# PHYSICAL ASPECTS OF PARTICULAR SUPERSPACES AND SUPERSYMPLECTIC STRUCTURES

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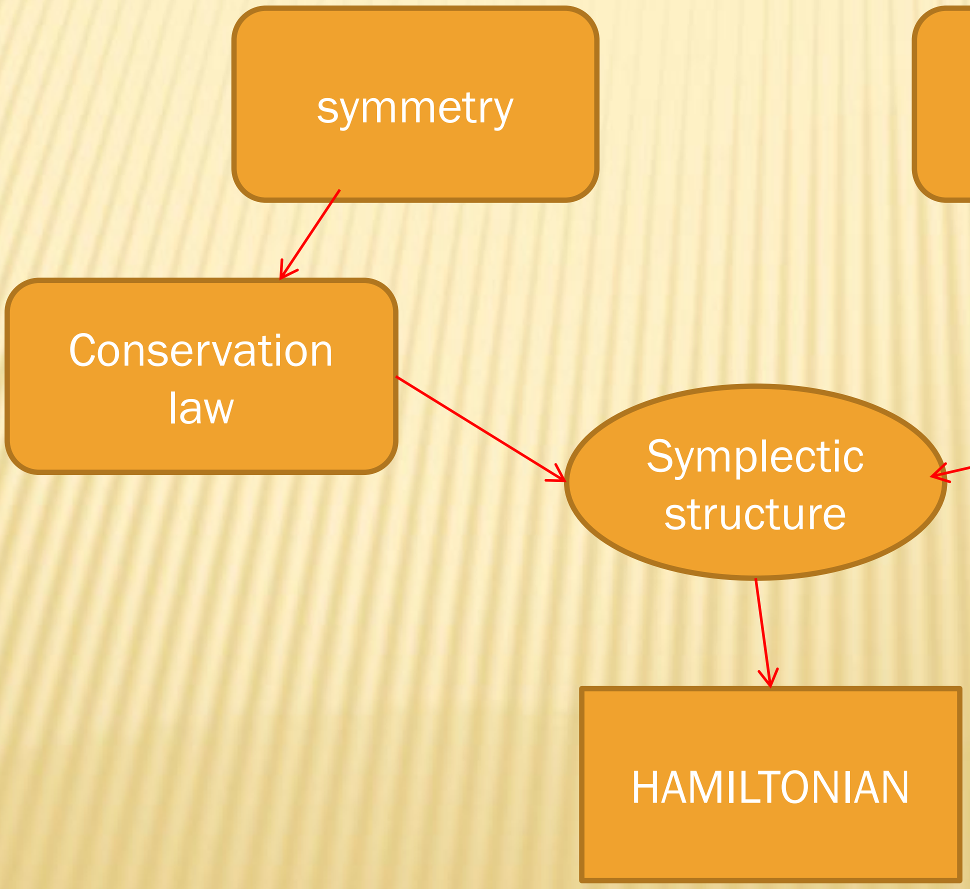
In this talk (if the time permit us) we will make some remarks concerning the physical consequences arising from some mathematical models based in non-degenerate Riemannian superspaces. Emphasis is made on the relation of such models with with the underlying supersymplectic structure and hints towards a quantum gravity theory

## CLASSICAL STRUCTURES-INTEGRABILITY (MARSDEN AND RATIU, ARNOLD AND KHESIN) ↔ SUSY STRUCTURES-SUPER-INTEGRABILITY (BATCHELOR, ROTHSTEIN)



# ON QUANTIZATION (SOME TROUBLES…NON TREATED HERE)

- Not unique(?)
- Not related with the underlying symmetries of the physical system
- Based on prescriptions instead physical principles
- Etc etc….

- (e.g: models based in weakly reductive geometries are canonically quantized)

# OUTLINE

- Statistical/information theoretical results
- Spacetime discretization, quantum gravity and other issues
- Remarks (concluding?)

# INFORMATION METRIC FROM RIEMANNIAN SUPERSPACES.

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- The Fisher's information metric is introduced in order to find the real meaning of the probability distribution in classical and quantum systems described by Riemaniann non-degenerated superspaces. In particular, the physical rôle played by the coefficients a and $a*$ of the pure fermionic part of a genuine emergent metric solution, obtained in previous work is explored. To this end, two characteristic viable distribution functions are used as input in the Fisher definition: first, a Lagrangian generalization of the Hitchin Yang-Mills prescription and, second, the probability current associated to the emergent non-degenerate superspace geometry. We have found that the metric solution of the superspace allows establish a connexion between the Fisher metric and its quantum counterpart, corroborating early conjectures by Caianiello et al. This quantum mechanical extension of the Fisher metric is described by the $CP^1$ structure of the Fubini--Study metric, with coordinates a and $a*$

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# INTRODUCTION

- The problem of giving an unambiguous quantum mechanical description of a particle in a general spacetime has been repeatedly investigated.
- The introduction of supersymmetry provided a new approach to this question, however, some important aspects concerning the physical observables remain not completely understood, classically and quantically speaking.
- The superspace concept, on the other hand, simplify considerably the link between ordinary relativistic systems and `supersystems', extending the standard (bosonic) spacetime by means of a general (super)group manifold, equipped with also fermionic (odd) coordinates

# INTRODUCTION

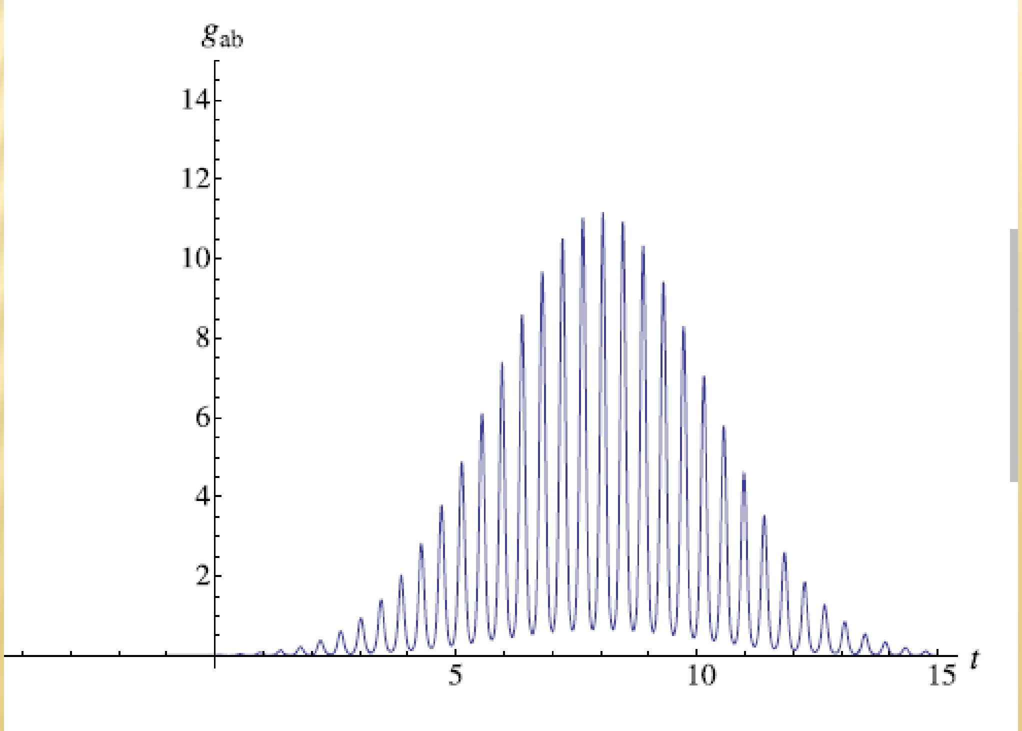
- In [1] we introduced, besides other supersymmetric quantum systems of physical interest, a particular N=1 superspace
- That was made with the aim of studying a superworld-line quantum particle (analogously to the relativistic case) and its relation with SUGRA theories [2]
- The main feature of this superspace is that the supermetric, which is the basic ingredient of a Volkov-Pashnev particle action [4]is invertible and non-degenerate, that is, of G4 type in the Casalbuoni's classification
- As shown in [2,3]the non-degeneracy of the supermetrics (and therefore of the corresponding superspaces) leads to important consequences in the description of physical systems.
- In particular, notorious geometrical and topological effects on the quantum states, namely, consistent mechanisms of localization and confinement, due purely to the geometrical character of the Lagrangian.
- Also an alternative to the Randall-Sundrum (RS) model without extra bosonic coordinates, can be consistently formulated in terms of such non degenerated superspace approach, eliminating the problems that the RS-like models present at the quantum level [2,3]


[1] D.J. Cirilo-Lombardo, Eur. Phys. J. C 72 (2012) 2079.
[2] D.J. Cirilo-Lombardo, Found. Phys. 37 (2007) 919; D.J. Cirilo-Lombardo, Found. Phys. 39 (2009) 373.
[3] D.J. Cirilo-Lombardo, Phys. Lett. B 661 (2008) 186.
[4] D.V. Volkov, A.I. Pashnev, Theoret. and Math. Phys. 44 (3) (1980) 770.

$$ds^2 = \omega^\mu \omega_\mu + \mathbf{a}\omega^\alpha \omega_\alpha - \mathbf{a}^* \omega^{\dot\alpha} \omega_{\dot\alpha},$$

$$\omega_\mu = dx_\mu - i(d\theta\sigma_\mu\bar\theta - \theta\sigma_\mu d\bar\theta),$$
$$\omega^\alpha = d\theta^\alpha, \qquad \omega^{\dot\alpha} = d\theta^{\dot\alpha}$$

| Space–time | $(5-D)$ gravity $+ \Lambda$ | Superspace $(1, d \mid 1)$ |
|---|---|---|
| Interval | $ds^2 = A(y)dx_{3+1}^2 - dy$ | $ds^2 = \omega^\mu \omega_\mu + \mathbf{a}\omega^\alpha \omega_\alpha - \mathbf{a}^* \omega^{\dot\alpha} \omega_{\dot\alpha}$ |
| Equation | $[-\partial_y^2 - m^2 e^{H\|y\|} + H^2 - 2H\delta(y)]u(y) = 0$ | $[\|a\|^2(\partial_0^2 - \partial_i^2) + \frac{1}{4}(\partial_\eta - \partial_\xi + i\partial_\mu(\sigma^\mu)\xi)^2 - \frac{1}{4}(\partial_\eta + \partial_\xi + i\partial_\mu(\sigma^\mu)\xi)^2 + m^2]_{cd}^{ab} g_{ab} = 0$ |
| Solution | $u(y) = ce^{-H\|y\|}, \quad H \equiv \sqrt{-\frac{2\Lambda}{3}} = \frac{\|T\|}{M^3}$ | $g_{ab}(x) = e^{-(\frac{m}{\|\mathbf{a}\|})^2 x^2 + c_1' x + c_2'} e^{\xi\varrho(x)} \|f(\xi)\|^2 \binom{\alpha}{\alpha^*}_{ab}$ |



# INTRODUCTION

- As the Lagrangian of this particular supermetric bring us localized states showing a Gaussian behaviour, it is of clear interest to analyze the probabilistic and information theoretical meaning of such a geometry.
- To this specific end, the Fisher metric [7](Fisher-Rao in the quantum sense [8])have been considered in several works in order to provide a geometrical interpretation of the statistical measures.
- Fisher's information measure (FIM) was advanced already in the 1920's decade, well before the advent of Information Theory (IT).
- Much interesting work has been devoted to the physical applications of FIM in recent times (see, for instance, [9]and references therein Also,a generalization of the Yang-Mills Hitchin proposal was made suggesting an indentification of the Lagrangian density with the Fisher probability distribution (P(θ)).
- However, this idea was explored from a variational point of view, in previous work by Plastino et al. [9,10,11]and in several geometrical ways by Brody and Hughston.
- This proposal brought a contribution to the line of works looking for a connexion between the spacetime geometry and quantum field theories.
- In the last decades it has been claimed that the above expectation is partially realized in the AdS/CFT (anti-de Sitter/Conformal Field Theory) correspondence which asserts that the equivalence of a gravitational theory (i.e., the geometry of spacetime) and a conformal quantum field theory at the boundary of spacetime certainly exists.

[7] R. Fisher, Math. Proc. Cambridge Philos. Soc. 22 (1925) 700.
[8] C.R. Rao, Bull. Calcutta Math. Soc. 37 (1945) 81.
[9] B. Roy Frieden, Science from Fisher Information: A Unification, Cambridge Univ. Press, ISBN 0-521-00911-1, 2004.
[10] L.P. Chimento, F. Pennini, A. Plastino, Phys. Lett. A 293 (2002) 133.
[11] B. Roy Frieden, A. Plastino, Phys. Lett. A 272 (2000) 326.

# FISHER'S METRIC AND HITCHIN'S PRESCRIPTION

- The Fisher information metric is a Riemannian metric for the manifold of the parameters of probability distributions.
- The Rao distance (geodesic distance in the parameter manifold) provides a measure of the difference between distinct distributions.

In the thermodynamic context, the Fisher information metric is directly related to the rate of change in the corresponding order parameters and can be used as an information-geometric complexity measure for classifying phase transitions, e.g., the scalar curvature of the thermodynamic metric tensor diverges at (and only at) a phase transition point (this issue will be analyzed in future work). In particular, such relations identify second-order phase transitions via divergences of individual matrix elements

# FISHER'S METRIC AND HITCHIN'S PRESCRIPTION

The Fisher-Rao information metric is given by

$$G_{ab}(\theta) = \int d^D x\, P(x;\theta)\, \partial_a \ln P(x;\theta)\, \partial_b \ln P(x;\theta)$$

where $x_\mu$($\mu$,$\nu$=0,...,D) are the random variables and $\theta_a$ (a,b=1,...,N) are the parameters of the probability distribution. Besides this, P(x;θ) must fulfill the normalization condition

$$\int d^D x P(x;\theta) = 1$$

# FISHER'S METRIC AND HITCHIN'S PRESCRIPTION

Hitchin proposed the use of the the squared field strength of Yang-Mills theory as a probability distribution.

A generalization of the Hitchin's proposal consist in identifying the probability distribution with the on-shell Lagrangian density of a field theory

$$P(x;\theta) \coloneqq -\mathcal{L}(x;\theta)\big|_{solution}.$$

# FISHER'S METRIC AND HITCHIN'S PRESCRIPTION

- Hitchin proposed an alternative definition of the moduli space geometry. For the set of instantons with a fixed instanton number Q, the Lagrangian density $\mathcal{L}_{YM}$ may be considered as a probability distribution functions: $\int_{\mathbb{R}4} \mathcal{L}_{YM} = Q$

Q. Hitchin then proposes to utilize so-called Fisher-Rao's information metric to describe the geometry of the instanton moduli space M.

# FISHER'S METRIC AND HITCHIN'S PRESCRIPTION

- First, we will follow the `generalized Hitchin prescription', identifying our Lagrangian (calculated at the solution) with the probability distribution.
- Second, we will introduce a new proposal: we will take the state probability current of the emergent metric solution of the superspace as being itself the probability distribution.
- The results of the two approaches will be compared in order to infer the physical meaning of the $a$ and $a*$ parameters appearing in the pure fermionic part of the superspace metric.

# EMERGENT METRIC SOLUTION

The model describes a free particle in a superspace with coordinates

$$z_A \equiv (x^\mu, \theta_\alpha, \bar\theta_{\dot\alpha})$$

The corresponding Lagrangian density is

$$\mathcal{L} = -m\sqrt{\omega^A \omega_A} = -m\sqrt{\overset{\circ}{\omega}_\mu \;\overset{\circ}{\omega}{}^\mu + \mathbf{a}\dot\theta^\alpha\dot\theta_\alpha - \mathbf{a}^*\dot{\bar\theta}^{\dot\alpha}\dot{\bar\theta}_{\dot\alpha}}\,.$$

$$\overset{\circ}{\omega}_\mu = \dot x_\mu - i(\dot\theta\;\sigma_\mu\bar\theta - \theta\,\sigma_\mu\;\bar\theta)$$

In coordinates

$$ds^2 = \dot z^A \dot z_A = \dot x^\mu \dot x_\mu - 2i\dot x^\mu(\dot\theta\sigma_\mu\bar\theta - \theta\sigma_\mu\dot{\bar\theta}) + (\mathbf{a} - \bar\theta^{\dot\alpha}\bar\theta_{\dot\alpha})\dot\theta^\alpha\dot\theta_\alpha - (\mathbf{a}^* + \theta^\alpha\theta_\alpha)\dot{\bar\theta}^{\dot\alpha}\dot{\bar\theta}_{\dot\alpha}$$

# EMERGENT METRIC SOLUTION

The `squared' solution with three compactified dimensions (λ spin fixed) is

$$g_{AB}(t) = e^{A(t)+\xi\varrho(t)} g_{AB}(0),$$

where the initial values of the metric components are given by

$$g_{ab}(0) = \langle\psi(0)| \begin{pmatrix} a \\ a^{\dagger} \end{pmatrix}_{ab} |\psi(0)\rangle,$$

or, explicitly,

$$g_{\mu\nu}(0) = \eta_{\mu\nu}\,, \qquad g_{\mu\alpha}(0) = -i\sigma_{\mu\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\,, \qquad g_{\mu\dot{\alpha}}(0) = -i\theta^{\alpha}\sigma_{\mu\alpha\dot{\alpha}}\,,$$

$$g_{\alpha\beta}(0) = (a - \bar{\theta}^{\dot{\alpha}}\bar{\theta}_{\dot{\alpha}})\epsilon_{\alpha\beta}\,, \qquad g_{\dot{\alpha}\dot{\beta}}(0) = -(a^{*} + \theta^{\alpha}\theta_{\alpha})\epsilon_{\dot{\alpha}\dot{\beta}}\,.$$

# EMERGENT METRIC SOLUTION

The bosonic and spinorial parts of the exponent in the superfield solution are, respectively,

$$A(t) = -\left(\frac{m}{|\mathbf{a}|}\right)^2 t^2 + c_1 t + c_2,$$

and

$$\begin{aligned}
\xi\varrho(t) &= \xi(\phi_\alpha(t) + \bar{\chi}_{\dot{\alpha}}(t)) \\
&= \theta^\alpha\left(\overset{\circ}{\phi}_\alpha \cos(\omega t/2) + \tfrac{2}{\omega}Z_\alpha\right) - \bar{\theta}^{\dot{\alpha}}\left(-\overset{\circ}{\bar{\phi}}_{\dot{\alpha}} \sin(\omega t/2) - \tfrac{2}{\omega}\bar{Z}_{\dot{\alpha}}\right) \\
&= \theta^\alpha \overset{\circ}{\phi}_\alpha \cos(\omega t/2) + \bar{\theta}^{\dot{\alpha}} \overset{\circ}{\bar{\phi}}_{\dot{\alpha}} \sin(\omega t/2) + 4|\mathbf{a}|Re(\theta Z),
\end{aligned}$$

where $\varphi\alpha$, $Z\alpha$, are constant spinors, $\omega \approx 1/|a|$ and the constant $c_1$, due to physical reasons and the chirality restoration of the superfield solution should be taken purely imaginary

# IMPORTANT PROPIERTIES OF THE SUPERFIELD SOLUTIONS RELATED WITH THE PARAMETERS

- Gaussian solutions
- consistent mechanisms of localization and confinement.
- an alternative to the Randall-Sundrum (RS) model without extra bosonic coordinates.
- Chirality restoration in the system.

# FISHER INFORMATION METRIC FROM RIEMANNIAN SUPERSPACES

- Fisher method considers a family of probability distributions, characterized by certain number of parameters. The metric components are then defined by considering derivatives in different `directions' in the parameters space.
- That is, measuring `how distant' two distinct set of parameters put apart the corresponding probability distributions.
- In the following we will calculate the Fisher information metric corresponding to a generalized Hitchin `on-shell' Lagrangian prescription. In our case, the parameters of interest in the metric solution are $a$ and $a*$, which could indicate the residual effects of supersymmetry given that they survive even when `turning off' all the fermionic fields.

# GENERALIZED HITCHIN PRESCRIPTION FOR THE PROBABILITY DISTRIBUTION

- Following the generalized Hitchin prescription we identify the probability ditribution with Lagrangian evaluated at solution $g_{AB}(t) = e^{(A(t)+\xi\varrho(t))} g_{AB}(0).$

  the probability distribution density takes the form

$$P(z^A, a, a^*) \coloneqq -\mathcal{L}\big|_{g_{AB}(t)} = e^{\frac{1}{2}(A(t)+\xi\varrho(t))}\mathcal{L}_0,$$

where

$$\mathcal{L}_0 \equiv \mathcal{L}(g_{ab}(0)) = m\sqrt{\overset{\circ}{\omega}{}^{\nu}\overset{\circ}{\omega}{}^{\mu} + \mathbf{a}\dot{\theta}^{\alpha}\dot{\theta}_{\alpha} - \mathbf{a}^{*}\dot{\bar{\theta}}^{\dot{\alpha}}\dot{\bar{\theta}}_{\dot{\alpha}}}\,.$$

# GENERALIZED HITCHIN PRESCRIPTION FOR THE PROBABILITY DISTRIBUTION

- After calculate the a and a∗ derivatives of our probability distribution, we can now write down the Fisher's metric components

$$G_{aa} = \int dx^4 P^{-1}\left[\frac{\partial P}{\partial \mathbf{a}}\right]^2 = \frac{\mathcal{L}_0}{16}\int dt\left(\frac{\mathbf{a}^*}{|\mathbf{a}|}\Xi(t;|a|) + \frac{2m^2}{\mathcal{L}_0^2}\dot{\theta}^{\alpha}\dot{\theta}_{\alpha}\right)^2 e^{\frac{1}{2}(A(t)+\xi\varrho(t))}$$

$$G_{a^*a^*} = \int dx^4 P^{-1}\left[\frac{\partial P}{\partial \mathbf{a}^*}\right]^2 = \frac{\mathcal{L}_0}{16}\int dt\left(\frac{\mathbf{a}}{|\mathbf{a}|}\Xi(t;|a|) - \frac{2m^2}{\mathcal{L}_0^2}\dot{\bar{\theta}}^{\dot{\alpha}}\dot{\bar{\theta}}_{\dot{\alpha}}\right)^2 e^{\frac{1}{2}(A(t)+\xi\varrho(t))}$$

$$G_{aa^*} = g_{a^*a} = \int dx^4 P^{-1}\frac{\partial P}{\partial \mathbf{a}}\frac{\partial P}{\partial \mathbf{a}^*} = \frac{\mathcal{L}_0}{16}\int dt\left(\Xi(t;|a|)^2 + 4\frac{m^4}{\mathcal{L}_0^4}\dot{\theta}^{\alpha}\dot{\theta}_{\alpha}\dot{\bar{\theta}}^{\dot{\alpha}}\dot{\bar{\theta}}_{\dot{\alpha}}\right)e^{\frac{1}{2}(A(t)+\xi\varrho(t))}$$

where

$$\begin{aligned}\Xi(t;|a|) &\equiv \frac{\partial(A(t)+\xi\varrho(t))}{\partial|\mathbf{a}|}\\ &= \frac{2m^2}{|\mathbf{a}|^3}t^2 + \frac{\omega^2 t}{2}\left(\theta^{\alpha}\ \overset{\circ}{\phi}_{\alpha}\ \sin(\omega t/2) - \bar{\theta}^{\dot{\alpha}}\ \overset{\circ}{\bar{\phi}}_{\dot{\alpha}}\ \cos(\omega t/2)\right) + 4Re(\theta Z),\end{aligned}$$

# STATE PROBABILITY CURRENT AS DISTRIBUTION

- identifying the state probability density (zero component of the probability current) of the solution as the probability density itself.
- The zero component of the probability current can be obtained by making

$$j_0(t) = 2E^2 g_{ab}(t) g^{ab}(t).$$

$K_0 \equiv 32E^2|\alpha|^2$,

$$j_0(t) = \frac{1}{16}\mathcal{K}_0 e^{-2\left(\frac{m}{|\mathbf{a}|}\right)^2 t^2 + 2c_2 + 2\xi\varrho(t)},$$

# STATE PROBABILITY CURRENT AS DISTRIBUTION

- taking $P \equiv j_0(t)$

$$G_{aa} = \int dx^4 P^{-1} \left[ \frac{\partial P}{\partial \mathbf{a}} \right]^2 = \left( \frac{\mathbf{a}^*}{|\mathbf{a}|} \right)^2 \mathcal{I}_\Xi$$

$$G_{a^*a^*} = \int dx^4 P^{-1} \left[ \frac{\partial P}{\partial \mathbf{a}^*} \right]^2 = \left( \frac{\mathbf{a}}{|\mathbf{a}|} \right)^2 \mathcal{I}_\Xi$$

$$G_{aa^*} = G_{a^*a} = \int dx^4 P^{-1} \frac{\partial P}{\partial \mathbf{a}} \frac{\partial P}{\partial \mathbf{a}^*} = \mathcal{I}_\Xi,$$

where $\mathcal{I}_\Xi$ corresponds to the integral of the temporal part

$$\mathcal{I}_\Xi \equiv \frac{1}{16} \mathcal{K}_0 \int dt \ \Xi(t;|a|)^2 e^{-2\left(\frac{m}{|\mathbf{a}|}\right)^2 t^2 + 2c_2 + 2\xi\varrho(t)}.$$

# THE $B_0$ PART OF THE FISHER SUPERMETRIC: GENERALIZED HITCHIN PRESCRIPTION WITH ZERO FERMIONS

When putting all fermion to zero, the derivative of the time dependent exponential and the $L_0$ initial value `on-shell' Lagrangian reduce, respectively, to

$$\Xi(t;|a|)|_{\theta=\chi=0} = \frac{2m^2}{|\mathbf{a}|^3}t^2 \qquad \text{and} \qquad \mathcal{L}_0|_{\theta=\chi=0} = m\sqrt{\dot{x}^\mu \dot{x}_\mu} = m,$$

In that case, and writing the complex parameters as a=|a|exp(iφ), the metric components take the simple form

$$G_{ab} = \mathcal{I}(m, c_1, c_2, |\mathbf{a}|)\begin{pmatrix} e^{-i2\phi} & 1 \\ 1 & e^{i2\phi} \end{pmatrix},$$

where indices a,b take values in {a,a∗}, and the prefactor is the integral of the time varying factor, that can be easily performed to obtain

$$\mathcal{I}(m, c_1, c_2, |\mathbf{a}|) \equiv \frac{1}{4}\left(\frac{m^5}{|\mathbf{a}|^6}\right)\int dt\ t^4\, e^{-\left(\frac{m}{|\mathbf{a}|}\right)^2 t^2 + c_1 t + c_2}$$

$$= \frac{\sqrt{\pi}}{64}\left[\left(\frac{c_1}{m}\right)^4|\mathbf{a}|^3 + 12\left(\frac{c_1}{m}\right)^2|\mathbf{a}| + 12|\mathbf{a}|^{-1}\right]e^{\frac{1}{4}\left(\frac{c_1}{m}\right)^2|\mathbf{a}|^2 + c_2}$$

# THE $B_0$ PART OF THE FISHER SUPERMETRIC: STATE PROBABILITY CURRENT WITH ZERO FERMIONS

- although the fermions are turning out, the gaussian (localized)behaviour of the solution remains due the complex fermionic coefficients a. This is very important for the phenomenological point of view because the localized behavior of the solution remains also after the susy breaking

$$j_0|_{\theta=\chi=0} = \frac{1}{16}\mathcal{K}_0 e^{-2\frac{m^2}{|\mathbf{a}|^2}t^2+2c_2},$$

$$G_{ab} = \mathcal{J}(m, E, |\alpha|, c_2, |\mathbf{a}|)\begin{pmatrix} e^{-i2\phi} & 1 \\ 1 & e^{i2\phi} \end{pmatrix}$$

now the prefactor is obtained performing the integral

$$\begin{aligned} \mathcal{J}(m, E, |\alpha|, c_2, |\mathbf{a}|) &\equiv \frac{1}{4}\left(\frac{\mathcal{K}_0 m^4}{|\mathbf{a}|^6}\right)\int dt\, t^4\, e^{-2\left(\frac{m}{|\mathbf{a}|}\right)^2 t^2+2c_2} \\ &= \frac{3\sqrt{2\pi}}{4}\frac{E^2|\alpha|^2}{m}e^{2c_2}|\mathbf{a}|^{-1}, \end{aligned}$$

# THE $B_0$ PART OF THE FISHER SUPERMETRIC: COMPARISON

$$\mathcal{I}(m, c_1, c_2, |\mathbf{a}|) \equiv \frac{1}{4}\left(\frac{m^5}{|\mathbf{a}|^6}\right)\int dt\ t^4\, e^{-\left(\frac{m}{|\mathbf{a}|}\right)^2 t^2 + c_1 t + c_2}$$

$$= \frac{\sqrt{\pi}}{64}\left[\left(\frac{c_1}{m}\right)^4 |\mathbf{a}|^3 + 12\left(\frac{c_1}{m}\right)^2 |\mathbf{a}| + 12|\mathbf{a}|^{-1}\right] e^{\frac{1}{4}\left(\frac{c_1}{m}\right)^2 |\mathbf{a}|^2 + c_2}$$

$$\mathcal{J}(m, E, |\alpha|, c_2, |\mathbf{a}|) \equiv \frac{1}{4}\left(\frac{\mathcal{K}_0 m^4}{|\mathbf{a}|^6}\right)\int dt\, t^4\, e^{-2\left(\frac{m}{|\mathbf{a}|}\right)^2 t^2 + 2c_2}$$

$$= \frac{3\sqrt{2\pi}}{4}\frac{E^2|\alpha|^2}{m} e^{2c_2}|\mathbf{a}|^{-1},$$

- Note that this last expression presents only the $|a|^{-1}$ singular term. This is precisely due to the lack of the `free wave' (linear in t) term in the exponential factor, which leads to a complete departure of the Gaussian behaviour shown by remaining in common just the singular term

# INFORMATION METRIC AND GEOMETRICAL LAGRANGIANS

Note that the Fisher metric can be rewritten in the form

$$G_{ab}(\theta) = \int d^D x\, P(x;\theta)\, \partial_a \ln P(x;\theta)\, \partial_b \ln P(x;\theta)$$
$$= 4\int d^D x\, \partial_a P^{1/2}(x;\theta)\, \partial_b P^{1/2}(x;\theta).$$

The appearance of the square root of the probability density P above, naturally leads to the identification

$$P^{1/2} \equiv \mathcal{L}_g \quad \Rightarrow \quad P = ds^2$$

the P function is related to the line element that define the geometrical (superspace in our case) Lagrangian of the theory.
Therefore, this is a first approach to connect the two "distances": the Rao distance in the probability parameters manifold, and the geometric space-time distance

# P AS THE CURRENT OF PROBABILITY: THE QUANTUM CORRESPONDENCE

Consider a Hilbert space $\mathcal{H}$ with a symmetric inner product $G_{ij}$. For instance, we can have in mind the case $\mathcal{H} = \mathbf{L}_2(\mathcal{R})$, where $\mathcal{R} = \mathbb{R}^{2m}$ (e.g.: the phase space of a classical dynamical system, the configuration space spin systems, etc.)

$$\begin{aligned} G_{ab}(\theta) &= \int d^D x\, P(x;\theta)\,\partial_a \ln P(x;\theta)\,\partial_b \ln P(x;\theta) \\ &= 4\int d^D x\; \partial_a P^{1/2}(x;\theta)\,\partial_b P^{1/2}(x;\theta). \end{aligned}$$

puts in evidence the clear possibility of mapping the probability density function $P(x;\theta)$ on $\mathcal{R}$ to $\mathcal{H}$ by forming the square-root.

As we propose

$$j_0 = \frac{1}{16}\mathcal{K}_0 e^{-2\left(\frac{m}{|\mathbf{a}|}\right)^2 t^2 + 2c_2 + 2\xi\varrho(t)} \equiv P(x;\theta),$$

The metric components take then the form

$$\begin{aligned} G_{ab}(\theta) &= 4\int d^D x\; \partial_a\, P^{1/2}(x;\theta)\,\partial_b\, P^{1/2}(x;\theta) \\ &\rightarrow 4\int d^D x\, \partial_a g_{AB}(x;\mathbf{a},\mathbf{a}^*)\,\partial_b\, g^{AB}(x;\mathbf{a},\mathbf{a}^*) \\ &= 4\int d^D x\, \partial_a\, g_{AB}(x;\mathbf{a},\mathbf{a}^*)\,\partial_b\, g_{CD}(x;\mathbf{a},\mathbf{a}^*)\eta^{(AB)(CD)}. \end{aligned}$$

# P AS THE CURRENT OF PROBABILITY: THE QUANTUM CORRESPONDENCE

the quantum `crossover' is

$$G_{ab}(\theta) = 4\partial_a g_{AB}(x;\mathbf{a},\mathbf{a}^*)\,\partial_b g_{CD}(x;\mathbf{a},\mathbf{a}^*)\eta^{(AB)(CD)}$$
$$\equiv \langle \partial_a g_{AB}(x;\mathbf{a},\mathbf{a}^*)\,\partial_b g_{CD}(x;\mathbf{a},\mathbf{a}^*)\rangle.$$

We can also introduce the following Hermitian metric tensor

$$\widetilde{G}_{ab}(\theta) = \langle \partial_a g_{AB}(x;\mathbf{a},\mathbf{a}^*)\,\partial_b g_{CD}(x;\mathbf{a},\mathbf{a}^*)\rangle$$
$$-\langle \partial_a g_{AB}(x;\mathbf{a},\mathbf{a}^*)\, g_{CD}(x;\mathbf{a},\mathbf{a}^*)\rangle\langle g_{AB}(x;\mathbf{a},\mathbf{a}^*)\partial_b g_{CD}(x;\mathbf{a},\mathbf{a}^*)\rangle,$$

since its real part can be ***exactly*** rewritten as

$$\mathrm{Re}\,\widetilde{G}_{ab}(\theta) = \Big\langle \partial_a L^{1/2}_{AB}(x;\mathbf{a},\mathbf{a}^*)\,\partial_b L^{1/2}_{CD}(x;\mathbf{a},\mathbf{a}^*)\Big\rangle_{HS},$$

# P AS THE CURRENT OF PROBABILITY: THE QUANTUM CORRESPONDENCE

where $L_{AB}$ (non-diagonal representation) is given by

$$L_{AB} = \int \frac{d^2\alpha}{\pi}\left[\int \frac{d^2 w}{\pi}\int \frac{d^2\alpha'}{\pi} e^{-\left(\frac{m}{\sqrt{2}|\mathbf{a}|}\right)^2[(\alpha+\alpha^*)-B]^2+D} e^{\xi\varrho\left(\alpha'+\alpha'^*\right)}|f(\xi)|^2 \times \right.$$

$$\left.\begin{pmatrix}\alpha' \\ \alpha^{*\prime}\end{pmatrix}_{(m+n)AB} e^{\frac{|w|^2}{4}} e^{\frac{i}{2}\left[\left(\alpha-\alpha'\right)w^*+\left(\alpha^*-\alpha^{*\prime}\right)w\right]}\right]|\Psi_m(\alpha)\rangle\langle\Psi_n(\alpha')|$$

with $m,n = 1/4, 3/4\,(m \neq n)$, and $\langle , \rangle_{GS}$ standing for the customary inner product in the Banach space of the Gram-Schmidt operators in $\mathcal{H}$. The complex numbers $\alpha$ and $\alpha^*$ in the exponential factor are the *eigenvalues* of the coherent states.

the corresponding Gram-Schmidt operator reads

$$G = \int \frac{d^2\alpha}{\pi}\left[\int \frac{d^2 w}{\pi} e^{\frac{|w|^2}{4}} e^{\frac{i}{2}\left[\left(\alpha-\alpha'\right)w^*+\left(\alpha^*-\alpha^{*\prime}\right)w\right]}\right]|\Psi_m(\alpha)\rangle\langle\Psi_n(\alpha')|$$

In summary, our results come to realize the conexion conjectured by Caianiello.

# CONCLUDING REMARKS

- In this work we have analyzed several aspects of the geometrical meaning of the Fisher's metric definition. A new generalization of the Hitchin prescription for constructing the Fisher's information metric was presented, taking a geometrical Lagrangian as probability distribution ($P \leftrightarrow L$). The results were confronted with a completely different prescription: to take as probability distribution the state probability density of an emergent metric (coherent) state, which is a solution for a non-degenerated superspace obtained in a previous work ($P \leftrightarrow j_0$). We then analyze the bosonic ($B_0$) part of the Fisher supermetric by putting all fermionic fields to zero, in order to compare our solutions with those in the literature. The main results of this research can be summarized as follows:

# CONCLUDING REMARKS I

i) The choice of the complex constants a and $a*$ as our set of (physically meaningful) parameters is based on that they are responsible for the localized Gaussian behaviour of the physical states. This lead to a Fisher's metric on a complex 2-dimensional manifold presenting notably different behaviours in the two approaches. In the first one ($P \leftrightarrow L$) the Gaussian behaviour of the metric state solution gab was preserved while in the second one ($P \leftrightarrow j_0$) it was completely lost. However, it is important to remark that, in both cases, the ultralocal characteristic behavior of gab preserved through a singular ($|a|^{-1}$) term.

ii) In principle, it should be possible to relate, from the quantum point of view, the a- a^{∗} complex manifold (Fisher's) metric with an invariant metric on a Kähler or on a projective Hilbert space ($CP^1$).

iii) the function P, in sharp contrast with the Hitchin's proposal, an be put in direct relation with the spacetime line element $ds^2$ by making the identification P^{1/2}≡L_{g} in the same Fisher's formula.

iv) we demonstrate that, using the probability current $j_0$ as the probability density P, the quantum counterpart of the Fisher's metric can be exactly implemented, and all the quantum operators involved in the geometrical correspondence, exactly constructed, as already inferred on a general basis by Caianiello et al.

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# Quantum gravity: Physics from supergeometries

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We show that the metric (line element) is the first geometrical object to be associated to a discrete (quantum) structure of the spacetime without necessity of black hole-entropy-area arguments, in sharp contrast with other attempts in the literature. To this end, an emergent metric solution obtained previously in [*Phys. Lett. B* 661 (2008) 186–191] from a particular non-degenerate Riemannian superspace is introduced. This emergent metric is described by a physical coherent state belonging to the metaplectic group $\mathrm{Mp}(n)$ with a Poissonian distribution at lower $n$ (number basis) restoring the classical thermal continuum behavior at large $n$ ($n \to \infty$), or leading to non-classical radiation states, as is conjectured in a quite general basis by means of the Bekenstein–Mukhanov effect. Group-dependent conditions that control the behavior of the macroscopic regime spectrum (thermal or not), as the relationship with the problem of area/entropy of the black hole are presented and discussed.

*Keywords*: Supermanifolds; Riemannian geometry; quantum gravity.

Mathematics Subject Classification 2010: 83C45

# INTRODUCTION

- Gravity $\Leftrightarrow$ quantum theory (new concepts:strings, LQG)
- S.T quantum structure $\Leftrightarrow$ discretization
- Fundamental scale (minimal length) $\Leftrightarrow$ naturally metric
- Now $\Rightarrow$ LENGTH AREA (strings, LQG)

- AREA/ENTROPY (BH)

  Spin networks/LQG Dynamical triangulations/string theory

Quantum Gravity Unification

1) MACROSCOPICAL REG.(thermal or not)

2) CONSISTENT FORMULATION (length)

3)CORRECT SPECTRUM (limits for n)

# Superspace and discrete spacetime structure

$$\sum_m |m\rangle\langle m| = 1,$$

$$g_{ab}(0) = \sum_{n,m} \langle\psi(0)|m\rangle\langle m|L_{ab}|n\rangle\langle n|\psi(0)\rangle$$

$$g_{ab}(t) = \underbrace{e^{A(t)+\xi\rho(t)}}_{f(t)} \sum_{n,m} \langle\psi(0)|m\rangle\langle n|\psi(0)\rangle\langle m|L_{ab}|n\rangle$$

$$\langle m|L_{ab}|n\rangle = \langle m|\begin{pmatrix} a \\ a^\dagger \end{pmatrix}_{ab} |n\rangle = \begin{pmatrix} \langle m|n-1\rangle\sqrt{n} \\ \langle m|n+1\rangle\sqrt{n+1} \end{pmatrix}_{ab} = \begin{pmatrix} \delta_{m,n-1}\sqrt{m} \\ \delta_{m,n+1}\sqrt{m+1} \end{pmatrix}_{ab}$$

$$g_{ab}(0) = \sum_{n,m} \langle\psi(0)|m\rangle \begin{pmatrix} \delta_{m,n-1}\sqrt{m} \\ \delta_{m,n+1}\sqrt{m+1} \end{pmatrix}_{ab} \langle n|\psi(0)\rangle$$

$$g_{ab}(0) = \sum_n \sqrt{n}\langle\psi(0)|n-1\rangle\langle n|\psi(0)\rangle \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{ab} + \sum_m \sqrt{n+1}\langle\psi(0)|n+1\rangle\langle n|\psi(0)\rangle \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{ab}$$

$$\psi(0)\rangle = A|\alpha_+\rangle + B|\alpha_-\rangle$$

Poissonian distribution for the coherent states

$$P_\alpha(n) = |\langle n|\alpha\rangle|^2 = \frac{\alpha^n e^{-\alpha}}{n!}$$ obeying $$\sum_{n=0}^{\infty} P_\alpha(n) = 1, \quad \sum_{n=0}^{\infty} n\, P_\alpha(n) = \alpha$$

it differs with the individual distributions coming from each one of the two irreducible representations of the metaplectic group Mp(2) (spanning even and odd n respectively):

$$\sum_{n=0}^{\infty} P_{\alpha_+}(2n) = e^{-\alpha}\cosh(\alpha), \quad \sum_{n=0}^{\infty} P_{\alpha_-}(2n+1) = e^{-\alpha}\sinh(\alpha) \rightarrow \sum_{n=0}^{\infty} \left(P_{\alpha_+}(n) + P_{\alpha_-}(n)\right) = 1$$

$$|\alpha_+\rangle \equiv |\Psi_{1/4}(0,\xi,q)\rangle = \sum_{k=0}^{+\infty} f_{2k}(0,\xi)|2k\rangle = \sum_{k=0}^{+\infty} f_{2k}(0,\xi)\frac{(a^\dagger)^{2k}}{\sqrt{(2k)!}}|0\rangle$$

$$|\alpha_-\rangle \equiv |\Psi_{3/4}(0,\xi,q)\rangle = \sum_{k=0}^{+\infty} f_{2k+1}(0,\xi)|2k+1\rangle = \sum_{k=0}^{+\infty} f_{2k+1}(0,\xi)\frac{(a^\dagger)^{2k+1}}{\sqrt{(2k+1)!}}|0\rangle$$

$$g_{ab}(t) = \frac{f(t)}{2}\sum_{m}\left\{[P_{\alpha_+}(2m)\cdot 2m + P_{\alpha-}(2m+1)\cdot(2m+1)]\begin{pmatrix}1\\0\end{pmatrix}_{ab} + \right.$$

$$\left. + [P_{\alpha_+^*}(2m)\cdot 2m + P_{\alpha_-^*}(2m+1)\cdot(2m+1)]\begin{pmatrix}0\\1\end{pmatrix}_{ab}\right\}$$

this expression is the core of our discussion: it shows explicitly the discrete structure of the spacetime as the fundamental basis for a consistent quantum field theory of gravity. By the other hand, when we reach the limit n→∞ the metric solution goes to the continuum due:

$$\sum_{n=0}^{\infty}[P_{\alpha_+}(2m)\cdot 2m + P_{\alpha-}(2m+1)\cdot(2m+1)] = \sum_{n=0}^{\infty}[P_{\alpha_+}(2m+2)\cdot(2m+2) + P_{\alpha_-}(2m+1)\cdot(2m+1)] = \alpha e^{-|\alpha|}(\cosh(\alpha)+\sinh(\alpha)) = \alpha$$

and similarly for the lower part (spinor down) of above equation

$$\sum_{n=0}^{\infty} \left[ P_{\alpha_+}(2m) \cdot 2m + P_{\alpha_-}(2m+1) \cdot (2m+1) \right] = \alpha^*$$

Consequently, when the number of levels increase the metric solution goes to the continuum "manifold" general relativistic behaviour:

$$g_{ab}(t)_{n\to\infty} \to \frac{f(t)}{2} \left\{ \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{ab} + \alpha^* \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{ab} \right\} = f(t) \langle \psi(0) | \begin{pmatrix} a \\ a^\dagger \end{pmatrix}_{ab} | \psi(0) \rangle$$

# CONCLUDING REMARKS II

- 1) Emergent nature of the spacetime.
- 2) Independence of the discretization method.
- 3) Consistent suitable transition to the macroscopic (classical, semiclassical, etc.) regime.
- 4) Total and absolute independence of particular solutions or other arguments involving particular geometries (e.g. black-hole/area and the entropy ) .
- 5) Solutions, arguments involving particular geometries, etc. of the previous point, must be reached by the quantum gravity theory but not depending them at the fundamental level.