Polynomial Integrals in Dilaton Gravity

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New models in gravitation and cosmology come from

- Inflation
- Dark energy

Recent observations suggest that the models can be complicated

- BICEP2: two stages of inflation?
- BOSS: not ΛCDM?

E.g. Wiggly Whipped Inflation scenarios[1]:

$$V(\phi) = \gamma\psi^p + \lambda[(\psi - \phi_0)^q + \psi_{01}^q]\theta(\psi - \psi_0),$$

$$V(\phi) = \gamma\psi^p + \lambda(\psi - \psi_0)^p\theta(\psi - \psi_0),$$

ψ_0 — transition value of inflaton.

[1] D. K. Hazra, A. Shafieloo, G. F. Smoot and A. A. Starobinsky, *Wiggly Whipped Inflation*, arXiv:astro-ph.CO/1405.2012.

So there is a motivation to study a variety of models. Dilaton gravity provides useful technique.

A fairly general higher-dimensional field theories can be reduced to two-dimensional effective DGS models by taking into account their space-time symmetries. We specify the following generic action with $\mu = (t, r)$:

$$\mathcal{L} = \sqrt{-g}\left[\phi R(g) + W(\varphi)(\nabla_\mu \phi)^2 + Z(\phi)\nabla_\mu \psi \nabla^\mu \psi + X(\phi;\psi)\right]$$

where ϕ is the dilaton field, and ψ is a scalar field. Here we can describe *both* cosmological and static states, considering effective theory of some (may be unknown) fundamental theory.

See how this works for *Affine Generalization of Gravity* (the gravity theory in a non - Riemannian space with a symmetric connection): The action is proportional to tensor density which is a square root of $\det(s_{ij} + \lambda a_{ij})$, where s_{ij} and a_{ij} are symmetric and antisymmetric parts of Ricci tensor. Reformulated in terms of standard GR there arise effective action

$$\mathcal{L} = \sqrt{-g}\left[-2\Lambda\left[\det(\delta_i^j + \lambda f_i^j)\right]^{\nu} + R(g) + c_a m^2\, g^{ij} a_i a_j\right],$$

where $\nu \equiv 1/(D-2)$ and c_a is also a parameter depending on D. g_{ij} is metric and f_{ij} — field tensor of additional (due to non-Riemannian space) massive vector field a_i. Spherical dimensional reduction from $D=4$ to $D=2$ provides dilaton gravity with

$$X(\varphi;\psi) = 2\phi^{-1/2} - 2\Lambda\sqrt{\phi}\left[1 + \psi^2/\lambda^2\Lambda^2\phi^2\right]^{1/2} \; Z(\phi) = -1/m^2\phi\,.$$

Switching to one-dimensional models in Dilaton gravity, describing static and cosmological (not necessarily FRW) states with metric

$$ds^2 = h(\tau)(-d\tau^2 + d\sigma^2)$$

we can write

$$L = s\Big[h^{-1}\dot{h}\dot{\phi} + W(\phi)\dot{\varphi}^2 + Z(\varphi)\dot{\psi}^2\Big] - s^{-1}h\,X(\varphi,\psi)\,, \qquad (1)$$

where s is a Lagrangian multiplier; variation with respect to it leads to Hamiltonian constraint:

$$H = s\,\Big[h^{-1}\dot{h}\dot{\phi} + W(\phi)\dot{\phi}^2 + Z(\phi)\dot{\psi}^2\Big] + s^{-1}h\,X(\phi,\psi) = 0\,.$$

In what follows we prefer variables

$$\dot{F} = h^{-1}\dot{h} + W(\phi)\dot{\phi}, \quad \dot{\varphi} = s\,\dot{\phi}\,.$$

The choice of dynamical variables may depend on the underlying theory. Consider a toy-model[2] with conformal invariance $g_{\mu\nu} \to e^{-2\sigma(x)} g_{\mu\nu}$, $\psi \to e^{\sigma(x)}\psi$:

$$\mathcal{L} = Na^3 \left[\frac{1}{2}\frac{\psi^2}{12} R - \frac{\dot{\psi}^2}{2N^2} - \frac{\lambda}{4}\psi^4 \right]$$

In the gauge $\psi = \sqrt{6}$ it has a de-Sitter inflationary solution $a = e^{\sqrt{3\lambda}t}$, while in gauge $a = 1$ it describes the *conformon* field dynamics in flat space: $\psi'' = \lambda\psi^3$. The Hamiltonian constraints will look like

$$H = -\frac{1}{12a} p_a^2 + 9\lambda a^3 = 0, \quad \text{or} \quad H = -\frac{1}{2} p_\psi^2 + \frac{\lambda}{4}\psi^4 = 0.$$

[2]R. Kallosh and A. Linde, *Hidden Superconformal Symmetry of the Cosmological Evolution*, JCAP **1401**, 020 (2014); arXiv:hep-th/1311.3326.

One of the important steps during the models investigation is a search for the first integrals what improves our knowledge about the space of solutions and may reveal hidden symmetries. We are focusing on this subject in our talk.

Universality of Dilaton gravity formalism requires a method for the first integrals search which should

- deal with a variety of gauge/variable choices in diverse models
- take into account the presence of Hamiltonian constraint.

For this we refined the *Whittaker's program* of searching for the first integrals which are polynomial in momenta:

$$\text{if } \; H = a_i(q)p^i \; \text{ and } \; I = b_j(q)p^j \; \text{ then } \; \{H, I\} = c_k(q)p^k.$$

It is convenient to work in space of polynomials.

In the symmetric algebra representation the polynomial of d-th order in momenta p^i read as

$$I_d = b_0 + b^i p_i + b^{ij} p_i p_j + b^{ijk} p_i p_j p_k + \ldots \equiv \sum_{\mathrm{n}=0}^{d} b_{\mathrm{n}}(q) p^{\odot n}\,,$$

where $\odot$ is a symmetrized tensor product and $p^{\odot n}$ is n-th tensor power of p_i with respect to this product. Equivalently, we may consider vectors $I_d = (b_0, .., b_{\mathrm{n}}, 0, \ldots) \in S(p^*)$ with coordinates in symmetric tensors. The key object is a Schouten bracket which assigns for two symmetric tensors of ranks k and n the symmetric tensor of rank $(n+k-1)$:

$$\mathcal{L}_{b_{\mathrm{n}}} a_{\mathrm{k}} \equiv \mathrm{Sym}\left(\sum_{r=1}^{n}\sum_{m_r=1}^{N} b^{m_1..m_r..m_n}\partial_{m_r} a^{l_1..l_k} - \sum_{r=1}^{k}\sum_{l_r=1}^{N}\partial_{l_r} b^{m_1..m_n} a^{l_1..l_r..l_k}\right)$$

The Poisson bracket then acts on $S(p^*)$ as

$$\{H_g, I_d\} = Y_{g+d-1}, \quad \text{where} \quad c_{\mathrm{n}} = \sum_{\substack{k=0..d,\\ m=0..g,\\ k+m-1=n}} \mathcal{L}_{b_{\mathrm{k}}} j_{\mathrm{m}} \,.$$

And a first integral should satisfy the condition

$$\{H_g, I_d\} = V_{d-1} \odot H_g, \quad \text{vanishing on shell!}$$

Normally the condition $\{H_g, I_d\} = Y_{g+d-1} = 0$ is required, which implies the vanish of all coefficients of Y_{g+d-1}, say $c_{\mathrm{n}} = 0$, $n = 0..(g + d - 1)$. Now the conditions on c_n are

$$c_{\mathrm{n}} = \sum_{m+k=n} v_{\mathrm{m}} \odot a_{\mathrm{k}}, \quad n = 0..(d + g - 1),$$

containing free tensor coefficients v_{m}.

For the most common quadratic Hamiltonians

$$H_2 = a^{ij}(q)p_i p_j + U(q) \equiv a_2 p^{\odot 2} + a_0$$

we can derive a compact equation

$$\{H_2, I_{2s+\varepsilon}\}|_{H_2=0} = 0, \quad \text{if} \quad \sum_{n=0}^{s+\varepsilon} (-a_0)^n (a_2)^{\odot(s+\varepsilon-n)} \odot c_{2\mathrm{n}+1-\varepsilon} = 0\,,$$

where

$$c_{\mathrm{n}} = \mathcal{L}_{b_{\mathrm{n}+1}} a_0 + \mathcal{L}_{b_{\mathrm{n}-1}} a_2 .$$

Note that in this case we also can use grading on $S(p^*)$. Since H_2 belongs to an even subspace (containing only tensor components with even rank), the set of first integrals I_d also decomposes into even and odd subspaces: $I_d \to I_{2s+\varepsilon}$.

- Well suited for analytic calculus programs (e.g. Maple).
- May consider classes of Hamiltonians: when the structure of kinetic part a^{ij} and the shape of potential U are known yet not exactly specified.
- May consider classes of symmetries: when the shape of first integral is specified the set of allowed Hamiltonians with desired structure will be found.

In what follows we solve equations for linear integrals, $a_2 c_0 - a_0 c_2 = 0$:

$$a^{ij}\partial_b U - U\mathcal{L}_b a^{ij} = 0,$$

and quadratic integrals, $a_2 \odot c_1 - a_0 c_3 = 0$:

$$\mathrm{Sym}\left[a^{lm}(b^{ik}\partial_k U - a^{ik}\partial_k \mathcal{U}) - U(b^{ik}\partial_k a^{lm} - \partial_k b^{il} a^{km})\right] = 0\,.$$

Now for one-dimensional Dilaton Gravity Hamiltonian

$$H = \dot{F}\dot{\varphi} - sZ\dot{\psi}^2 + s^{-1}\Omega X e^F = 0$$

we have $q^i = (F, \varphi, \psi)$ and the momenta $p_i = (\dot{\varphi}, \dot{F}, -2sZ\dot{\psi})$. We see that the in such coordinates the structure of Hamiltonian kinetic term, $a^{12} = a^{21} = 1/2$, $a^{33} = -1/4sZ$ is defined up to function sZ, which is gauge and coupling of scalar kinetic term to dilaton. Obviously, we can also choose the gauge $s = 1/Z$, which ensures us that for any Dilaton Gravity there will be a constant kinetic part, which leads to the following *allowed linear integrals*:

$$\begin{aligned}
b_F &= C_6\psi^2/2 + (C_4 F + C_7)\psi + C_1 F^2/2 + C_9 F + C_8,\\
b_\varphi &= C_1\psi^2/2 + (C_2 + C_4\varphi)\psi + C_6\varphi^2/2 + C_5\varphi + C_3,\\
b_\psi &= C_4\psi^2/2 + (C_1 F + C_6\varphi + C_5 + C_9)\psi/2 + (C_4\varphi + C_2)F/2 +\\
&\quad + C_7\varphi/2 + C_{10},
\end{aligned}$$

No other variants! Then solving the equation

$$(\partial_b + C_1 F + C_6\varphi + 2C_4\psi + C_5 + C_9)U = 0$$

we can found what integrals do exist for specified systems.

The number of equations in components rapidly grows with the increase of polynomials order. Thus for quadratic integrals we consider some special cases.

Consider, for instance, the quadratic integrals which does not depend on the momenta component p_1. This reduces the number of equations, so we can find two allowed quadratic integrals:

$$
\begin{aligned}
&H = p_1 p_2 - C_3 \beta'(\varphi) p_3^2/4 + [C_1 \psi^2 + C_2 \psi + C_4 - C_1 C_3 \beta(\varphi)] \beta'(\varphi) e^F \\
&I_2^{(1)} = p_2^2/\beta'^2 - 2 C_1 C_3 e^F, \quad I_2^{(2)} = 2 p_2 p_3/\beta' + (4 C_1 \psi + 2 C_2) e^F, \\
&\{H, I_2^{(1)}\} = 2 H p_2 \beta''/\beta'^3, \ \{H, I_2^{(2)}\} = 2 H p_3 \beta''/\beta'^2, \ \{I_2^{(1)}, I_2^{(2)}\} = 0.
\end{aligned}
$$

Thus we found in Dilaton Gravity an integrable model with quite non-trivial potential which is quadratic in scalar field ψ.

Conclusion and Outlook

- Easy to obtain negative results, i.e. find that something does not go. For example, we found that affine gravity even in most simple cases does not admit integrable analogues of the Schwarzschild solution.
- Next we will apply this approach to the models describing inflationary scenarios.

THANK YOU!