Dunkl angular momenta algebra and Calogero–Moser systems

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Plan of the talk

1. Root systems and associated Calogero-Moser systems
2. Integrability through Dunkl operators
3. Angular (spherical) Calogero-Moser systems
4. Dunkl angular momenta algebra
5. $gl(n)$ version

Root systems and Coxeter groups

Let $V=\mathbb{R}^n$, let (u,v) be the standard bilinear form in V. Let $\mathcal{R}\subset V$ be a Coxeter root system. That is

- $\forall\alpha\in\mathcal{R}\quad s_\alpha\mathcal{R}=\mathcal{R}$,
  where s_α is orthogonal reflection with respect to the hyperplane $(\alpha,x)=0$.
- If $\alpha,\beta\in\mathcal{R}$ and $\alpha\sim\beta$ then $\alpha=\pm\beta$.

Let $W=\langle s_\alpha|\alpha\in\mathcal{R}\rangle$ be the corresponding Coxeter group.
By Chevalley's theorem $\mathbb{C}[x_1,\dots,x_n]^W\cong\mathbb{C}[P_1,\dots,P_n]$, where P_i are homogeneous polynomials.
If $\mathcal{R}$ is irreducible then $\mathcal{R}$ is equivalent to one of
$A_r, B_r, D_r, E_6, E_7, E_8, F_4, H_3, H_4, I_{2m}$.

Example

$\mathcal{R} = A_{n-1}$: vectors $\pm(e_i - e_j)$, $\quad 1 \le i < j \le n$.
$W = S_n$ - symmetric group.

$\mathcal{R} = B_n$: vectors $\pm(e_i \pm e_j)$, $\quad 1 \le i < j \le n \quad$ and $\quad \pm e_i$, $\quad 1 \le i \le n$.

Suppose $g : \mathcal{R} \to \mathbb{C}$ is W-invariant. Let $g_\alpha = g(\alpha)$, $\alpha \in \mathcal{R}$.
There are corresponding Calogero-Moser type integrable systems [Olshanetsky, Perelomov'77]:

$$H = \Delta - \sum_{\alpha \in \mathcal{R}_+} \frac{g_\alpha(g_\alpha - 1)(\alpha, \alpha)}{(\alpha, x)^2},$$

where exactly one of the roots $\pm\alpha$ enters $\mathcal{R}_+$. The case $\mathcal{R} = A_{n-1}$ gives the Calogero–Moser Hamiltonian

$$H = \Delta - \sum_{i<j}^{n} \frac{2g(g-1)}{(x_i - x_j)^2}.$$

Rational Cherednik algebra

Let $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$, $\mathbb{C}[y] = \mathbb{C}[y_1, \ldots y_n]$. The rational Cherednik algebra [Etingof, Ginzburg'00]:

$$\mathcal{H} = \mathcal{H}_g(W) = < \mathbb{C}[x], \mathbb{C}[y], \mathbb{C}W > /(relations).$$

Relations: $x_i x_j = x_j x_i$, $y_i y_j = y_j y_i$, $wp(x) = p(w^{-1}x)w$, $wp(y) = p(w^{-1}y)w$, $x_i y_j - y_j x_i - \delta_{ij} = \sum a_\alpha s_\alpha$ for some $a_\alpha \in \mathbb{C}$.
Equivalently, define the algebra by its faithful representation on functions:
x_i acts by multiplication; $w(f(x)) = f(w^{-1}(x))$ for $w \in W$;
y_i acts as Dunkl operator ∇_i:

$$\nabla_i = \partial_{x_i} - \sum_{\alpha \in \mathcal{R}_+} \frac{g_\alpha(\alpha, e_i)}{(\alpha, x)} s_\alpha.$$

Note that Dunkl operators commute $[\nabla_i, \nabla_j] = 0$.
As a vector space $\mathcal{H} \cong \mathbb{C}[x] \otimes \mathbb{C}[y] \otimes \mathbb{C}W$.

Integrability of Calogero-Moser systems

Let $P(x) \in \mathbb{C}[x]^W$. Define $L_P = Res\, P(\nabla)-$ restriction of $P(\nabla)$ to invariant functions, it is a differential operator.
Then [Heckman'91]

- $[L_P, L_Q] = 0$ for any $P, Q \in \mathbb{C}[x]^W$.
- Let $P(x) = x^2 = \sum_{i=1}^n x_i^2$. Then

$$L_P = L_{x^2} = H = \Delta - \sum_{\alpha \in \mathcal{R}_+} \frac{g_\alpha(g_\alpha - 1)(\alpha, \alpha)}{(\alpha, x)^2},$$

the Calogero-Moser operator associated with $\mathcal{R}$.

Angular Calogero–Moser systems

H_Ω is obtained by separating spherical and radial coordinates:

$$H = \partial_r^2 + \frac{N-1}{r}\partial_r - \frac{H_\Omega}{r^2}.$$

- Calogero'71
- Intertwiners of quantum systems at integer coupling, F'03
- Extensive study of integrals, superintegrability, derivation from matrix model, Hakobyan, Karakhanyan, Krivonos, Lechtenfeld, Nersessian, Saghatelian, Yeghikyan'09-14
- Eigenfunctions, Dunkl, Xu'01; F, Lechtenfeld, Polychronakos'13

Angular Calogero–Moser systems through Dunkl operators

Define Dunkl angular momenta

$$M_{ij} = x_i \nabla_j - x_j \nabla_i.$$

Let $\mathbf{M}^2 = \sum_{i<j}^n M_{ij}^2$.

Theorem

$$Res\mathbf{M}^2 = H_\Omega + const.$$

To take care of the constant, we modify $\widetilde{\mathbf{M}}^2 = \mathbf{M}^2 - S(S-n+2)$, where $S = \sum_{\alpha \in \mathcal{R}_+} g_\alpha s_\alpha$. Then

$$Res\widetilde{\mathbf{M}}^2 = H_\Omega.$$

Dunkl angular momenta algebra, the centre

Define the algebra $\mathcal{H}_g^{so(n)}(W)$ generated by the operators M_{ij} and $\mathbb{C}W$. It is a deformation of the skew product of the algebra $\mathcal{M}$ generated by the usual angular momenta operators and the group algebra $\mathbb{C}W$.

Theorem

The centre of the algebra $\mathcal{H}_g^{so(n)}(W)$ is generated by $\widetilde{\mathbf{M}}^2$ and constants.

Remark

In type A_{n-1} the Casimir element by Kuznetsov'96

Remark

$\mathcal{H}_g^{so(n)}(S_n) \not\subset \mathcal{H}_g^{so(n+1)}(S_{n+1})$ *for $g \neq 0$. It makes it not possible to define a complete family of commuting elements by taking the central generators for different n.*

Poincaré-Birkhoff-Witt theorem

Let $\mathcal{V}$ be a vector space. Let $\alpha : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ be antisymmetric, bilinear. Consider the $\mathbb{C}$-algebra

$$U(\mathcal{V}) = <\mathcal{V}>/(\text{relations}),$$

where relations are $xy - yx = \alpha(x, y)$, $x, y \in \mathcal{V}$.
$U(\mathcal{V})$ has filtration by the degree, $F_0 \subset F_1 \subset \ldots \subset F_i \subset \ldots = U(\mathcal{V})$.
Define associated graded algebra $grU(\mathcal{V}) = \oplus F_i/F_{i-1}$. Let $S(\mathcal{V}^*)$ be the polynomial algebra on $\mathcal{V}$ and consider the natural surjection $\varphi : S(\mathcal{V}^*) \to U(\mathcal{V})$. Then $U(\mathcal{V})$ is said to have the PBW property if φ is isomorphism, i.e. $Ker\varphi = 0$.
$U(\mathcal{V})$ satisfies PBW if and only if $\mathcal{V}$ is a Lie algebra, i.e. α satisfies the Jacobi identity.

Theorem (Etingof, Ginzburg'00)

The algebra $\mathcal{H}_g(W)$ has PBW property. Equivalently, as a vector space $\mathcal{H}_g(W) \cong \mathbb{C}[x] \otimes \mathbb{C}[y] \otimes \mathbb{C}W$.

Theorem

The algebra $\mathcal{H}_g^{so(n)}(W)$ has PBW property.

Equivalently, a basis is given by a basis in the algebra $\mathcal{M} \ltimes \mathbb{C}W$, where $\mathcal{M}$ is the algebra of the usual angular momenta.

Explicitly, a basis has the form

$$M_{i_1 j_1}^{n_1} \dots M_{i_k j_k}^{n_k} \sigma \qquad \text{with} \quad i_s < j_s, \quad n_s > 0, \quad k \geq 0, \quad \sigma \in W$$

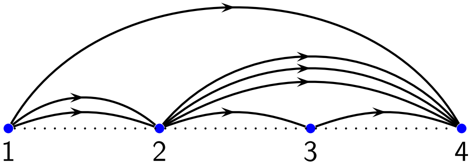
with the ordering

$$i_1 \leq \dots \leq i_k, \qquad \text{and} \qquad i_s = i_{s+1} \;\Rightarrow\; j_s < j_{s+1},$$

and non-crossing condition $i_s < i_{s'} < j_s \quad \Rightarrow \quad j_{s'} \leq j_s$.

Graphical interpretation

Graphical interpretation of a sample monomial, which does not contain intersecting angular momentum bonds:

$M_{12}^2 M_{14} M_{23} M_{24}^3 M_{34} =$ 

Quadratic property

We describe all the defining relations of the generators of the algebra $\mathcal{H}_g^{so(n)}(W)$. Let $M_{\xi\eta} = \sum \xi_i \eta_j M_{ij}$ for $\xi, \eta \in \mathbb{C}^n$.
Let

$$S_{\xi\eta} = (\xi, \eta) + \sum_{\alpha \in \mathcal{R}_+} \frac{2g_\alpha (\alpha, \xi)(\alpha, \eta)}{(\alpha, \alpha)} s_\alpha \in \mathbb{C}W.$$

Theorem

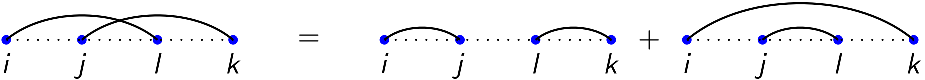
The Dunkl angular momenta satisfy

$$[M_{\xi\eta}, M_{\varphi\psi}] = M_{\xi\psi} S_{\eta\varphi} + M_{\eta\varphi} S_{\xi\psi} - M_{\xi\varphi} S_{\eta\psi} - M_{\eta\psi} S_{\xi\varphi},$$

$$M_{\xi\eta} M_{\varphi\psi} + M_{\eta\varphi} M_{\xi\psi} + M_{\varphi\xi} M_{\eta\psi} = M_{\varphi\xi} S_{\eta\psi} + M_{\xi\eta} S_{\varphi\psi} + M_{\eta\varphi} S_{\xi\psi}$$

for any $\xi, \eta, \varphi, \psi \in \mathbb{C}^n$.

Graphically, the non-crossing condition is illustrated as follows:



$+\quad [$ first-order terms in M_{ij} $]$

The homogeneous part is the Plücker relations for the Grassmanian of two-dimensional planes.

$gl(n)$ version

The algebra $H_g^{gl(n)}(W)$ is defined to be generated by the operators $E_{ij} = x_i \nabla_j$ and by $\mathbb{C}W$.

Theorem

$H_g^{gl(n)}(W)$ is a PBW algebra with quadratic defining relations

$$E_{ij}E_{kl} - E_{il}E_{kj} = E_{il}S_{kj} - E_{ij}S_{kl},$$

$$[E_{ij}, E_{kl}] = E_{il}S_{jk} - S_{il}E_{kj} + [S_{kl}, E_{ij}].$$

A basis is given by

$$E_{i_1 j_1}^{n_1} \dots E_{i_k j_k}^{n_k} \sigma, \qquad \sigma \in W,$$

where $i_1 \leq \ldots \leq i_k$ and $j_1 \leq \ldots \leq j_k$.

Theorem

The centre of $H_g^{gl(n)}(W)$ is generated by constants and

$$\rho = \sum_{i=1}^{n} E_{ii} + \sum_{\alpha \in \mathcal{R}_+} g_\alpha s_\alpha .$$

Consider another representation of the algebra where $E_{ij} \to \frac{1}{2}(x_i - \nabla_i)(x_j + \nabla_j)$. Then

$$-2Res(\rho) \to H - \mathbf{x}^2 + const,$$

the Calogero–Moser operator in the harmonic confinement.

Further directions

- Liouville integrability
- Further study of algebras $H_g^{so(n)}(W)$, $H_g^{gl(n)}(W)$, and their classical versions. Representation theory, relations with singularities
- Geometrical interpretations of algebras $H_g^{so(n)}(W)$, $H_g^{gl(n)}(W)$ via Cherednik algebras for varieties with finite group actions (Etingof'04)