf-Verlinde-Verlinde equation etc.). As $d > 1$ is physically more interesting, it is natural to wonder what happens beyond $d = 1$. This invokes nonrelativistic conformal algebras.

- Interesting mathematical problems are around. In particular, but for $l = 1$, it is not known whether the l–conformal Galilei algebra or its Newton-Hooke counterpart can be obtained by a nonrelativistic contraction from relativistic conformal algebra $so(p+1, q+1)$, $d = p + q$.

Newton-Hooke algebra

(Anti) de Sitter algebra$(\eta_{AB} = \mathrm{diag}(-,+,+,+,\mp))$

$$[M_{AB}, M_{CD}] = \eta_{AC} M_{BD} + \eta_{BD} M_{AC} - \eta_{BC} M_{AD} - \eta_{AD} M_{BC}$$

Another basis $(M_{AB} \to (M_{\alpha\beta}, P_\alpha = M_{\alpha 4}/R), \quad \alpha = 0,1,2,3)$

$$[M_{\alpha\beta}, M_{\gamma\delta}] = \eta_{\alpha\gamma} M_{\beta\delta} + \eta_{\beta\delta} M_{\alpha\gamma} - \eta_{\beta\gamma} M_{\alpha\delta} - \eta_{\alpha\delta} M_{\beta\gamma},$$

$$[M_{\alpha\beta}, P_\gamma] = \eta_{\alpha\gamma} P_\beta - \eta_{\beta\gamma} P_\alpha, \quad [P_\alpha, P_\beta] = \mp \frac{1}{R^2} M_{\alpha\beta}$$

Non–relativistic contraction (by analogy with relativistic mechanics)

$$M_{\alpha\beta} \to (M_{ij},\ M_{0i} = cK_i), \qquad P_\alpha \to (P_i,\ P_0 = cM + H/c), \qquad \boxed{R \to c\tilde{R}}$$

yields the (centrally extended) Galilei algebra with one structure relation altered

$$[H, P_i] = \mp \frac{1}{\tilde{R}^2} K_i$$

The flat space limit $\tilde{R} \to \infty$ yields the Galilei algebra.

In addition to the time translations consider the dilatations and the special conformal transformations

$$H = \partial_t, \qquad D = t\partial_t + l x_i \partial_i, \qquad K = t^2\partial_t + 2l t x_i \partial_i.$$

These form $so(2,1)$ for arbitrary l.

l-conformal Galilei algebra (J. Negro, M. del Olmo, A. Rodriguez-Marco, 1997)

$$[H, D] = H, \qquad [H, C_i^{(n)}] = nC_i^{(n-1)},$$

$$[H, K] = 2D, \qquad [D, K] = K,$$

$$[D, C_i^{(n)}] = (n-l)C_i^{(n)}, \qquad [K, C_i^{(n)}] = \boxed{(n-2l)}\, C_i^{(n+1)},$$

$$[M_{ij}, C_k^{(n)}] = -\delta_{ik}C_j^{(n)} + \delta_{jk}C_i^{(n)}, \quad [M_{ij}, M_{kl}] = -\delta_{ik}M_{jl} - \delta_{jl}M_{ik} + \dots$$

Realization in spacetime

$$C_i^{(0)} = P_i = \partial_i, \qquad C_i^{(1)} = K_i = t\partial_i, \qquad \dots \qquad , \qquad C_i^{(n)} = t^n\partial_i$$

The algebra is finite–dimensional provided $n = 0, 1, \dots, 2l$ which means that l is half–integer. $1/l$ is called the dynamical exponent. $C_i^{(n)} = t^n\partial_i$ are called the generators of accelerations.

l-conformal Newton-Hooke algebra (A.G., I. Masterov, 2011)

$$[H,D]=H\mp\frac{2}{\tilde{R}^2}K, \qquad [H,C_i^{(n)}]=nC_i^{(n-1)}\pm\frac{(n-2l)}{\tilde{R}^2}C_i^{(n+1)},$$

$$[H,K]=2D, \qquad [D,K]=K,$$

$$[D,C_i^{(n)}]=(n-l)C_i^{(n)}, \qquad [K,C_i^{(n)}]=(n-2l)C_i^{(n+1)},$$

$$[M_{ij},C_k^{(n)}]=-\delta_{ik}C_j^{(n)}+\delta_{jk}C_i^{(n)}, \qquad [M_{ij},M_{kl}]=-\delta_{ik}M_{jl}-\delta_{jl}M_{ik}+\ldots$$

Remarks:

- l–conformal Newton–Hooke algebra and l–conformal Galilei algebra are isomorphic

$$H \quad \rightarrow \quad H\mp\frac{1}{\tilde{R}^2}K$$

Warning: in dynamical realizations this implies change of the Hamiltonian

- The flat space limit $\tilde{R}\rightarrow\infty$ yields the l–conformal Galilei algebra

- Realization in spacetime $(\Lambda < 0)$

$$H = \partial_t, \qquad D = \frac{1}{2}\tilde{R}\sin(2t/\tilde{R})\partial_t + l\cos(2t/\tilde{R})x_i\partial_i$$

$$K = -\frac{1}{2}\tilde{R}^2(\cos(2t/\tilde{R}) - 1)\partial_t + l\tilde{R}\sin(2t/\tilde{R})x_i\partial_i$$

$$C_i^{(n)} = \tilde{R}^n\big(\tan(t/\tilde{R})\big)^n\big(\cos(t/\tilde{R})\big)^{2l}\partial_i$$

- Realization in spacetime $(\Lambda > 0)$

$$H = \partial_t, \qquad D = \frac{1}{2}\tilde{R}\sinh(2t/\tilde{R})\partial_t + l\cosh(2t/\tilde{R})x_i\partial_i$$

$$K = \frac{1}{2}\tilde{R}^2(\cosh(2t/\tilde{R}) - 1)\partial_t + l\tilde{R}\sinh(2t/\tilde{R})x_i\partial_i$$

$$C_i^{(n)} = \tilde{R}^n\big(\tanh(t/\tilde{R})\big)^n\big(\cosh(t/\tilde{R})\big)^{2l}\partial_i$$

Symmetries are used to integrate equations of motion. The number of functionally independent integrals of motion needed to integrate a differential equation correlates with its order. The presence of the generators of accelerations in the nonrelativistic conformal algebras in general leads to higher derivative formulations (J. Lukierski, P.C. Stichel, W.J. Zakrzewski, J. Gomis, K. Kamimura, C. Duval, P.A. Horváthy, K. Andrzejewski, J. Gonera, P. Maslanka, P. Kosinski)

Realization without higher derivatives (J. Lukierski, E. Ivanov, S. Fedoruk, A.G., I. Masterov)

$$\ddot{\rho} = \frac{\gamma^2}{\rho^3} - \frac{\rho}{R^2}, \qquad \rho^2 \frac{d}{dt}\left(\rho^2 \frac{d}{dt}\vec{\chi}_p\right) + (p\gamma)^2 \vec{\chi}_p = 0$$

where γ is a coupling constant and $p = 1, 3, \ldots, 2l$ for half-integer l and $p = 2, 4, \ldots, 2l$ for integer l. In this formulation the acceleration generators prove to be functionally dependent on the others.

The objective is to construct a higher derivative dynamical realization of the l-conformal Newton–Hooke algebra. In particular, we will unravel the l-conformal Newton-Hooke symmetry of the Pais–Uhlenbeck oscillator.

Pais–Uhlenbeck (PU) oscillator in arbitrary dimension

$$\prod_{k=1}^{n}\left(\frac{d^2}{dt^2}+\omega_k^2\right)\vec{x}(t)=0, \qquad 0<\omega_1<\cdots<\omega_n.$$

The structure of the general solution

$$\vec{x}(t)=\sum_{k=1}^{n}(\vec{\alpha}_k\cos{(\omega_k t)}+\vec{\beta}_k\sin{(\omega_k t)})$$

where $\vec{\alpha}_k$ and $\vec{\beta}_k$ are constants, suggests that the PU oscillator is equivalent to a set of decoupled isotropic oscillators.
The new variables

$$\vec{x}_i=\sqrt{\rho_i}\prod_{k\neq i}\left(\frac{d^2}{dt^2}+\omega_k^2\right)\vec{x}, \qquad \rho_i=\frac{(-1)^{i+1}}{\prod_{k\neq i}(\omega_k^2-\omega_i^2)}$$

where $i=1,\ldots,n$, bring the PU oscillator to decoupled isotropic oscillators

$$L=-\frac{1}{2}\vec{x}\prod_{k=1}^{n}\left(\frac{d^2}{dt^2}+\omega_k^2\right)\vec{x}=\frac{1}{2}\sum_{k=1}^{n}(-1)^{k+1}\left(\dot{\vec{x}}_k^2-\omega_k^2\vec{x}_k^2\right)$$

- Analysis of solutions

Consider the dilatation transformation from the Newton–Hooke group $\left(l = \frac{2n-1}{2}\right)$

$$D = \sin(\tilde{\omega} t)\partial_t + \frac{2n-1}{2}\tilde{\omega}\cos(\tilde{\omega} t) x_i \partial_i$$

Given a solution $\vec{x}(t) = \sum_{k=1}^{n}(\vec{\alpha}_k \cos(\omega_k t) + \vec{\beta}_k \sin(\omega_k t))$ of the equation of motion, let us require the transformed function

$$\vec{x}'(t) = \vec{x}(t) + \frac{2n-1}{2}\epsilon\tilde{\omega}\cos(\tilde{\omega} t)\vec{x}(t) - \epsilon \sin(\tilde{\omega} t)\dot{\vec{x}}(t)$$

where ϵ is an infinitesimal parameter, to be a new solution

$$\vec{x}'(t) = \frac{\epsilon}{2}\left(\frac{2}{\epsilon}\sum_{k=1}^{n}\boxed{(\vec{\alpha}_k \cos(\omega_k t) + \vec{\beta}_k \sin(\omega_k t))} + \right.$$

$$\sum_{k=1}^{n}\vec{\alpha}_k\left(\left[\frac{(2n-1)}{2}\tilde{\omega} - \omega_k\right]\cos(\omega_k + \tilde{\omega})t + \left[\frac{(2n-1)}{2}\tilde{\omega} + \omega_k\right]\cos(\omega_k - \tilde{\omega})t\right)$$

$$\left. + \sum_{k=1}^{n}\vec{\beta}_k\left(\left[\frac{(2n-1)}{2}\tilde{\omega} - \omega_k\right]\sin(\omega_k + \tilde{\omega})t + \left[\frac{(2n-1)}{2}\tilde{\omega} + \omega_k\right]\sin(\omega_k - \tilde{\omega})t\right)\right)$$

The only way to match frequencies is to require

$$\omega_k + \tilde{\omega} = \omega_{k+1}, \qquad \omega_n = \frac{(2n-1)}{2}\tilde{\omega}$$

where $k = 1, \ldots, n-1$, which yields

$$\omega_j = (2j-1)\omega_1, \qquad \tilde{\omega} = 2\omega_1$$

with $j = 1, \ldots, n$.

- Niederer's transformation

$$t' = \frac{1}{\omega}\tan(\omega t), \qquad x'(t') = \left(\cos(\omega t)\right)^{-1} x(t)$$

relates the motion of a free particle to a half–period of the harmonic oscillator

$$\frac{d^2 x'(t')}{dt'^2} = 0 \quad \rightarrow \quad \frac{d^2 x(t)}{dt^2} + \omega^2 x(t) = 0$$

Conformal Newton–Hooke symmetry of the PU oscillator

A generalization of Niederer's transformation (A.G., I. Masterov, 2011)

$$t' = \frac{1}{\omega_1} \tan(\omega_1 t), \qquad x_i'(t') = (\cos(\omega_1 t))^{-2l} x_i(t)$$

where l is a (half)integer, relates conventional realizations of the l–conformal Galilei algebra and its Newton–Hooke counterpart.

Free particle obeying the higher derivative equation of motion

$$\frac{d^{2n} \vec{x}'(t')}{dt'^{2n}} = 0$$

is known to possess the l–conformal Galilei symmetry (J. Gomis, K. Kamimura).

Implementation of the Niederer's transformation yields the PU oscillator with special frequencies

$$\prod_{k=1}^{l+\frac{1}{2}} \left(\frac{d^2}{dt^2} + [(2k-1)\omega_1]^2 \right) \vec{x}(t) = 0$$

which confirms the constraint on the frequencies $\boxed{\omega_j = (2j-1)\omega_1,\ j = 1, \ldots, n}$ revealed above.

- Direct approach

Consider infinitesimal transformations

$$t' = t + \epsilon\psi(t), \qquad \vec{x}'(t') = \vec{x}(t) + \epsilon\vec{\phi}(t,x)$$

with $\psi(t)$, $\vec{\phi}(t,x)$ to be determined.

Let us demand the transformation to leave invariant the PU oscillator equation

$$\prod_{k=1}^{n}\left(\frac{d^2}{dt'^2} + \omega_k^2\right) x_i'(t') = (\delta_{ij} + \epsilon\lambda_{ij}(t,x)) \prod_{k=1}^{n}\left(\frac{d^2}{dt^2} + \omega_k^2\right) x_j(t)$$

where i is a spatial index and $\lambda_{ij}(t,x)$ is an invertible matrix to be fixed.

This requirement yields $\left(A^{(p)} = \frac{d^p A}{dt^p}\right)$

$$\sum_{k=0}^{n} \sigma_k^n \phi_i^{(2k)} - \sum_{k=1}^{n} \sigma_k^n \sum_{p=1}^{2k} C_{2k}^{p-1} \psi^{(2k-p+1)} x_i^{(p)} = \lambda_{ij}(t,x) \sum_{k=0}^{n} \sigma_k^n x_j^{(2k)}$$

$$\sigma_k^n = \sum_{i_1 < i_2 < \cdots < i_{n-k}}^{n} \omega_{i_1}^2 \ldots \omega_{i_{n-k}}^2, \qquad \sigma_n^n = 1$$

Gathering terms which have the same number of derivatives acting on x_i, one has

$$\psi(t) = a + b\sin(\tilde{\omega}t) + c\cos(\tilde{\omega}t)$$
$$\phi_i(t,x) = a_{ij}(t)x_j(t) + b_i(t)$$
$$b_i(t) = \sum_{k=1}^{n}(\mu_i^k\cos(\omega_k t) + \nu_i^k\sin(\omega_k t))$$
$$a_{ij}(t) = \frac{2n-1}{2}\dot{\psi}(t)\delta_{ij} + a_{ij}^0$$
$$\lambda_{ij} = -\frac{2n+1}{2}\dot{\psi}(t)\delta_{ij} + a_{ij}^0.$$
$$\omega_k = (2k-1)\omega_1,\ k = 1,\ldots,n$$

where a, b, c, μ_i^k, ν_i^k, a_{ij}^0 are constants (parameters of the transformations) and $\tilde{\omega}$ is the characteristic time.

The constraint on frequencies also follows from the closure of the algebra of the transformations.

The example of $l = \frac{3}{2}$

Conformal subalgebra

$$H = \frac{1}{2}(\vec{p}_2^2 + 9\omega_1^2\vec{x}_2^2) - \frac{1}{2}(\vec{p}_1^2 + \omega_1^2\vec{x}_1^2), \;\; K = -\frac{1}{2\omega_1^2}\left(A\sin(2\omega_1 t) + B\cos(2\omega_1 t)\right.$$

$$D = -\frac{1}{2\omega_1}\left(A\cos(2\omega_1 t) - B\sin(2\omega_1 t)\right),$$

$$A = \vec{p}_1\vec{p}_2 - 2\omega_1\vec{x}_1\vec{p}_1 - 3\omega_1^2\vec{x}_1\vec{x}_2, \qquad B = \vec{p}_1^2 - \omega_1^2\vec{x}_1^2 - \omega_1\vec{x}_1\vec{p}_2 - 3\omega_1\vec{x}_2\vec{p}_1.$$

Vector generators

$$\vec{C}^{(0)} = \omega_1\left(3\vec{p}_1\sin(\omega_1 t) - \vec{p}_2\cos(3\omega_1 t) + 3\omega_1\vec{x}_1\cos(\omega_1 t) - 3\omega_1\vec{x}_2\sin(3\omega_1 t)\right)$$

$$\vec{C}^{(1)} = -\vec{p}_1\cos(\omega_1 t) - \vec{p}_2\sin(3\omega_1 t) + \omega_1\vec{x}_1\sin(\omega_1 t) + 3\omega_1\vec{x}_2\cos(3\omega_1 t)$$

$$\vec{C}^{(2)} = \frac{1}{\omega_1}\left(\vec{p}_1\sin(\omega_1 t) + \vec{p}_2\cos(3\omega_1 t) + \omega_1\vec{x}_1\cos(\omega_1 t) + 3\omega_1\vec{x}_2\sin(3\omega_1 t)\right)$$

$$\vec{C}^{(3)} = \frac{1}{\omega_1^2}\left(-3\vec{p}_1\cos(\omega_1 t) + \vec{p}_2\sin(3\omega_1 t) + 3\omega_1\vec{x}_1\sin(\omega_1 t) - 3\omega_1\vec{x}_2\cos(3\omega_1 t)\right.$$

Summary

The Pais–Uhlenbeck oscillator in arbitrary dimension enjoys the l–conformal Newton–Hooke symmetry provided frequencies of oscillation form the arithmetic sequence $\omega_k = (2k-1)\omega_1$, where $k = 1, \ldots, n$, and l is the half–integer number $\frac{2n-1}{2}$.

Open problems

- A clear-cut physical interpretation of the arithmetic sequence for PU oscillator
- Interacting models with l–conformal Newton-Hooke symmetry are unknown
- Supersymmetric extensions
- But for $l = 1$, it is not known whether l–conformal Newton-Hooke algebra can be obtained by a nonrelativistic contraction from a relativistic conformal algebra $so(p+1, q+1)$, $d = p + q$