Exceptional Structures and Relativistic Quark Model

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- Yasemin Gurcan (CUNY)
- Amish Khalfan (CUNY)
- Levent Kurt (CUNY)

Symmetry is a wide-reaching concept that has been used in variety ways in physics. Originally it was used mainly to describe the arrangement of atoms in molecules and crystals (geometric symmetries.)

Over the course of 20th century and beyond, it has been considerably extended and covers some of the most fundamental ideas in physics.

Nature seems to favor some particularly unique and beautiful structures for the description of its inner secrets. They often appear in disguised broken-down form, so it is up to us to divine their existence from incomplete evidence: awareness of these structures is an important research tool.

There are four infinite families of simple Lie algebras:

$$\begin{aligned} A_n &\sim SU(n+1) \\ B_n &\sim SO(2n+1) \\ C_n &\sim Sp(2n) \\ D_n &\sim SO(2n) \end{aligned}$$

They describe spacetime rotation, quark and lepton charges and their associated Yang-Mills gauge structures.

Today, SU(N) gauge theories with N large are intensely studied.

With the advent of QM, Lie algebras and the groups they generate have found widespread uses in the description of physical systems. The quantum mechanical state of a particle is determined by labels. Some are continuous, such as particle's momentum (or position). Others are discrete, such as its spin and charges. All stem from irreducible unitary representations of Lie algebras. The continuous ones pertain to irreps of non-compact groups, and the discrete ones to compact groups. Mass and spin label the representations of the non-compact group of special relativity the Poincare group, and the color of a quark roam inside a representation of the compact color group SU(3). Moreover, their interactions are determined by dynamical structures based on these invariance groups.

1. Kinematic (space-time) symmetries.Examples are rotational invariance in non-relativistic quantum mechanics

$$H = \frac{p^2}{2m} + V(r)$$

$$H\psi(\vec{r}) = i\hbar\frac{\partial}{\partial t}\psi(\vec{r})$$

leading to SO(3) symmetry, and Lorentz invariance in relativistic quantum mechanics

$$[\gamma_\mu(i\partial_\mu - eA_\mu) + m]\psi(\vec{r}, t) = 0$$

which leads to SO(3,1) symmetry.

2. Dynamic (internal) supersymmetries.

Here we see development of two ideas:

a. there may exist in nature other symmetries in addition to space-time.

b. There may be symmetries of dynamical origin, related to special properties of the Hamiltonian (or Lagrangean) operator, rather than its space-time behavior.

Atomic Physics

Pauli (1926) $SO(4)$

Heisenberg (1932) $SU_T(2)$

Wigner (1937) $SU_{TS}(4) \supset SU_T(2) \otimes SU_S(2)$

Particle Physics

Nuclear Physics

Elliot (1958) $SU(3)$

Gell-Mann and Ne'eman (1962) $SU(3)$

Gürsey and Radicati (1964) $SU_{sf}(6) \supset SU_f(3) \otimes SU2)$

Arima and Iachello (1974) SU(6)

Supersymmetry:

In normal symmetry, symmetry operations transform separately fermions into fermions, bosons into bosons.

In supersymmetry, some of the symmetry operations transform bosons into fermions and vice versa. Introduction of SUSY led to other major developments in physics. SUSY is used in variety of ways. Particularly important are:

1. Kinematic (space-time) supersymmetries:
For example Wess-Zumino invariance. No experimental evidence for it yet.

2 Dynamic (internal) supersymmetries.

A Brief History of Dynamic SUSY in Physics

Miyazawa (1966) $SU(6/21)$

Particle Physics

Nuclear Physics

Balantekin, Bars & Iachello (1978) $SU(6/\Omega)$

Catto & Gürsey (1984) $SU(6/21)$

Is color related to octonions?

Is the quark structure a consequence of octonionic quantum mechanics?

Some consequences:

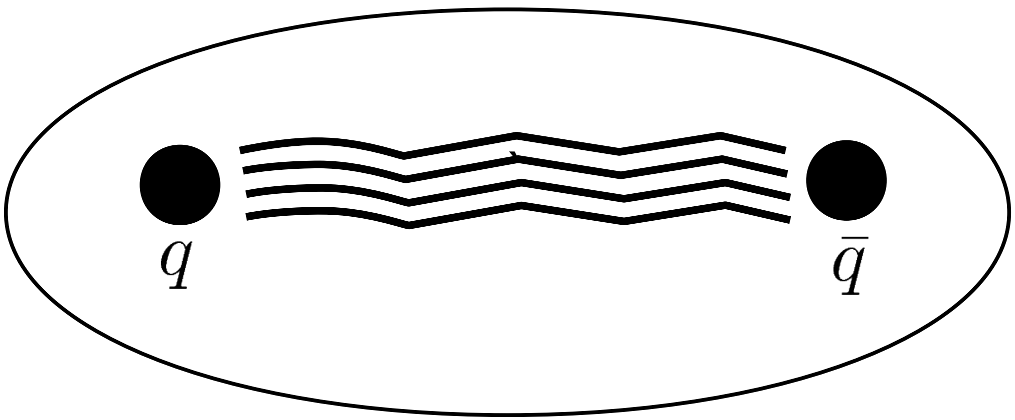
Since G_2 is the automorphism group of octonions

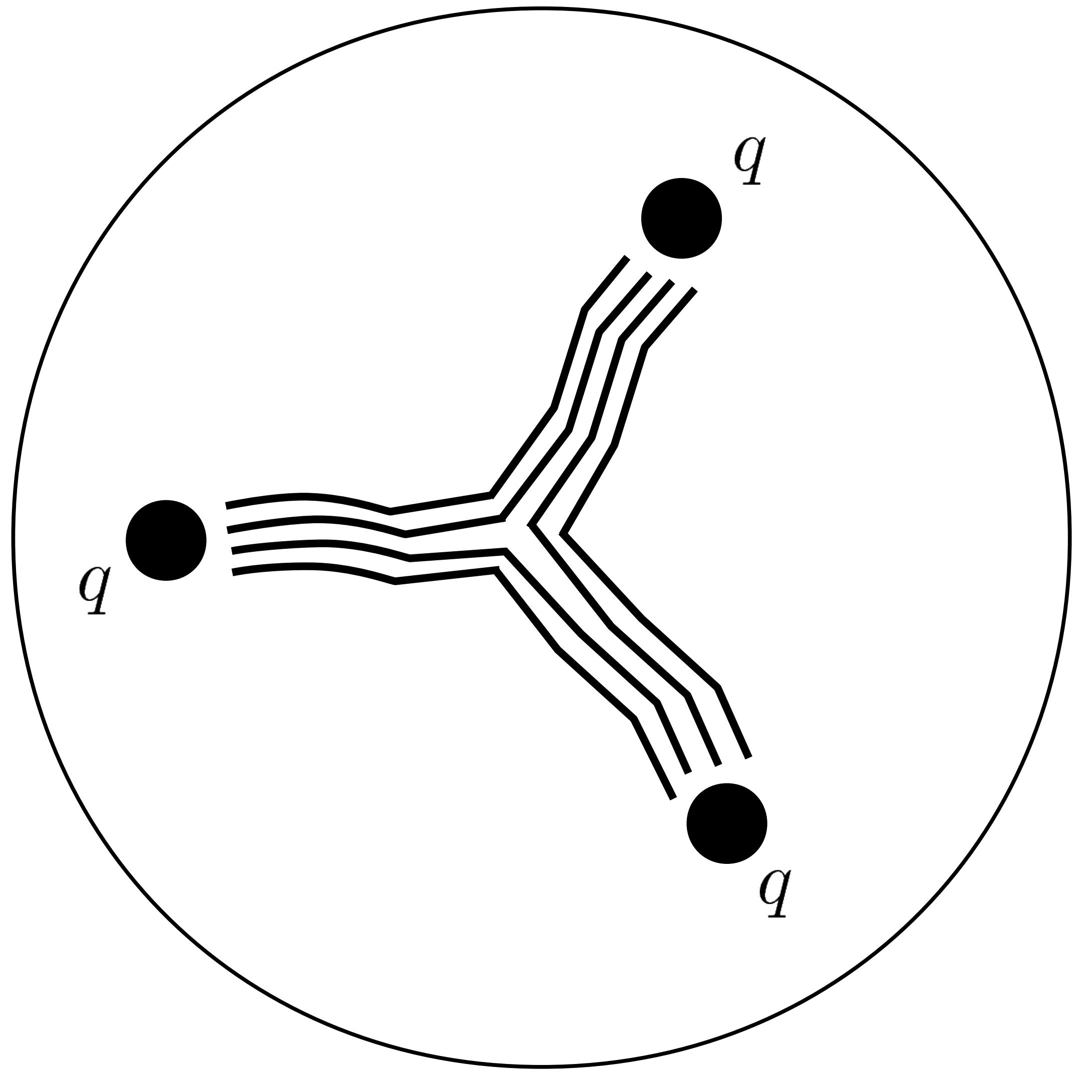
$G_2 = Aut(\Omega)$

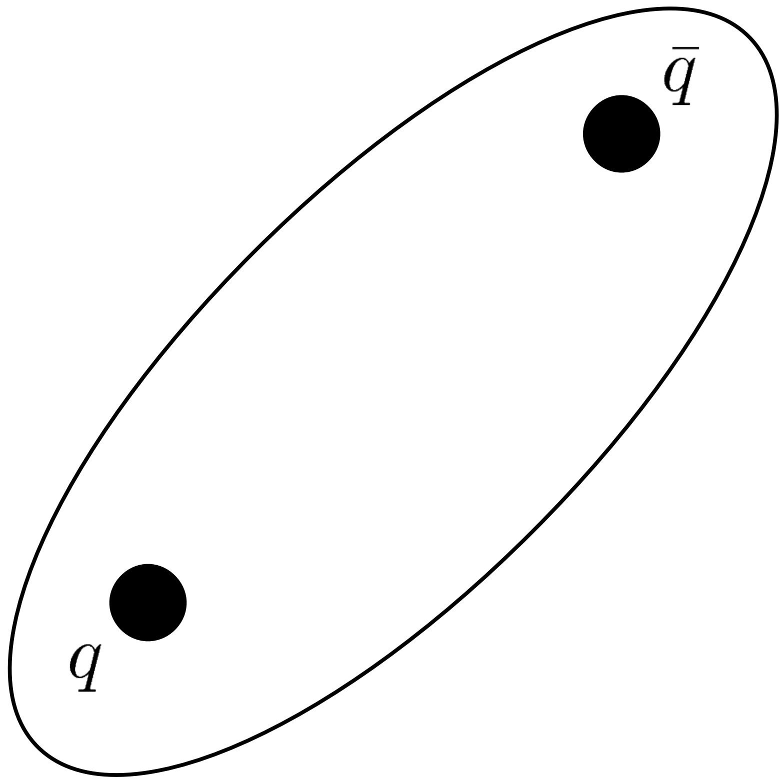
and it can be imbedded into $SO(7)$

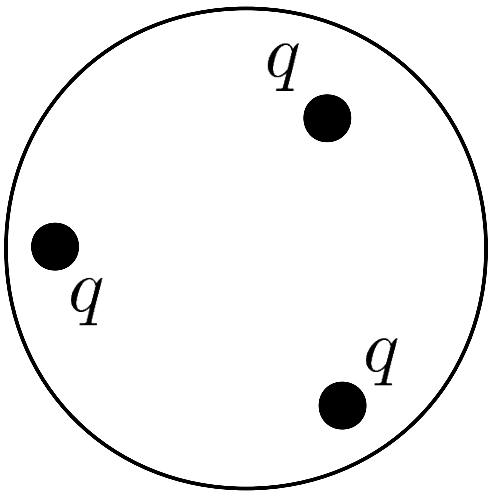
$SO(7) \supset G_2 \supset SU(3)$

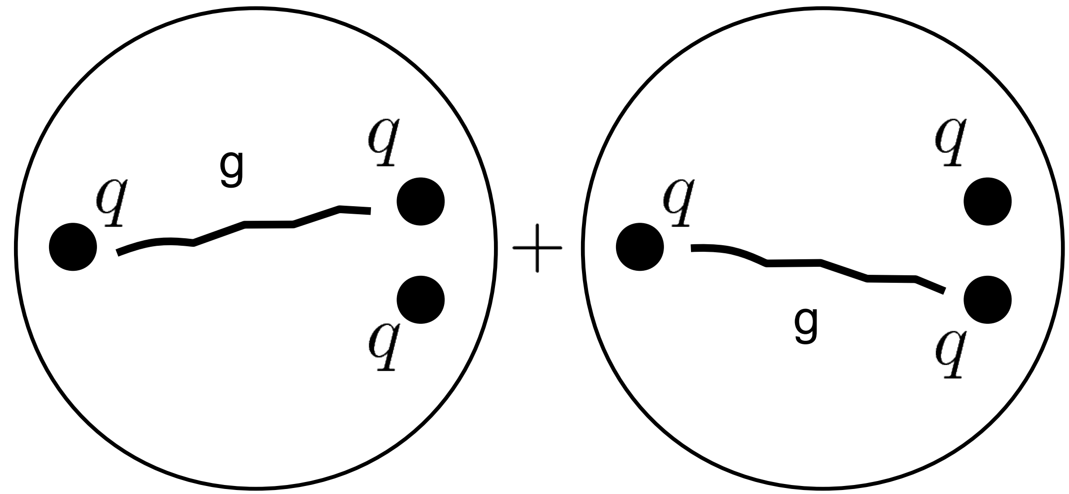
is $SO(7)$ a higher symmetry of strong interactions?


q
$\bar{q}$


q
q
q


$\bar{q}$
q


q
q
q


q
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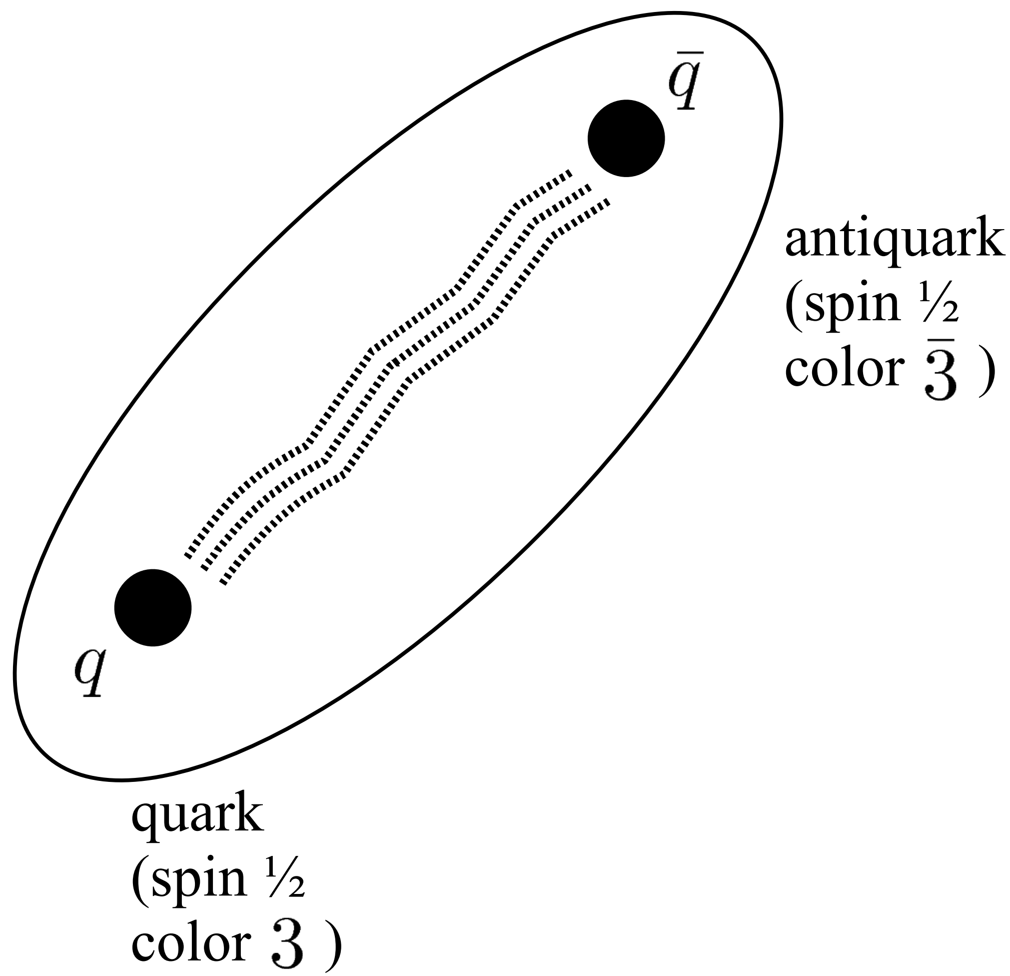
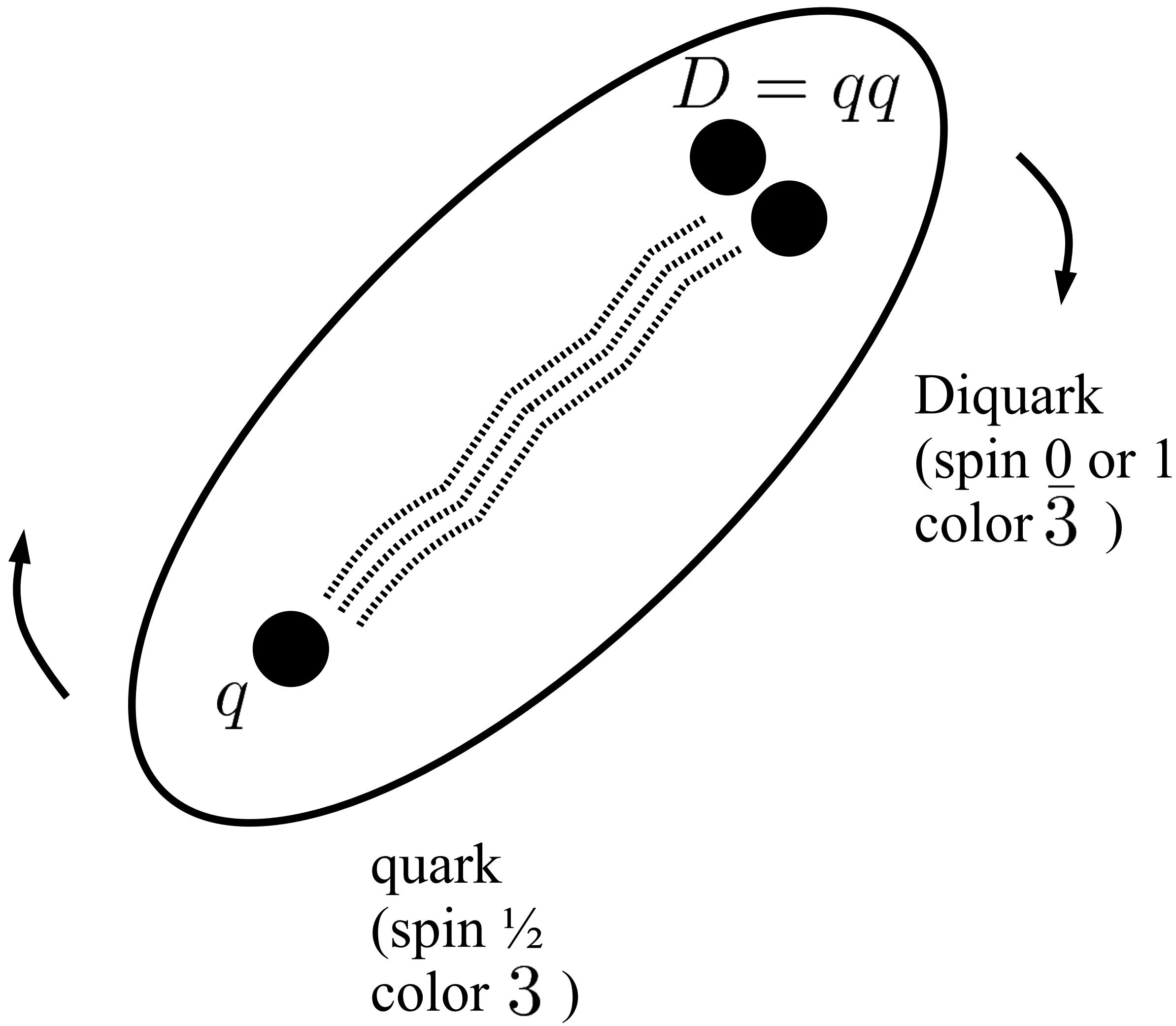
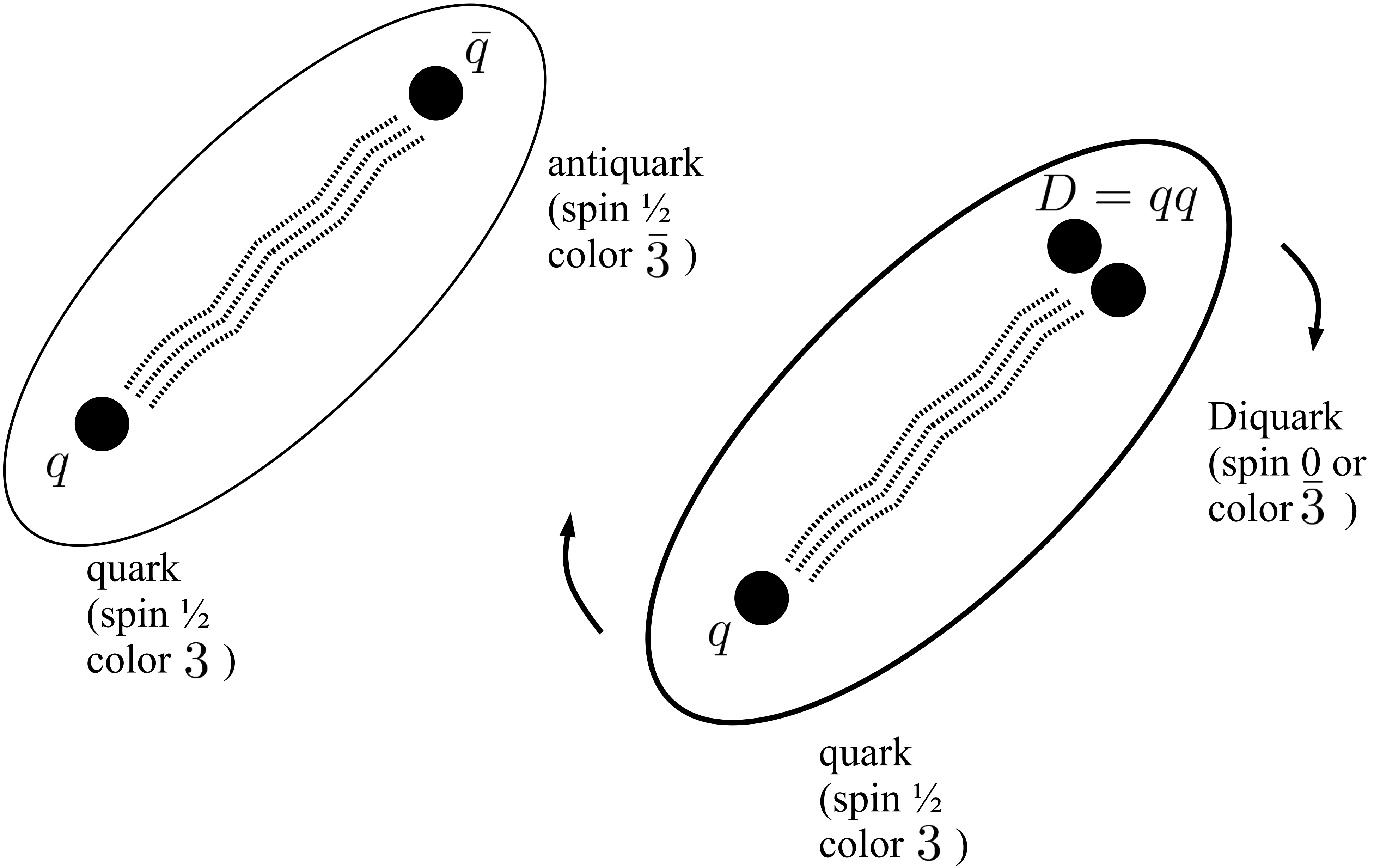
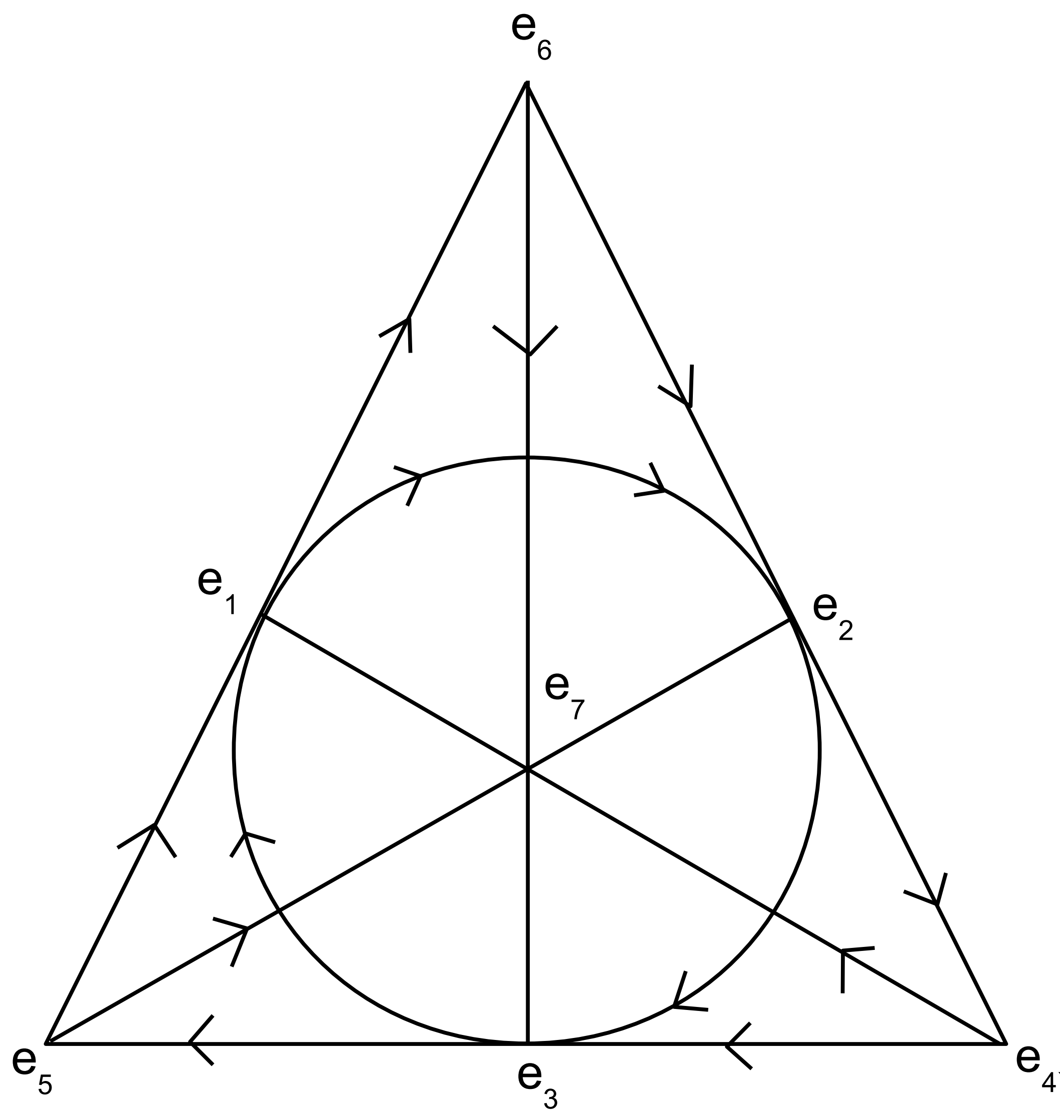

$\bar{q}$
antiquark
(spin ½
color $\bar{3}$)
q
quark
(spin ½
color 3)


$D = qq$
q
Diquark
(spin 0 or 1
color $\bar{3}$)
quark
(spin ½
color 3)


$\bar{q}$
q
antiquark
(spin ½
color $\bar{3}$)
quark
(spin ½
color 3)
$D = qq$
Diquark
(spin $\underline{0}$ or 1
color $\bar{3}$)
q
quark
(spin ½
color 3)

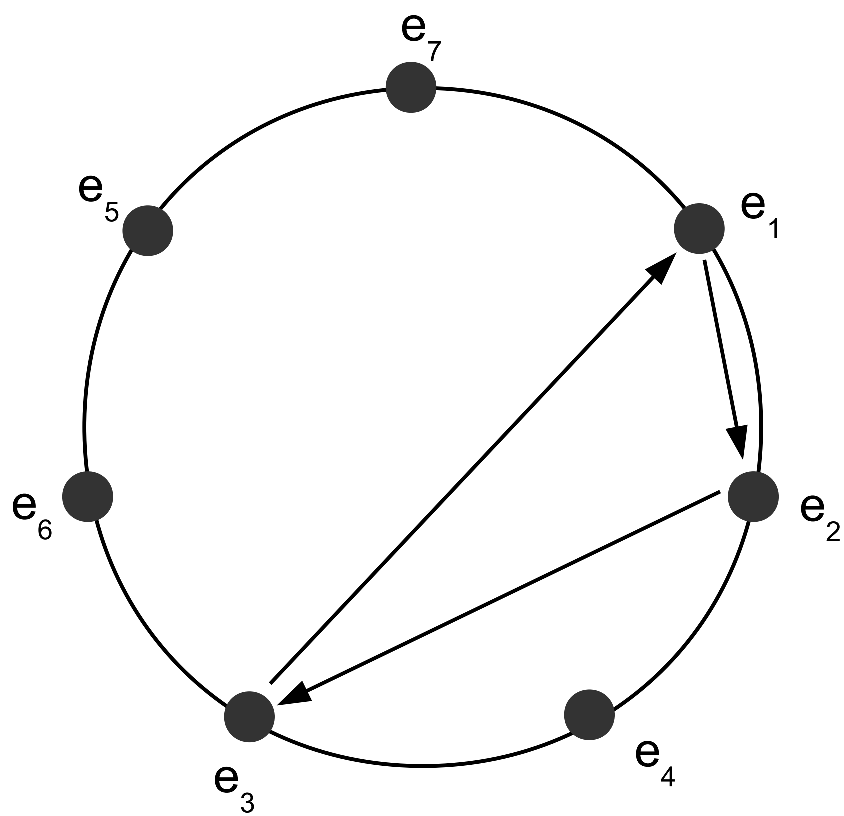
$$\text{Octonions: } 1, e_A \quad A = 1, ..., 7 \qquad e_A e_B = -\delta_{AB} + \epsilon_{ABC} e_C$$

$$\epsilon_{ABC} = 1 for ABC = 123, 516, 624, 435, 471, 673, 572$$



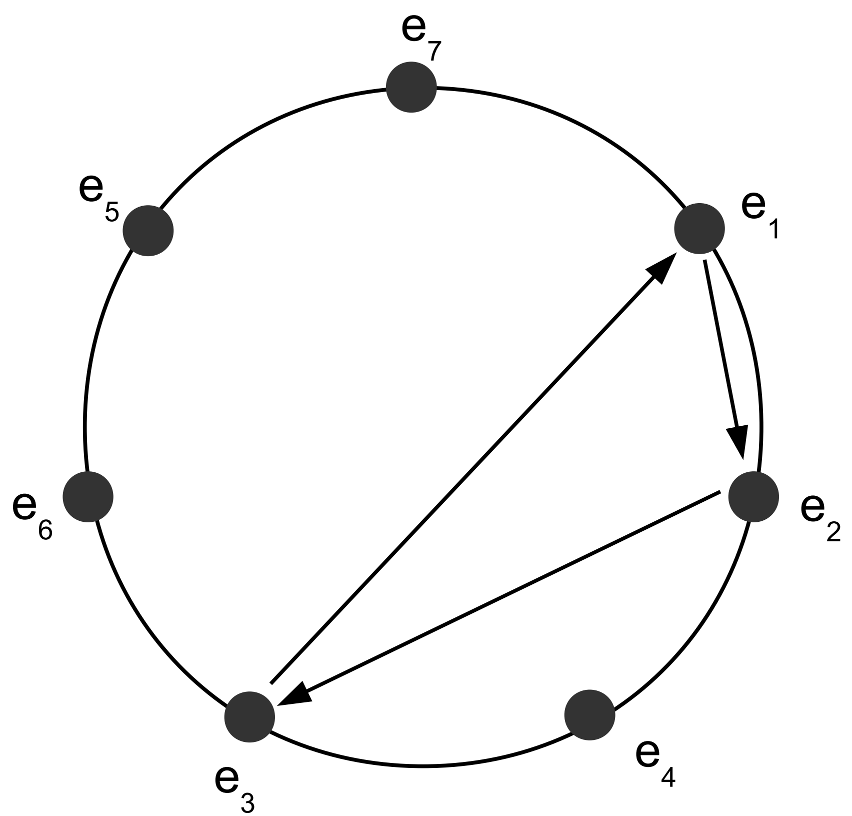
$$e_1 \, e_2 = e_3$$
$$e_2 \, e_1 = -e_3$$
$$[e_1, e_2] = 2e_3$$

Gürsey diagram



$$[x, y, z] = (xy)z - x(yz)$$

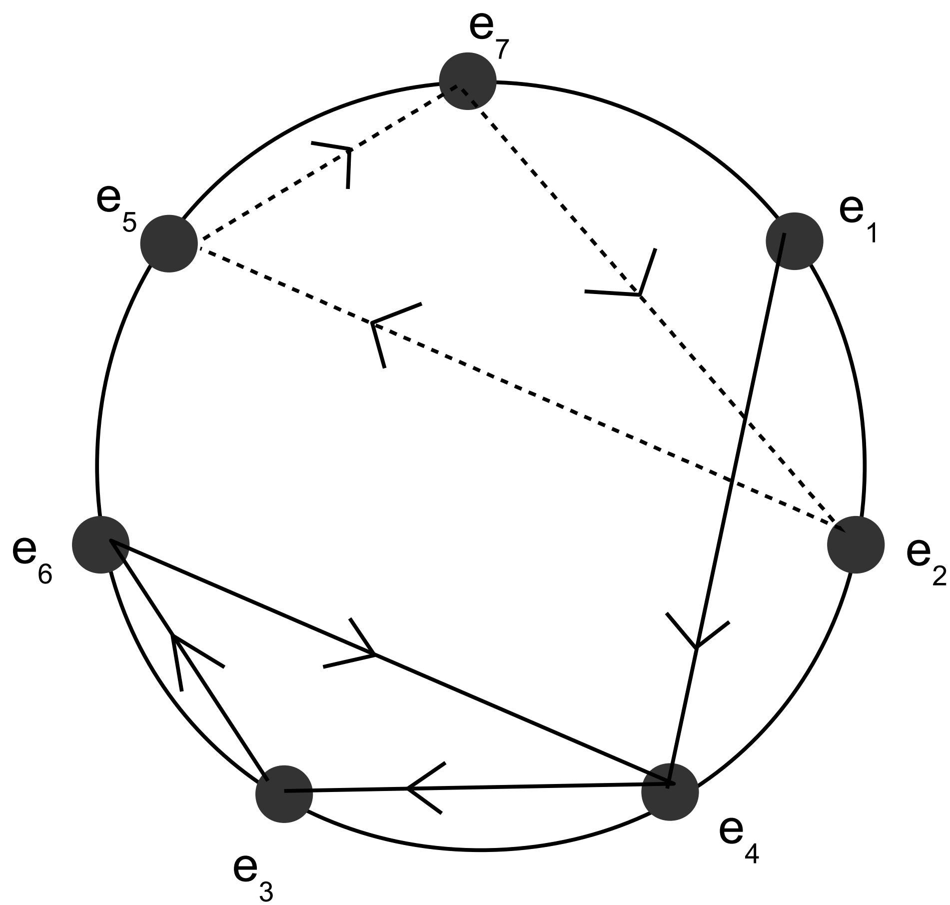
$$
\begin{aligned}
e_1 \; e_2 &= e_3 \\
e_2 \; e_1 &= -e_3 \\
[e_1, e_2] &= 2e_3
\end{aligned}
$$



$$[x, y, z] = (xy)z - x(yz)$$

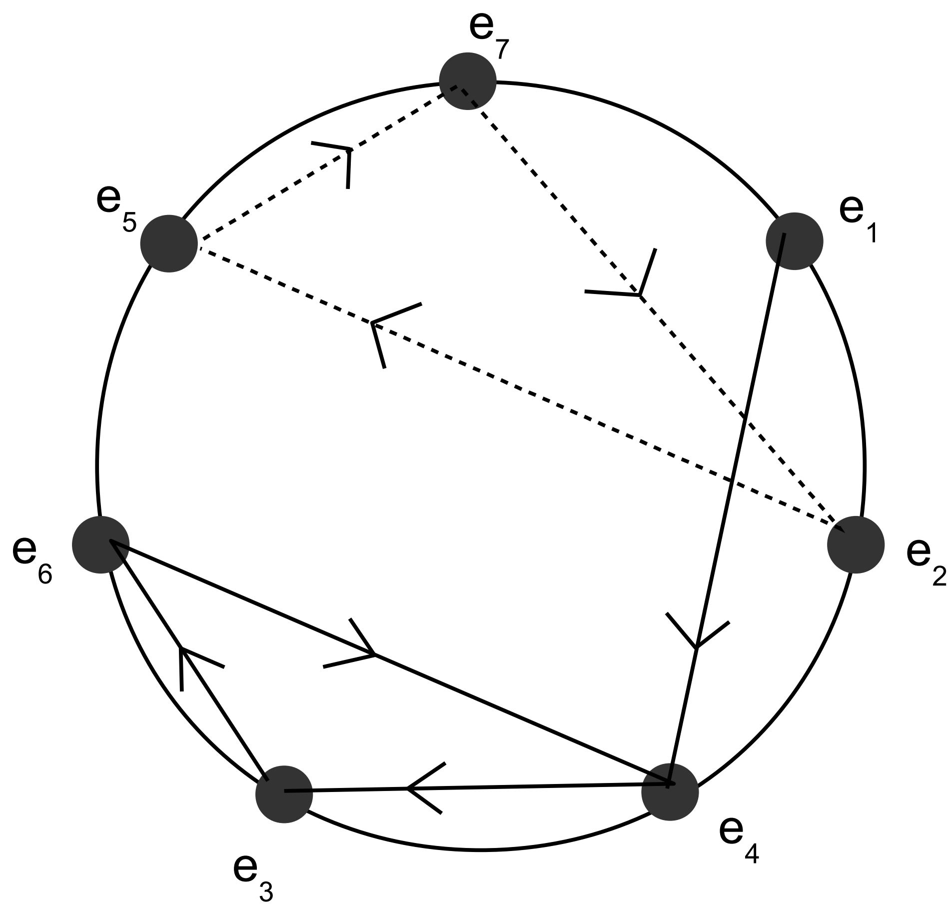
$$[e_4, e_6, e_5] = (e_4 e_6)e_5 - e_4(e_6 e_5) = 2e_7$$

Completion of Gürsey diagram



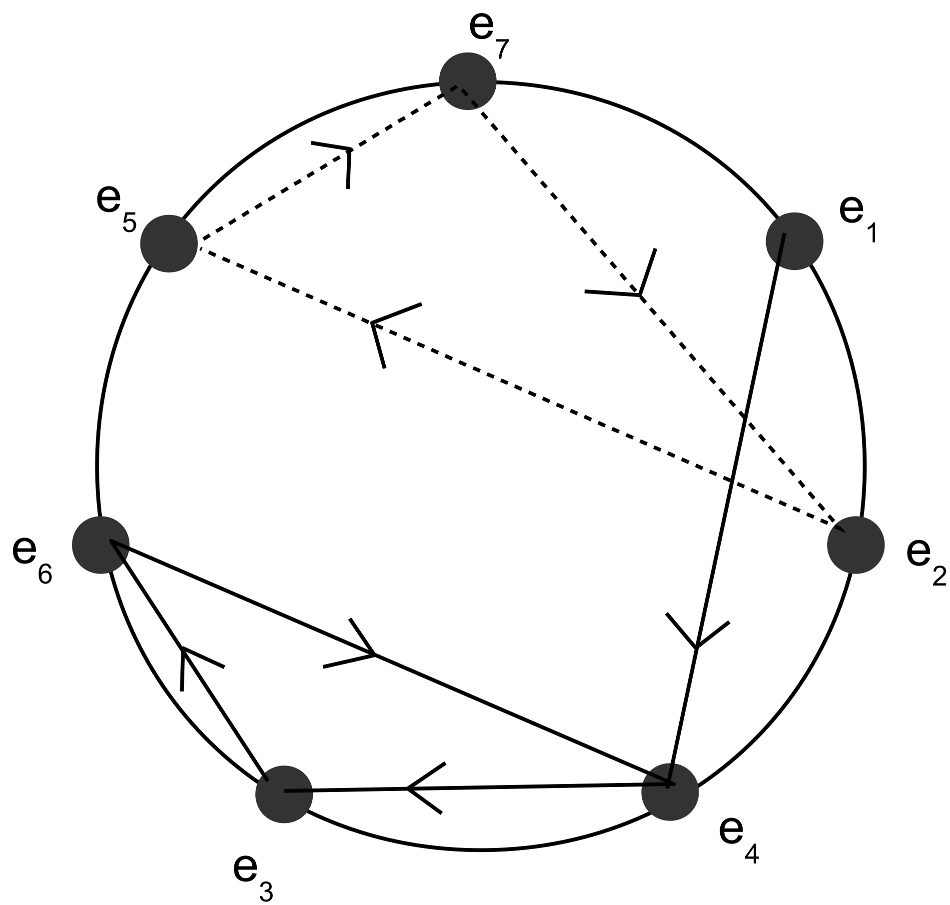
$$\frac{1}{2}[e_5, e_7] = e_2$$

$$\frac{1}{2}[e_7, e_1] = e_4$$



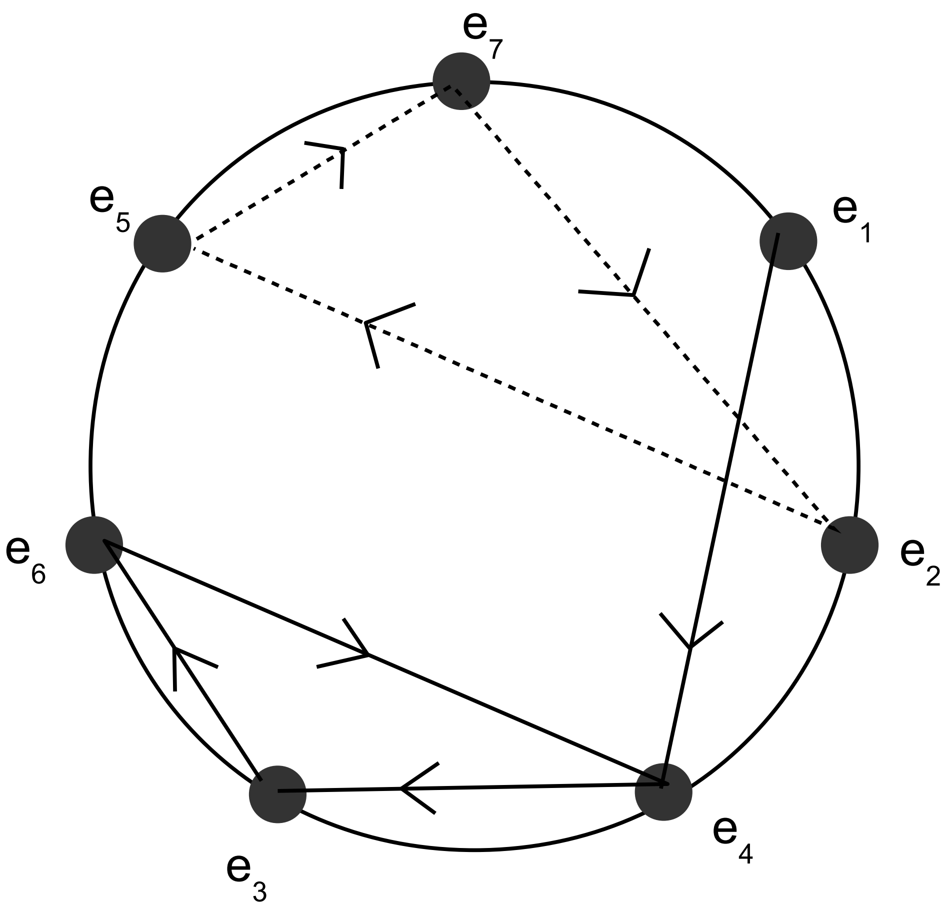
$$\frac{1}{2}[e_5, e_7] = e_2$$

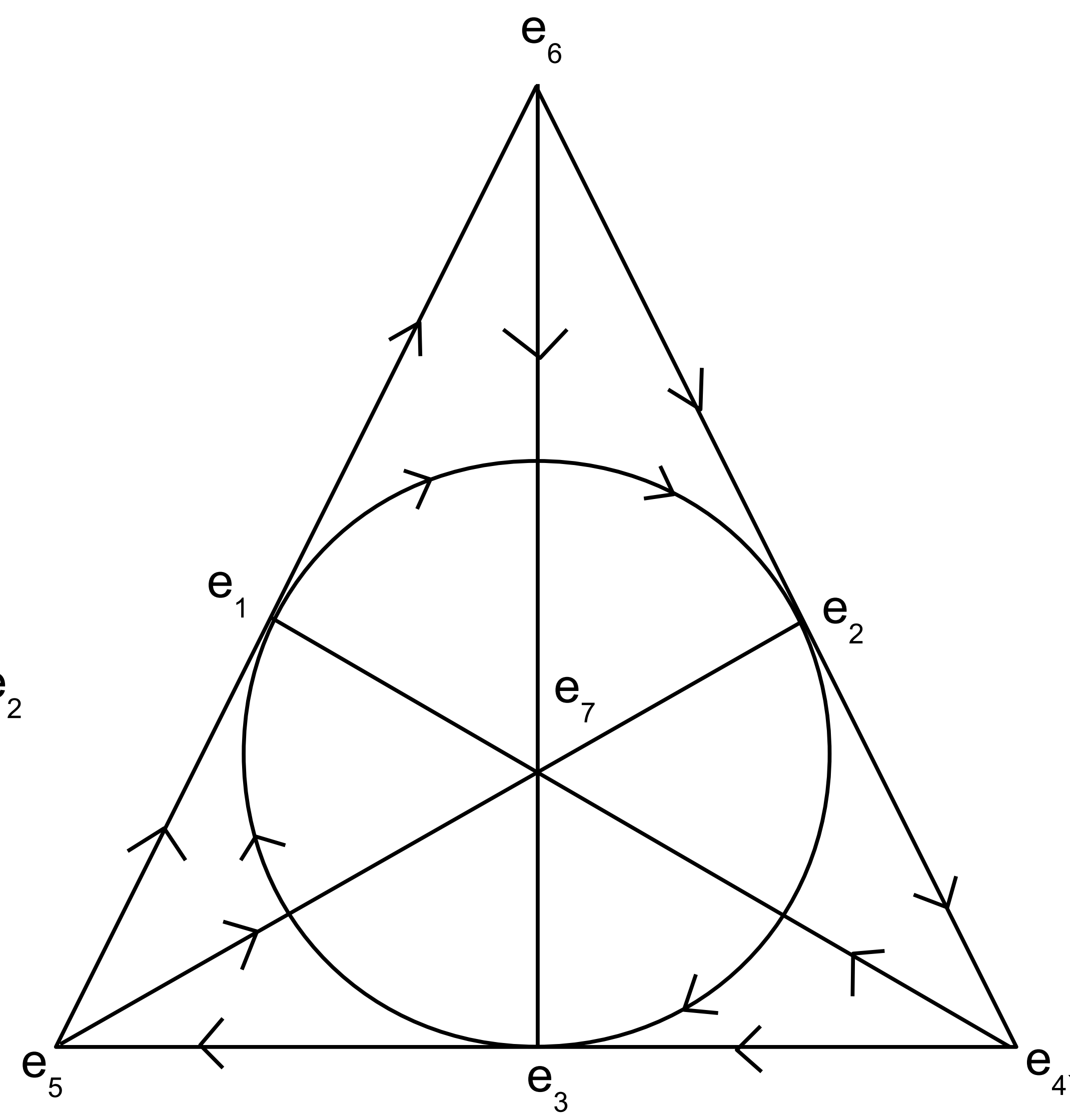
$$\frac{1}{2}[e_7, e_1] = e_4$$



$$\frac{1}{2}[e_1, e_4, e_3] = e_6$$

$$\frac{1}{2}[e_2, e_3, e_6] = e_5$$


e7
e5
e1
e6
e2
e3
e4


e6
e1
e2
e7
e5
e3
e4`

$$[x, y, z] = [y, z, x] = [z, x, y]$$

$$[x, y, z] = -[y, x, z] = -[x, z, y] = -[z, y, x]$$

Define a 4-index object $\psi_{\alpha\beta\mu\nu}$ related to the associator as

$$[e_\alpha, e_\beta, e_\mu] = 2\psi_{\alpha\beta\mu\nu} e_\nu$$

$\psi_{\alpha\beta\mu\nu} = 1$ for combinations $1346, 2635, 4567, 3751, 6172, 5214, 7423$

Duality property between $\epsilon_{\lambda\sigma\rho}$ and $\psi_{\alpha\beta\mu\nu}$ in R^7 is best seen in the following construction:

$$\left\{ \begin{array}{ccccccc} 5 & 7 & 1 & 2 & 4 & 3 & 6 \\ 7 & 1 & 2 & 4 & 3 & 6 & 5 \\ 2 & 4 & 3 & 6 & 5 & 7 & 1 \end{array} \right\} = \epsilon_{\lambda\sigma\rho}$$

$$\left\{ \begin{array}{ccccccc} 1 & 2 & 4 & 3 & 6 & 5 & 7 \\ 4 & 3 & 6 & 5 & 7 & 1 & 2 \\ 3 & 6 & 5 & 7 & 1 & 2 & 4 \\ 6 & 5 & 7 & 1 & 2 & 4 & 3 \end{array} \right\} = \psi_{\alpha\beta\mu\nu}$$

SPLIT OCTONION ALGEBRA

One can form a split Cayley algebra over the field of complex numbers with basis:

$$u_1 = \frac{1}{2}(e_1 + ie_4) \qquad u_1^* = \frac{1}{2}(e_1 - ie_4)$$

$$u_2 = \frac{1}{2}(e_2 + ie_5) \qquad u_2^* = \frac{1}{2}(e_2 - ie_5)$$

$$u_3 = \frac{1}{2}(e_3 + ie_6) \qquad u_3^* = \frac{1}{2}(e_3 - ie_6)$$

$$u_0 = \frac{1}{2}(1 + ie_7) \qquad u_0^* = \frac{1}{2}(1 - ie_7)$$

The automorphism group of the octonion algebra is the 14-parameter exceptional group G_2. The imaginary octonion units $e_\alpha, \alpha = 1, 2, \ldots, 7$ fall into its 7-dim representation.

Under the $SU(3)^c$ subgroup of the G_2 that leaves e_7 invariant, u_0 and u_0^* transform like singlets while u_j and u_j^* transform like a triplet and an antitriplet respectively. The multiplication table can now be written in a manifestly $SU(3)^c$ invariant manner:

$$u_0^2 = u_0 \qquad u_0 u_0^* = 0$$

$$u_0 u_j = u_j u_0^* = u_j \qquad u_0^* u_j = u_j u_0 = 0$$

$$u_i u_j = -u_j u_i = \epsilon_{ijk} u_k^*$$

$$u_i u_j^* = -\delta_{ij} u_0$$

To compactify our notation we write:

$$u_0 = \frac{1}{2}(1 + ie_7) \qquad u_0^* = \frac{1}{2}(1 - ie_7)$$

$$u_j = \frac{1}{2}(e_j + ie_{j+3}) \qquad u_j^* = \frac{1}{2}(e_j - ie_{j+3}) \qquad j = 1, 2, 3$$

$$MultiplicationTable:$$

	u_0	u_0^*	u_k	u_k^*
u_0	u_0	0	u_k	0
u_0^*	0	u_0^*	0	u_k^*
u_j	0	u_j	$\epsilon_{jki}u_i^*$	$-\delta_{jk}u_0$
u_j^*	u_j^*	0	$-\delta_{jk}u_o^*$	$\epsilon_{jki}u_i$

Note: u_i and u_j^* behave like fermionic creation and annihilation operators:

$$\{u_i, u_j\} = \{u_i^*, u_j^*\} = 0 \qquad \{u_i, u_j^*\} = -\delta_{ij}$$

Showing the three split units to be Grassmann numbers.
Being non-associative they give rise to an exceptional Grassmann algebra.

DYNAMICAL SUPERSYMMETRY

Under the color group $SU(3)^c$

$$q\bar{q}: \quad \mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$$

$$qq: \quad \mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}} \oplus \mathbf{6}$$

Under the spin-flavor group $SU_{sf}(6)$

$$q\bar{q}: \quad \mathbf{6} \otimes \bar{\mathbf{6}} = \mathbf{1} \oplus \mathbf{35}$$

$$qq: \quad \mathbf{6} \otimes \mathbf{6} = \mathbf{15} \oplus \mathbf{21}$$

DYNAMICAL SUPERSYMMETRY

Under the color group $SU(3)^c$

$q\bar{q}$: $\quad \mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8} \qquad u_j u_k^* = -\delta_{jk} u_0$

qq : $\quad \mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}} \oplus \mathbf{6} \qquad u_j u_k = \epsilon_{jki} u_i^*$

Under the spin-flavor group $SU_{sf}(6)$

$q\bar{q}$: $\quad \mathbf{6} \otimes \bar{\mathbf{6}} = \mathbf{1} \oplus \mathbf{35}$

qq : $\quad \mathbf{6} \otimes \mathbf{6} = \mathbf{15} \oplus \mathbf{21}$

If one re-writes qqq baryon as qD, where D is a diquark, the quantum numbers of D are:

for color, $\bar{3}$, since when combined with q must give a color singlet;
for spin-flavor, 21, since when combined with color must give antisymmetric wavefunctions.

But the quantum numbers of $\bar{q}$ are:

for color, $\bar{3}$,
and for spin-flavor, $\bar{6}$.

Thus $\bar{q}$ and D have the same color quantum numbers (color forces can not distinguish between $\bar{q}$ and D).

- Dimensions of Internal Degrees of Freedom of Quarks & Diquarks

	$\mathbf{SU_f(3)}$	$\mathbf{SU_s(2)}$	**dim**
q	$\square$	s=1/2	$3 \times 2 = 6$
D	$\square\square$	$s = 1$	$6 \times 3 = 18$
	$\begin{array}{c}\square\\\square\end{array}$	$s = 0$	$3 \times 1 = 3$

Thus there is an approximate dynamic supersymmetry in hadrons with supersymmetric partners

$$\psi = \begin{pmatrix} \bar{q} \\ D \end{pmatrix} \qquad \bar{\psi} = \begin{pmatrix} q \\ \bar{D} \end{pmatrix}$$

All hadrons can be obtained by combining ψ and $\bar{\psi}$: mesons are $q\bar{q}$, baryons are qD, antibaryons are $\bar{q}\bar{D}$, and exotic mesons are $D\bar{D}$.

Corresponding supersymmetry is $SU(6/21)$.

The confining energy associated with the Bohr radius for the bound state is obtained from the linear confining potential $S(r) = br$, so that the effective masses of the constituents become:

$$M_1 = m_1 + \frac{1}{2}S_0 \qquad M_2 = m_2 + \frac{1}{2}S_0 \qquad (S_0 = br_0)$$

For a meson m_1 and m_2 are the current quark masses while M_1 and M_2 can be interpreted as the constituent quark masses. Note that even in the case of vanishing quark masses associated with a perfect chiral symmetry, confinement results in non-zero constituent masses that spontaneously break the $SU(2) \times SU(2)$ symmetry of u, d quarks.

Simplified spin free Hamiltonian involving only the scalar potential:

$$H^2 = 4[(m + \frac{1}{2}br)^2 + P_r^2 + \frac{\ell(\ell+1)}{r^2}]$$

$$P_r^2 = -\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r}$$

$Potential\ model\ gives:$

$$\frac{9}{8}(m_\rho^2 - m_\pi^2) = m_\Delta^2 - m_N^2$$

$with\ an\ accuracy\ of\ 1\%\ of\ the\ experiment.$

$$\frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \left[\frac{E^2}{4} - \frac{1}{4}b^2(r + \frac{2m}{b})^2 - \frac{\ell(\ell+1)}{r^2}\right]\psi = 0$$

$$\psi = N r^\ell e^{-\frac{1}{4}br^2} F(\alpha = -n_r + 1, \gamma = \ell + \frac{2}{3}, x = \frac{1}{2}br^2) Y_\ell^m(\theta, \phi)$$

$$E_{min}^2 = 4\mu^2(\ell + \frac{1}{2})$$

$$
\begin{aligned}
N = \Bigg[& \frac{|\alpha|!(2\gamma-1)!(\gamma+|\alpha|-1)!\sqrt{\pi}}{2^{\gamma}b^{\gamma}(\gamma-\frac{1}{2})!(\gamma-1)!} \\
& -m\left\{ \sum_{p=0}^{|\alpha|} \frac{2^{\gamma+\frac{1}{2}}(\gamma+p-\frac{1}{2})!}{b^{\gamma+\frac{1}{2}}p!} \left(\frac{|\alpha|!(\frac{1}{2})!}{(|\alpha|-p)!(p-|\alpha|+\frac{1}{2})!} \right)^2 \right. \\
& + \sum_{n=0}^{n_r-1} \frac{(n_r-1)!(\ell+n_r-\frac{1}{2})!(2n)!(\ell+n)!(\ell+2n+1)}{2^{2n}(n!)^2(n_r-n-1)!(\ell+n+\frac{1}{2})!} \\
& \left. \times \sum_{k=n}^{n_\xi-1} \frac{(-1)^{k+|\alpha|+1}2^{n+\gamma+\frac{1}{2}}(-\frac{1}{2})!(n_\xi-n-1)!(n+k+\gamma-\frac{1}{2})!(n+k+\frac{1}{2})!}{b^{n+\gamma+\frac{1}{2}}(k+\frac{1}{2})!(n_\xi-k-1)!(\ell+k+1)!(n+k-|\alpha|+\frac{1}{2})!} \right\} \Bigg]^{-\frac{1}{2}}
\end{aligned}
$$

$$E_r^2 = 4b(\ell + 2n_r - \frac{1}{2})$$

$$E_\xi^2 = 4b(\ell + 2n_\xi + \frac{1}{2})$$

$$n_\xi \geq n_r$$

The observed hadronic $SU(6/21)$ supersymmetry is not exact. The breaking of this supersymmetry has two origins. First one is the q and D mass differences as well as mass differences among quarks. This leads to different intercepts for parallel Regge trajectories. The second breaking comes from the contribution to the potential from one gluon exchange.

This potential is a $4-$vector and is spin dependent. Since the quark and antiquark have spin $S=1/2$ and the diquark has $S=0$ or $S=1$, the spin dependent part of the $q-D$ potential is different from that of $q-\bar{q}$, causing supersymmetry breaking.

Another effect is the deviation of the Regge trajectories from linearity for low angular momenta, since the potential is no longer proportional to the distance, and quark masses can no longer be neglected.

To see the symmetry breaking effect, we consider a hadron that approximates a two-body system. The quark, antiquark, and diquark will acquire their effective masses under the influence of the effective potentials. There are also spin-dependent interactions among the quarks. Based on a semi-relativistic Hamiltonian for a quark-diquark system interacting with the same potentials $\tilde{S}(r)$ and the $\tilde{V}(r)$ *the* mass of a hadron will take the approximate form

$$m_{12} = m_1 + m_2 + K \frac{\boldsymbol{S}^{(1)} \cdot \boldsymbol{S}^{(2)}}{m_1 m_2},$$

where m_i and $\boldsymbol{S}^{(i)}(i=1,2)$ are respectively the constituent mass and the spin of a quark or a diquark. The spin-dependent Breit term will split the masses of hadrons of different spin values.

If we assume

$$m_q = m_{\bar{q}} = m$$

where m is the constituent mass of u or d quarks, and denote the mass of diquark as m_D, then this approximation gives

$$m_\pi = (m_{q\bar{q}})_{S=0} = 2m - K\frac{3}{4m^2},$$

$$m_\rho = (m_{q\bar{q}})_{S=1} = 2m + K\frac{1}{4m^2},$$

$$m_\Delta = (m_{qD})_{S=3/2} = m + m_D + K\frac{1}{2mm_D},$$

$$m_N = (m_{qD})_{S=1/2} = m + m_D - K\frac{1}{mm_D}.$$

Eliminating m, m_D and K, we obtain a mass relation

$$\frac{8}{3}\cdot\frac{2m_\Delta + m_N}{3m_\rho + m_\pi} = 1 + \frac{3}{2}\cdot\frac{m_\rho - m_\pi}{m_\Delta - m_N},$$

which agrees with experiment to 13%.

The supersymmetry based on the supergroup $U(6/21)$ acts on a quark and antidiquark situated at the same point x_1. At the point x_2 we can consider the action of a supergroup with the same parameters, or one with different parameters. In the first case we have a global symmetry. In the second case, if we only deal with bilocal fields, the symmetry will be represented by $U(6/21)\times U(6/21)$, doubling the Miyazawa supergroup.

On the other hand, if any number of points are considered, with different parameters attached to each point, we are led to introduce a local supersymmetry $U(6/21)$ to which we should add the local color group $SU(3)^c$. Since it is not a fundamental symmetry, we shall not deal with the local Miyazawa group here. However, the double Miyazawa supergroup is useful for bilocal fields since the decomposition of the adjoint representation of the $728-$dimensional Miyazawa group with respect to $SU(6)\times SU(21)$ gives

$$728=(35,1)+(1,440)+(6,21)+(\bar{6},\bar{21})+(1,1)$$

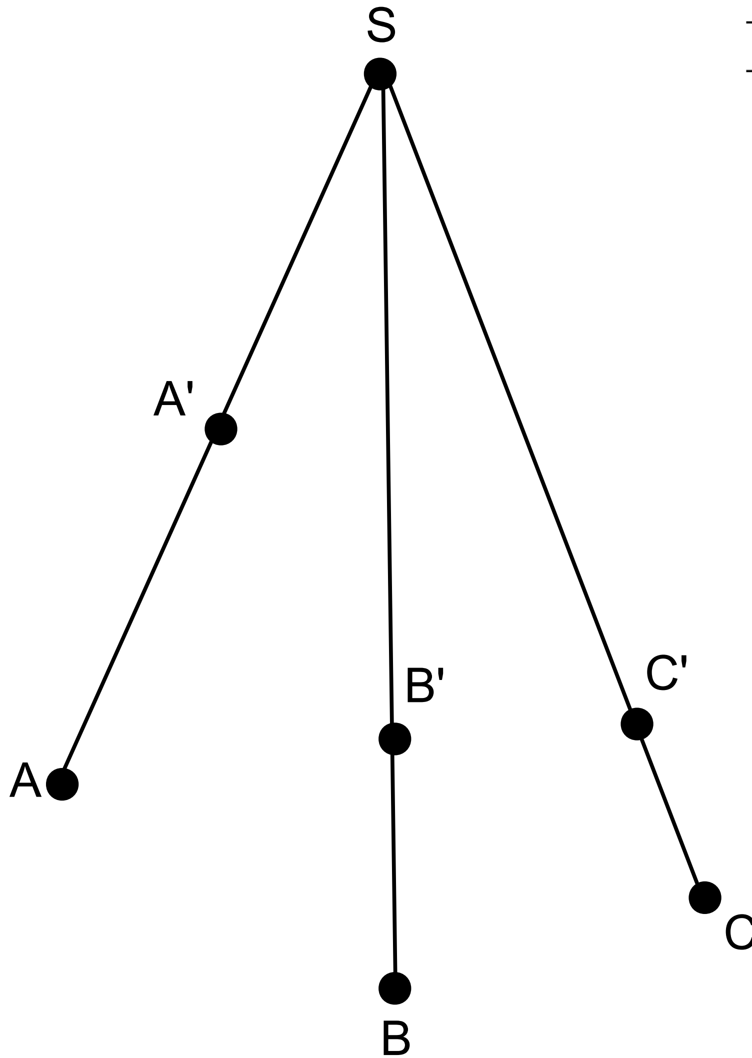
A further decomposition of the double Miyazawa supergroup into its field with respect to its c.o.m. coordinates leads to the decomposition of the $126-$dimensional cosets $(6,21)$ and $(21,6)$ into $56^{+}+70^{-}$ of the diagonal $SU(6)$.

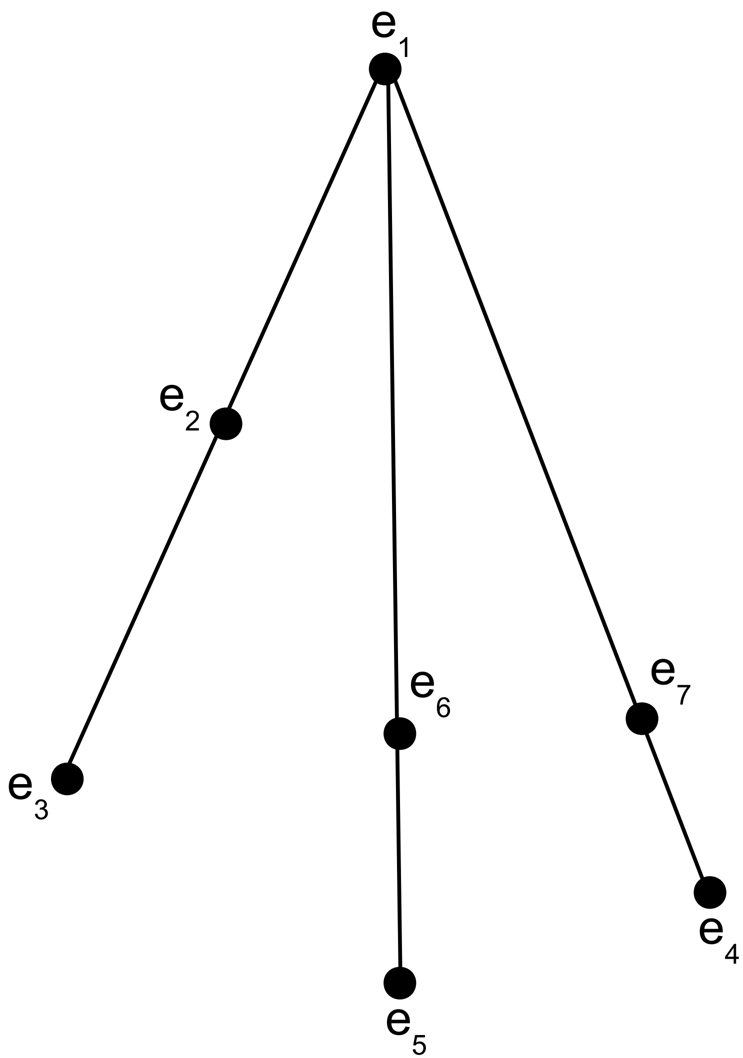
We would have a much tighter and more elegant scheme if we could perform such a decomposition from the start and be able to identify the $(1,21)$ part of the fundamental representation of $U(6/21)$ with the $21-$dimensional representation of the $SU(6)$ subgroup, which means going beyond the Miyazawa supersymmetry to asmaller supergroup having $SU(6)$ as a subgroup.

$RESULTS$

* Parallelism of Regge Trajectories
* Mass formulas $\pi - \rho$, $N - \Delta$ trajectories
* Existence of exotic mesons as $D\bar{D}$ states: $a_0(980)$, $f_0(980)$
* Multiquark states by $q \to \bar{D}$, $\bar{q} \to D$ transform

Desargues' Theorem





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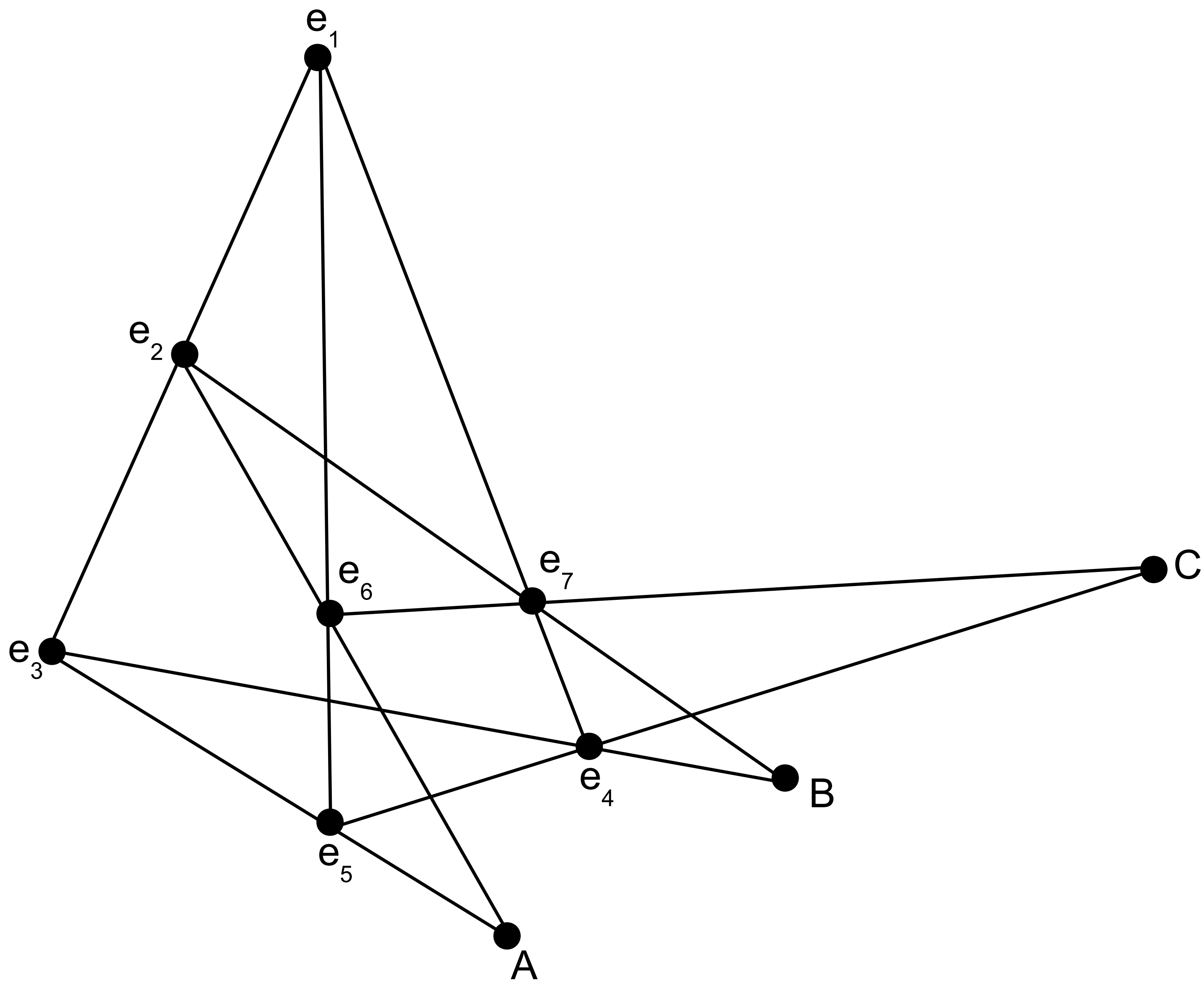
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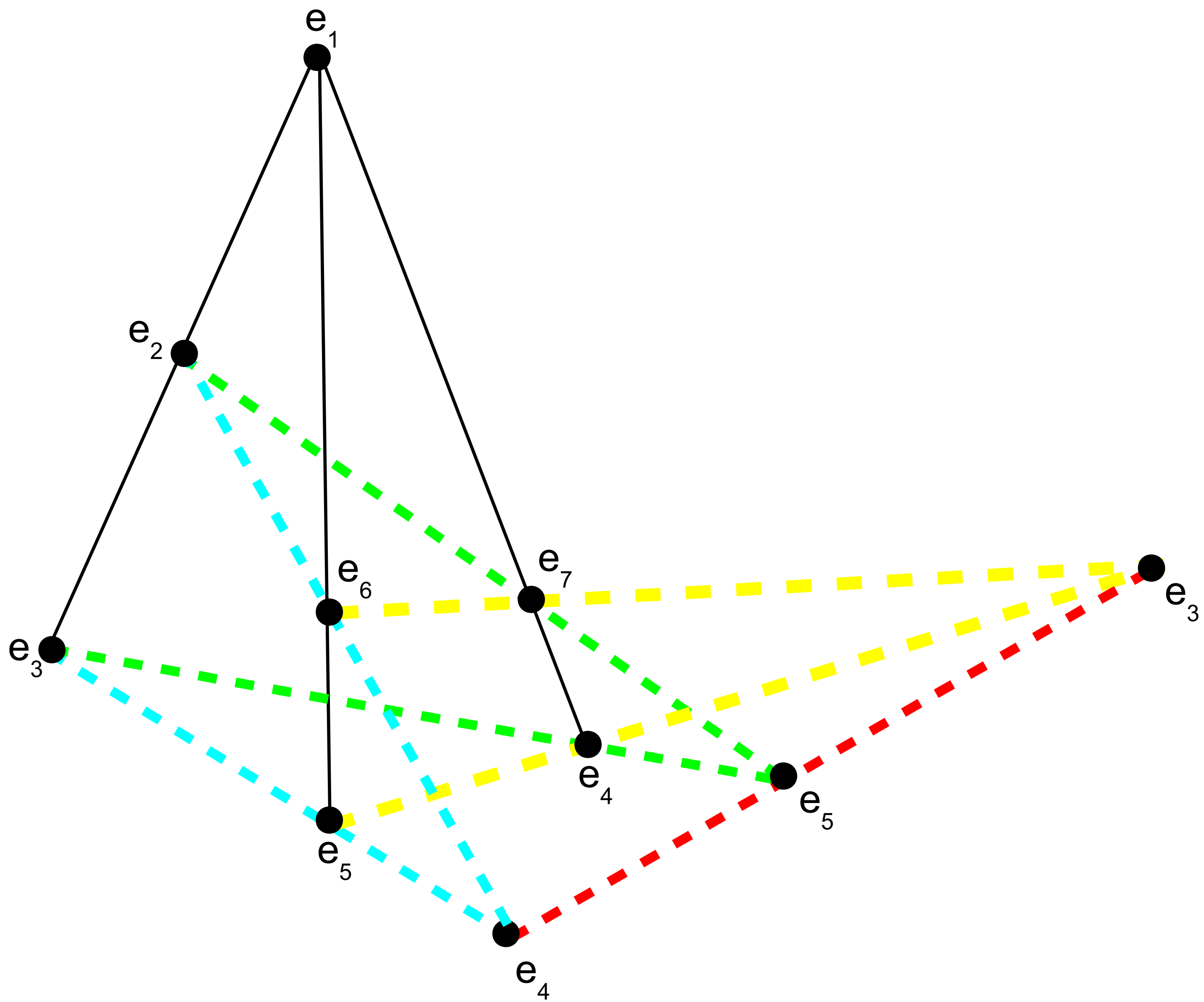

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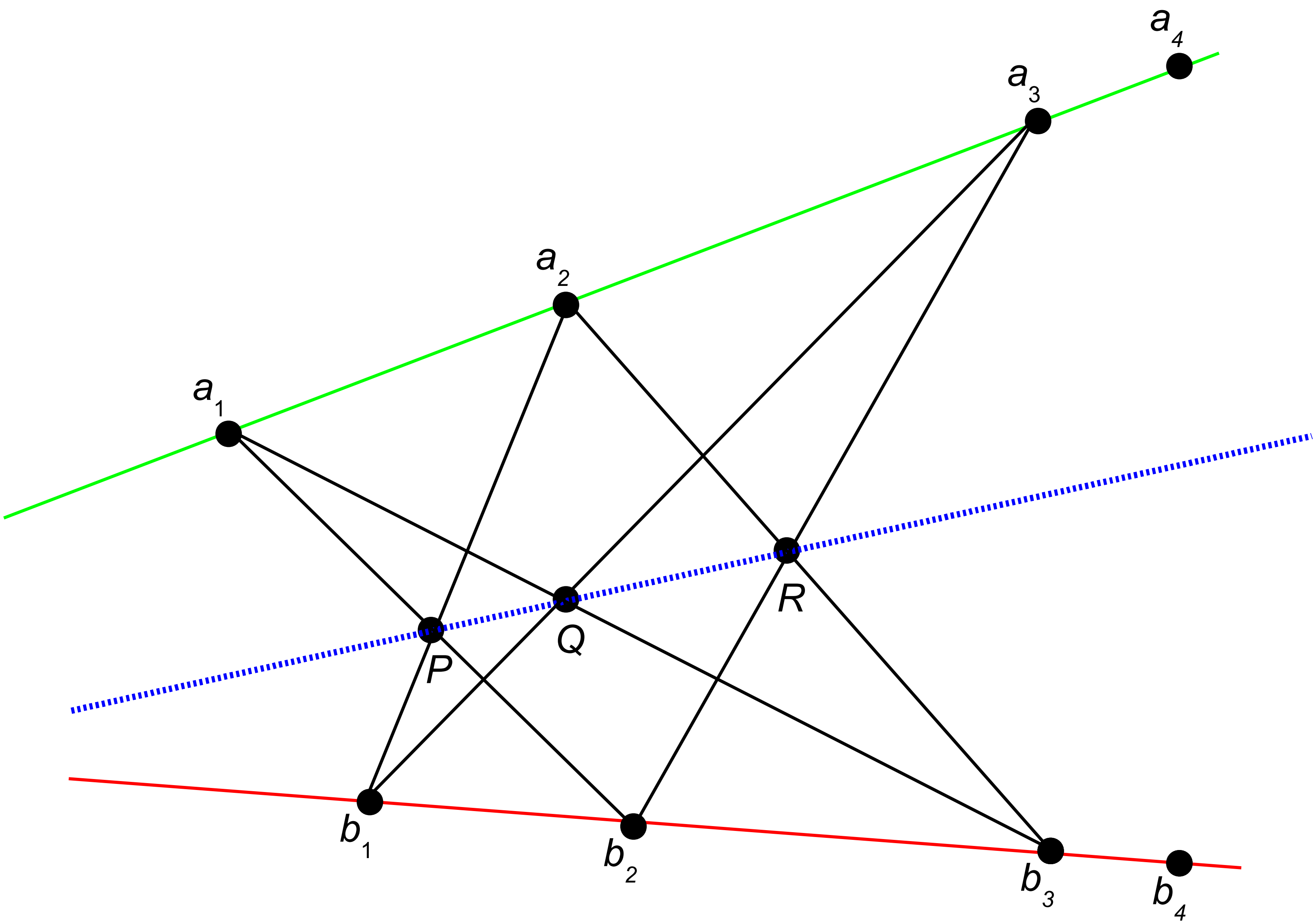
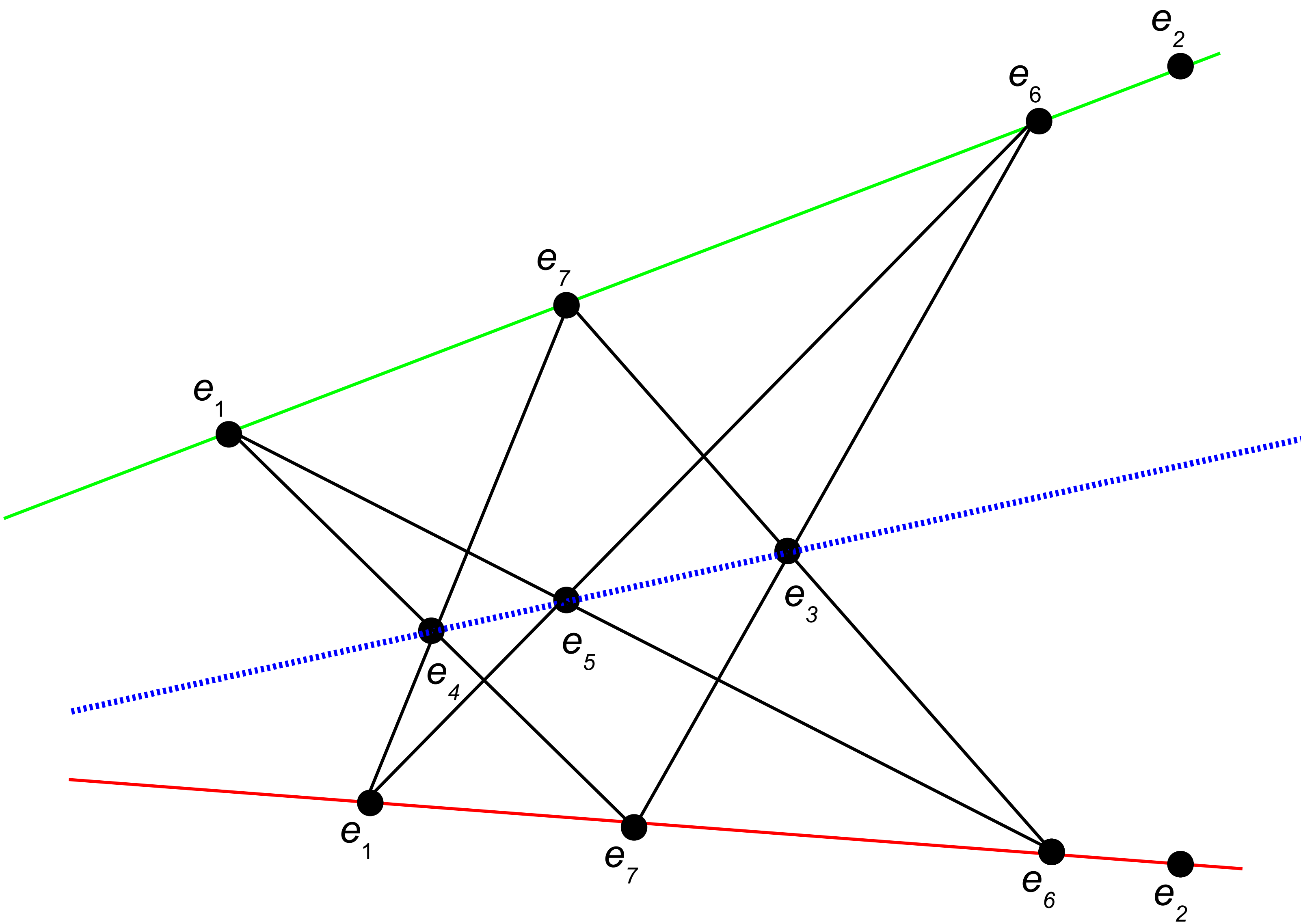
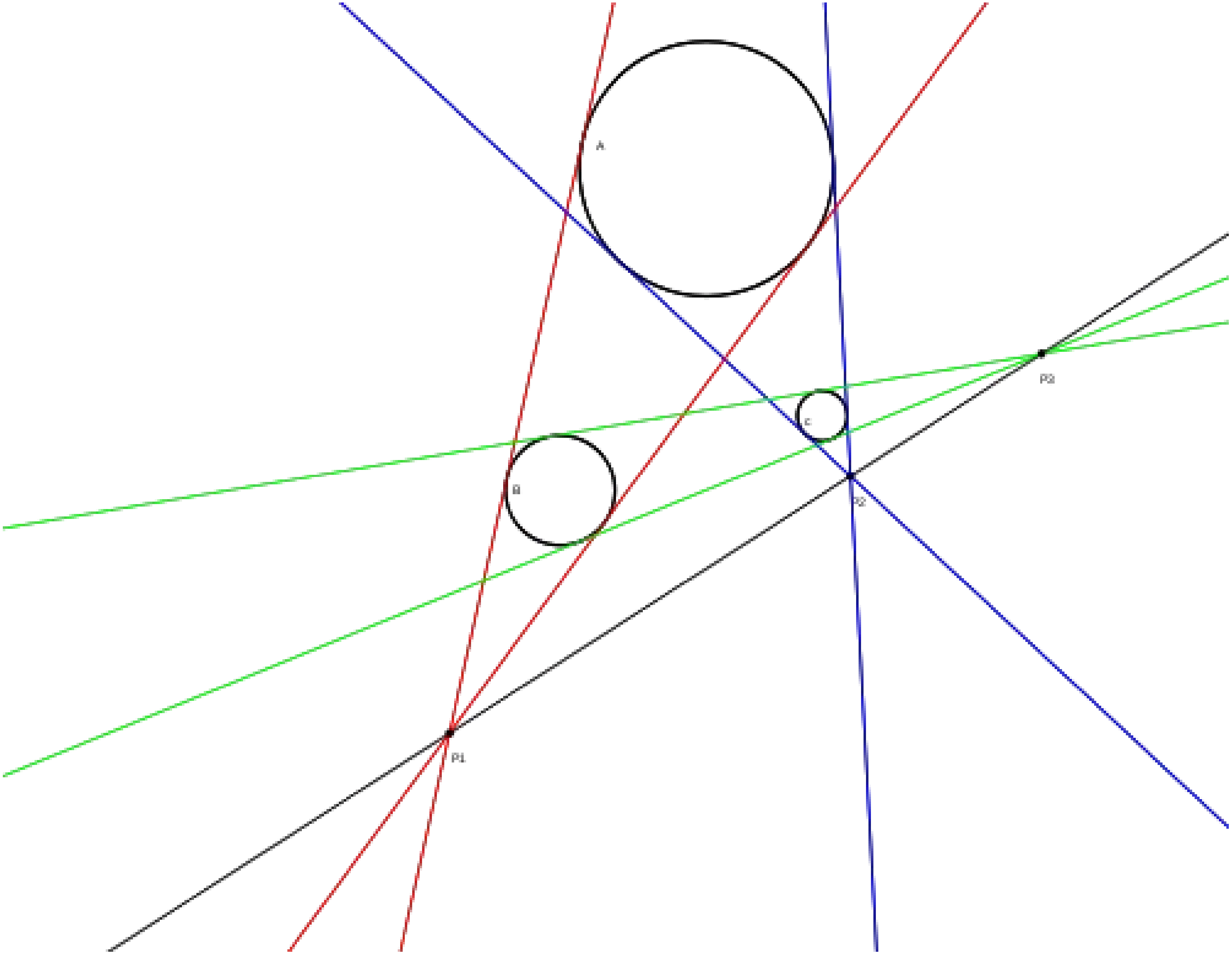
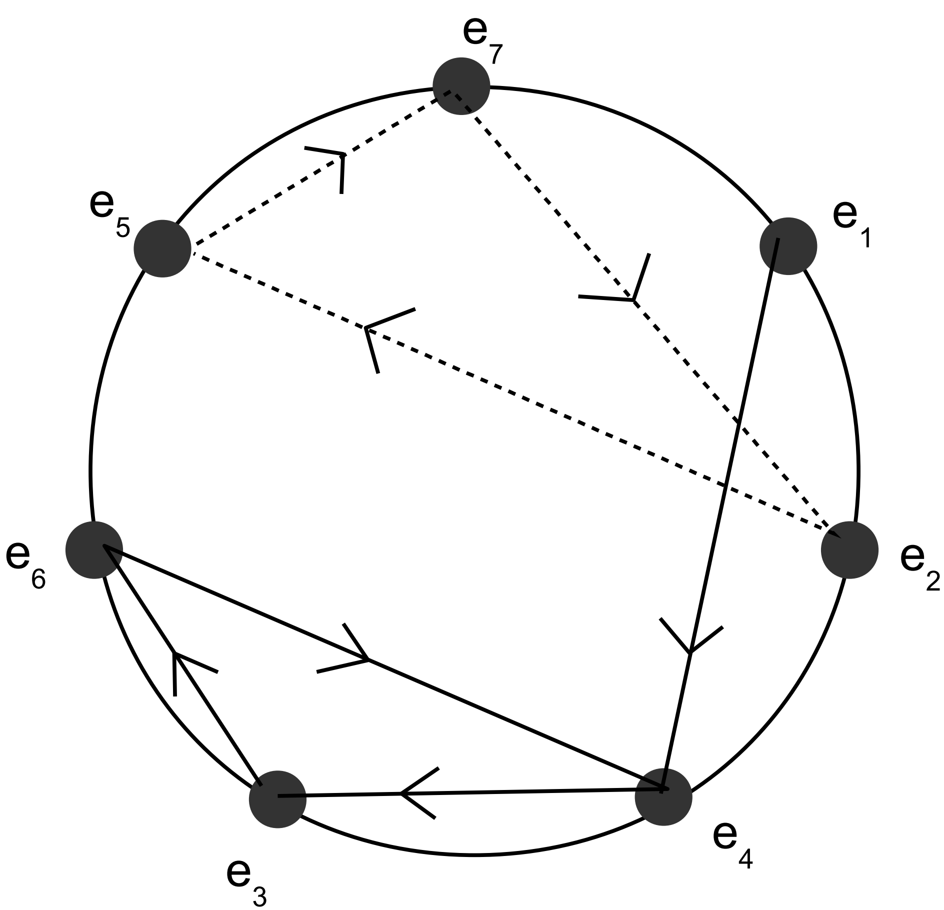
e4

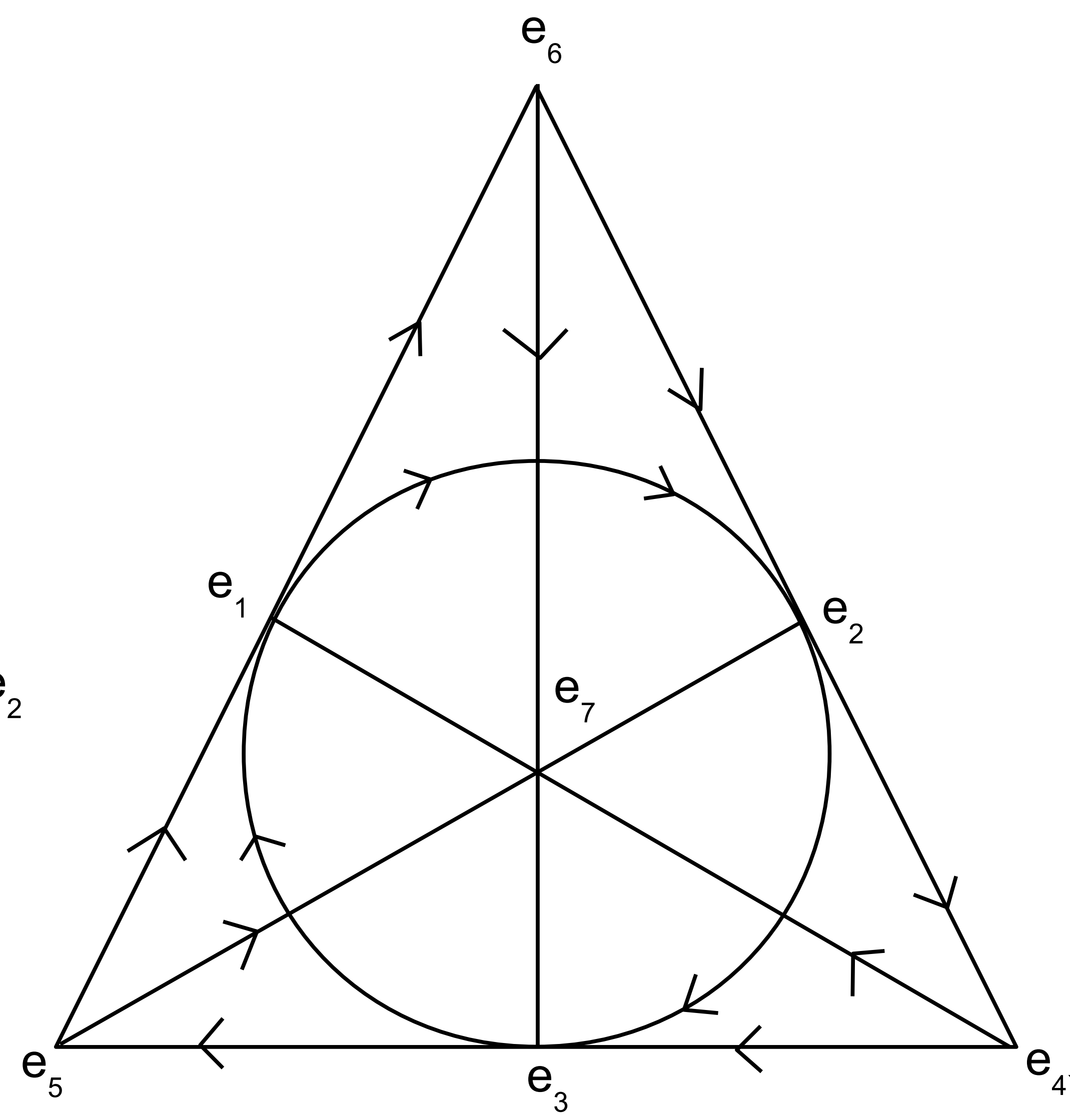
Pappus' Theorem




e_2
e_6
e_7
e_1
e_3
e_5
e_4
e_1
e_7
e_6
e_2


A
B
C
P1
P2
P3


e7
e5
e1
e6
e2
e3
e4


e6
e1
e2
e7
e5
e3
e4`

$\mathbf{Exceptional\,Groups : G_2, F_4, E_6, E_7, E_8}$

$\mathbf{Construction\,of\,the\,root\,lattices\,of\,E_8, E_9 = \hat{E}_8, or\,E_{10}}$

-Conway-Slone lattice associated with discrete Jordan algebras over octonions

-Association between superstring symmetries and lattices generated by discrete Jordan algebras

-Suggest all known superstring theories are related and originated from a more general theory related to Conway-Slone transhyperbolic group