OPE for Wilson loops/amplitudes in N=4 SYM

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Seminar on th. 1006.2788, 1303.1396, 1306.2058, 1402.3307, 1407.1736 B. Basso, A. Sever, P. Vieira, H. Maldacena, et. al.

Introduction.

- N=4 SYM is integrable model.
- Anomalous dimensions of local gauge invariant operators – Y-system .
- What about amplitudes (planar case) ?
- Much progress on the level of tree amplitudes and loop integrands,

BDS ansatz for 4 and 5 point amplitudes and strong coupling leading behavior via AdS/CFT.

- Amplitudes = (super) Wilson loops.
- Can we use integrability to actually compute finite parts of

amplitudes at finite values of coupling constant ?

Amplitudes in N=4 SYM

$$A^{(L)}_{MHV} = A^{\text{tree}}_{MHV}\,\mathcal{M}^{(L)}(s_{i,i+1}, s_{i\ldots i+2}, \ldots)$$

MHV – total helicity = n-4, NMHV - total helicity = n-6, etc. total helicity = n-4 Here we will consider MHV case only, generalizations to NMHV, etc cases are possible.

$$\mathcal{M}_n = \exp\left[-\frac{1}{8}\sum_{l=1}^{\infty} a^l \left(\frac{\gamma_K^{(l)}}{(l\epsilon)^2} + \frac{2\mathcal{G}_0^{(l)}}{l\epsilon}\right)\sum_{i=1}^{n}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{l\epsilon}\right] \times h_n$$

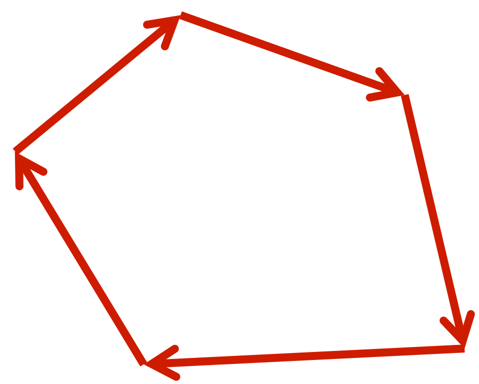
Exact answers for n=4,5 – BDS ansatz:

$$\mathcal{M}_n = \exp\left[\sum_{l=1}^{\infty} a^l f^{(l)}(\epsilon)\mathcal{M}_n^{(1)}(l\epsilon) + C^{(l)} + \mathcal{O}(\epsilon)\right]$$

Amplitudes and Wilson loops in N=4 SYM

$$\langle W_{C_n} \rangle = \frac{1}{N} \langle 0 | \mathrm{Tr} P \exp \left(ig \int_{C_n} d\tau A_\mu(x(\tau)) \dot{x}^\mu(\tau) \right) | 0 \rangle$$

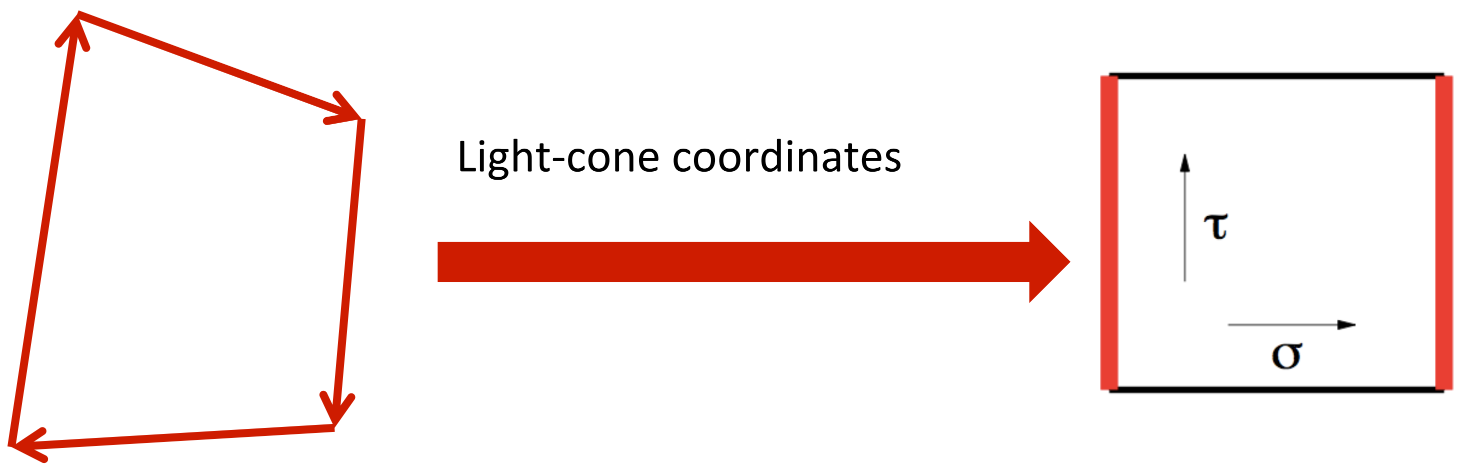
Dual conformal symmetry – Amplitudes/ (super)Wilson loops duality. Here MHV example is represented.



$$\mathcal{M}_n = \exp \left[-\frac{1}{8} \sum_{l=1}^{\infty} a^l \left(\frac{\gamma_K^{(l)}}{(l\epsilon)^2} + \frac{2\mathcal{G}_0^{(l)}}{l\epsilon} \right) \sum_{i=1}^{n} \left(\frac{\mu^2}{-s_{i,i+1}} \right)^{l\epsilon} \right] \times h_n$$

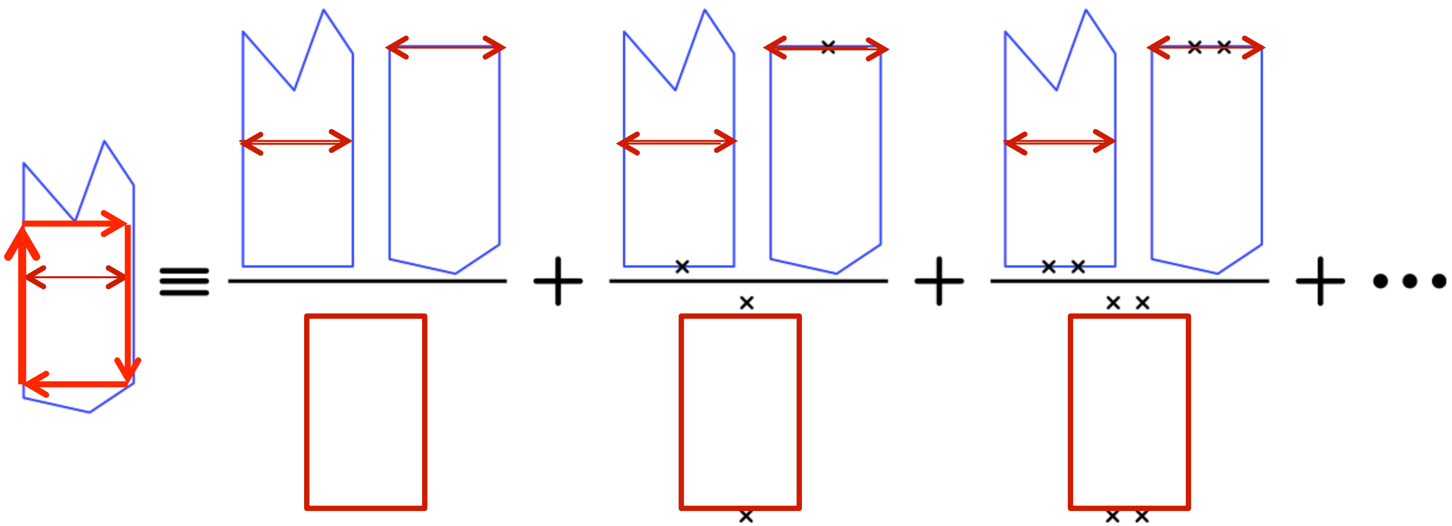
OPE for Wilson loops. Symmetries of the square.

Single Line	$D - M^{+-}$	M^{12}	$D + M^{+-}$	K^-	P_-	M^{+i}	K^+	K^i
Two Lines	$D - M^{+-}$	M^{12}	$D + M^{+-}$	K^-	P_-			
Square	$D - M^{+-}$	M^{12}	$D + M^{+-}$					
Name	$P, \quad \partial_\sigma$	$\tilde{S}, \quad \partial_\phi$	$E, \quad \partial_\tau$					



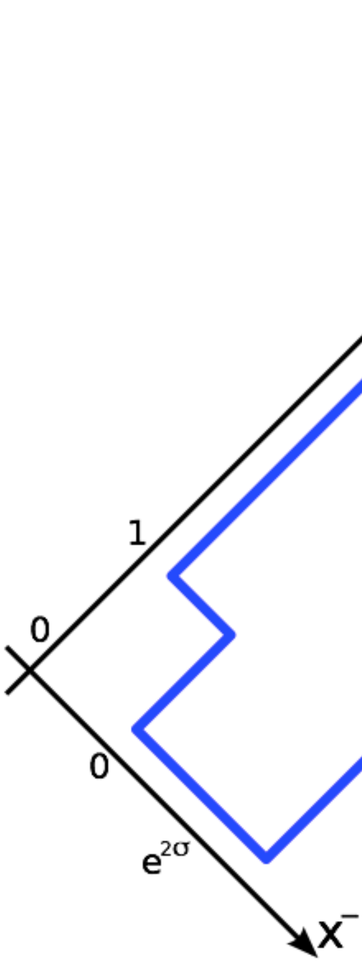
OPE for Wilson loops.

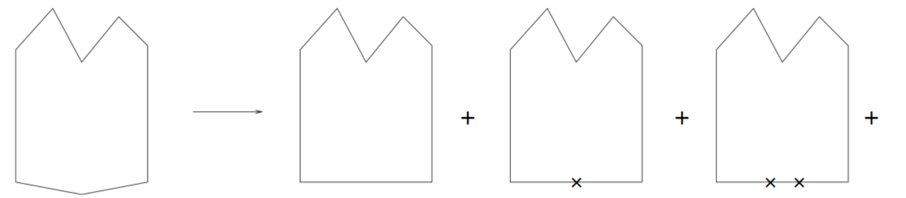
$$\langle W \rangle = \int dn e^{-\tau E_n + i p_n \sigma + i m_n \phi} C_n \qquad C_n = C_n^{\text{top}} C_n^{\text{bottom}}$$



$$\langle W \rangle(\tau, \sigma, \phi) \equiv \langle \text{top} | e^{-\tau E + i\sigma P + i\phi J} | \text{bottom} \rangle$$

2D example.





Large tau = collinear limit

$$u_i = u_{i+3} \equiv \frac{x^2_{i-1,i+1}x^2_{i-2,i+2}}{x^2_{i-1,i+2}x^2_{i+1,i-2}}$$

$$\frac{1}{u_2} = 1 + e^{2\tau}$$

$$\frac{1}{u_3} = 1 + (e^{-\tau} + e^{\sigma+i\phi})(e^{-\tau} + e^{\sigma-i\phi})$$

$$\frac{u_1}{u_2 u_3} = e^{2\sigma+2\tau}$$

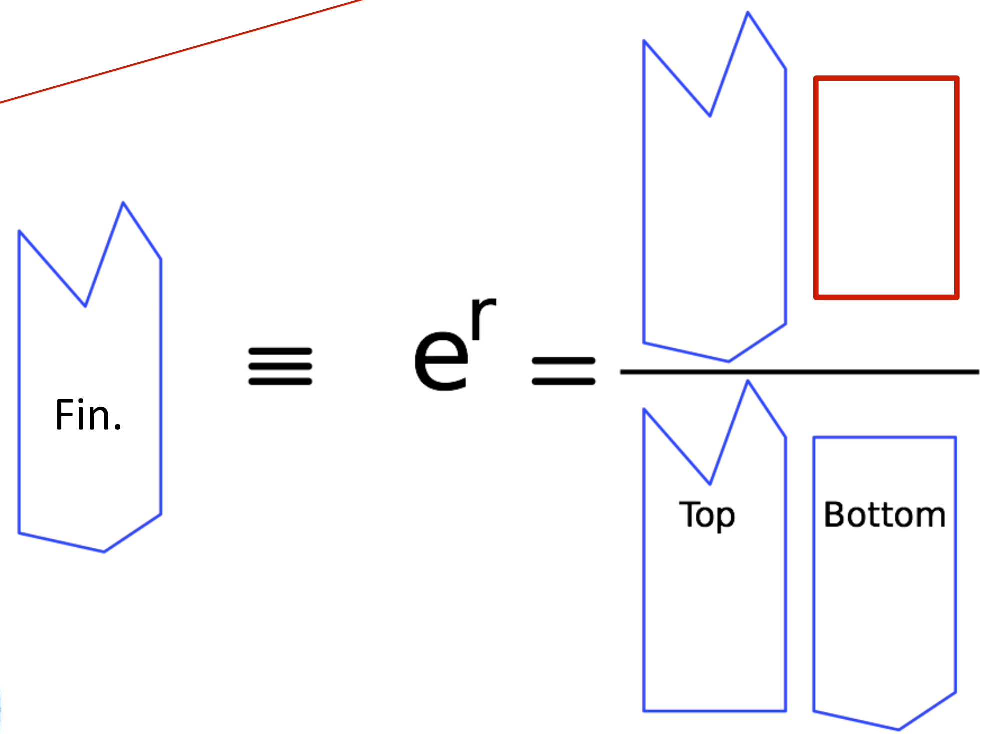
Typical contributing operator 2D case.

$$D^n_- F_{+-}$$

OPE for Wilson loops. Finite reminder.

$$\mathcal{M}_n = \exp\left[-\frac{1}{8}\sum_{l=1}^{\infty} a^l \left(\frac{\gamma_K^{(l)}}{(l\epsilon)^2} + \frac{2\mathcal{G}_0^{(l)}}{l\epsilon}\right)\sum_{i=1}^{n}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{l\epsilon}\right] \times h_n$$

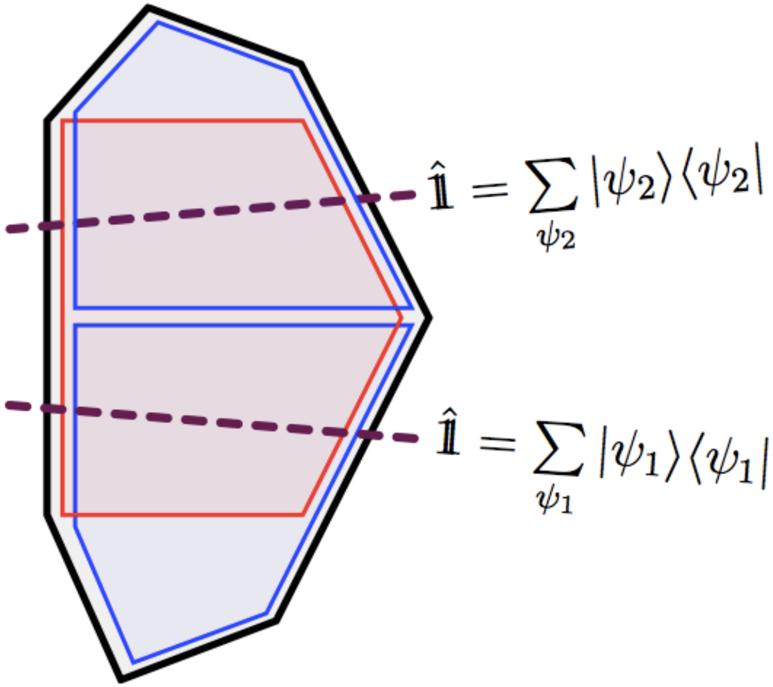
h=BDS X FiniteReminder



$$r = \log\left(\frac{\langle W\rangle\langle W^{\text{square}}\rangle}{\langle W^{\text{top}}\rangle\langle W^{\text{bottom}}\rangle}\right)$$

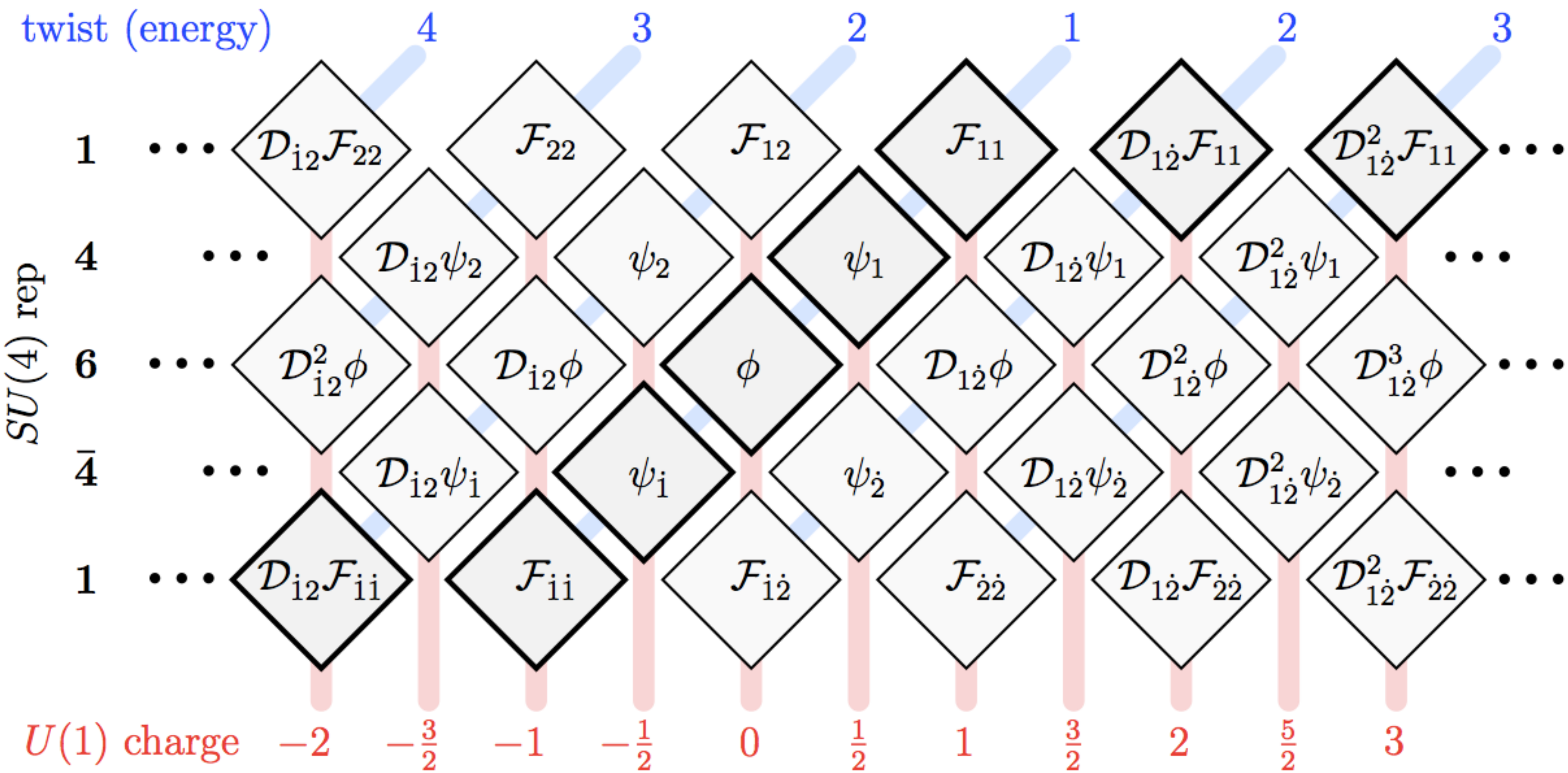
OPE for Wilson loops. N=4 SYM

$$\mathcal{W} = \sum_{\psi_i} \left[\prod_{i=1}^{n-5} e^{-E_i\tau_i + ip_i\sigma_i + im_i\phi_i} \right] P(0|\psi_1)\ P(\psi_1|\psi_2)\ \ldots\ P(\psi_{n-6}|\psi_{n-5})\ P(\psi_{n-5}|0)$$



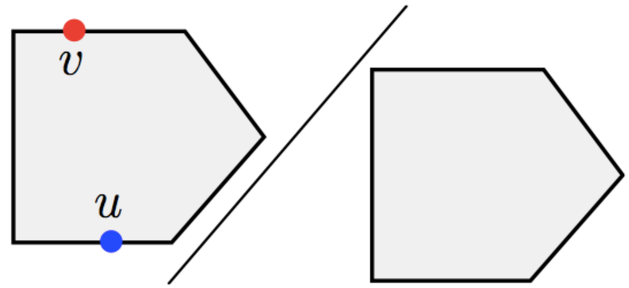
OPE for Wilson loops. N=4 SYM

$$\mathcal{W} = \sum_{\psi_i} \left[\prod_{i=1}^{n-5} e^{-E_i \tau_i + i p_i \sigma_i + i m_i \phi_i} \right] P(0|\psi_1)\ P(\psi_1|\psi_2)\ \ldots\ P(\psi_{n-6}|\psi_{n-5})\ P(\psi_{n-5}|0)$$

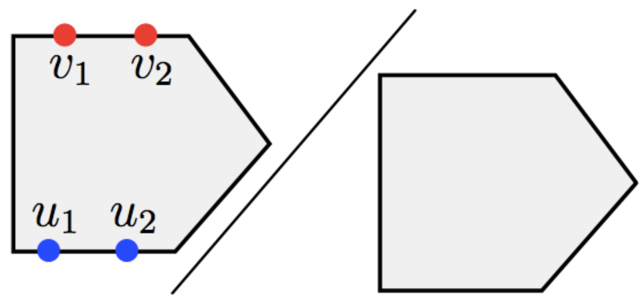


OPE for Wilson loops. N=4 SYM

$$P(u|v) =$$



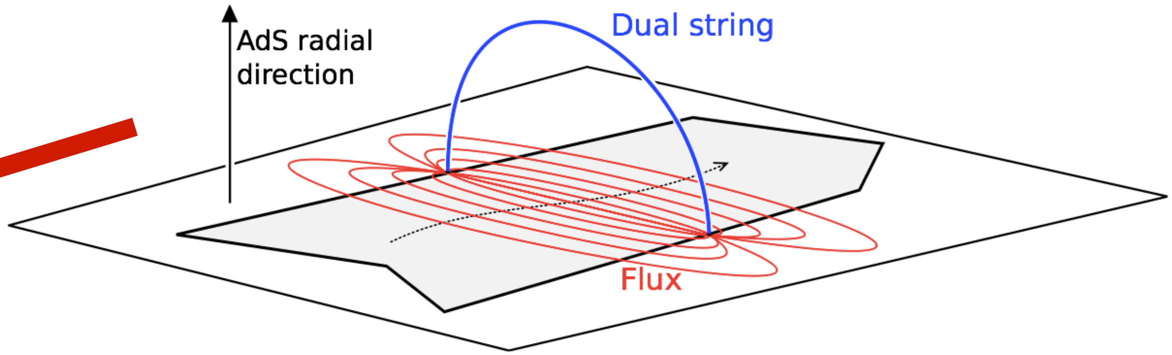
$$P(u_1, u_2|v_1, v_2) =$$

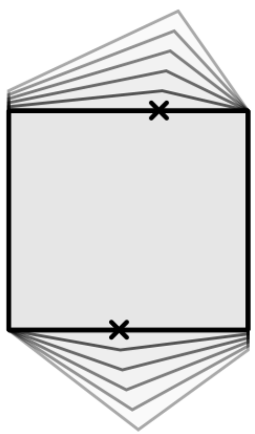


P – Pentagon transitions. Independent on kinematics, depends on operators anomalous dimension (string excitation energy)

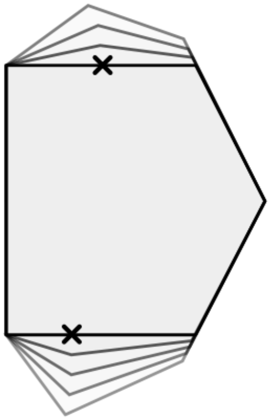
OPE for Wilson loops. N=4 SYM

How to actually compute Pentagon transitions ? P.T. analysis + AdS/CFT intuition – Pentagon transitions are related to 2D GPK string world sheet dynamics (form factors ?)





$\longrightarrow \mu(u)\,,$



$\longrightarrow P(u|v)$

OPE for Wilson loops. N=4 SYM

$$P(u|v) \equiv P_{FF}(u|v)$$

$$P(u|v)^2 = \left[\frac{f(u,v)}{g^2(u-v)(u-v-i)}\right]^\eta \frac{S(u,v)}{S(u^\gamma,v)}$$

One can conjecture the following ansatz for pentagon transitions (in all orders of g !!)

Here S are 2D S-matrixes of GPK string excitations (rather complicated functions)

$$f(u,v) = x^+x^-y^+y^- \left(1 - g^2/x^+y^-\right)\left(1 - g^2/x^-y^+\right) \left(1 - g^2/x^+y^+\right)\left(1 - g^2/x^-y^-\right),$$

$$x(u) = (u + \sqrt{u^2 - 4g^2})/2$$

OPE for Wilson loops. N=4 SYM

Other pentagon transitions with multiple operator insertions can be obtained as:

$$P(\mathbf{u}|\mathbf{v}) = \frac{\prod_{i,j} P(u_i|v_j)}{\prod_{i>j} P(u_i|u_j) \prod_{i<j} P(v_i|v_j)}$$

Similar picture for other operators types

OPE for Wilson loops. N=4 SYM P.T. example.

$$P(u|v) = -\frac{(u^2+\frac{1}{4})\Gamma(iu-iv)(v^2+\frac{1}{4})}{g^2\,\Gamma(\frac{3}{2}+iu)\Gamma(\frac{3}{2}-iv)} + O(g^0) \quad \operatorname*{Res}_{v=u} P_{aa}(u|v) = \frac{i}{\mu_a(u)}$$

$$\mathcal{W}_6^{\text{gluons}\,+} = g^2 \sum_{a=1}^{\infty} e^{-a\tau+ia\phi} \int \frac{du}{2\pi} \frac{(-1)^a \Gamma(\frac{a}{2}+iu)\Gamma(\frac{a}{2}-iu)}{(\frac{a^2}{4}+u^2)\Gamma(a)} e^{2iu\sigma} + O(g^4)$$

$$\tilde{u}_1 = \frac{1}{1+e^{-2\sigma}+e^{-\sigma-\tau+i\phi}}, \qquad \tilde{u}_3 = \frac{1}{1+e^{2\sigma}+e^{\sigma-\tau+i\phi}}$$

$$\mathcal{W}_6^{\text{gluons}\,+} = g^2(\pi^2/6 - \text{Li}_2(1-\tilde{u}_1) - \text{Li}_2(1-\tilde{u}_3) - \log(\tilde{u}_1)\log(\tilde{u}_3)) + O(g^4)$$

Large tau expansion, additional small parameter

$$e^{-2\tau}$$

Similar results for MHV n=6 2 and 3 (!) loops, perfect agreement, 1-2 loops NMHV

OPE for Wilson loops. N=4 SYM strong coupling.

$$P(u|v) \equiv P_{FF}(u|v)$$

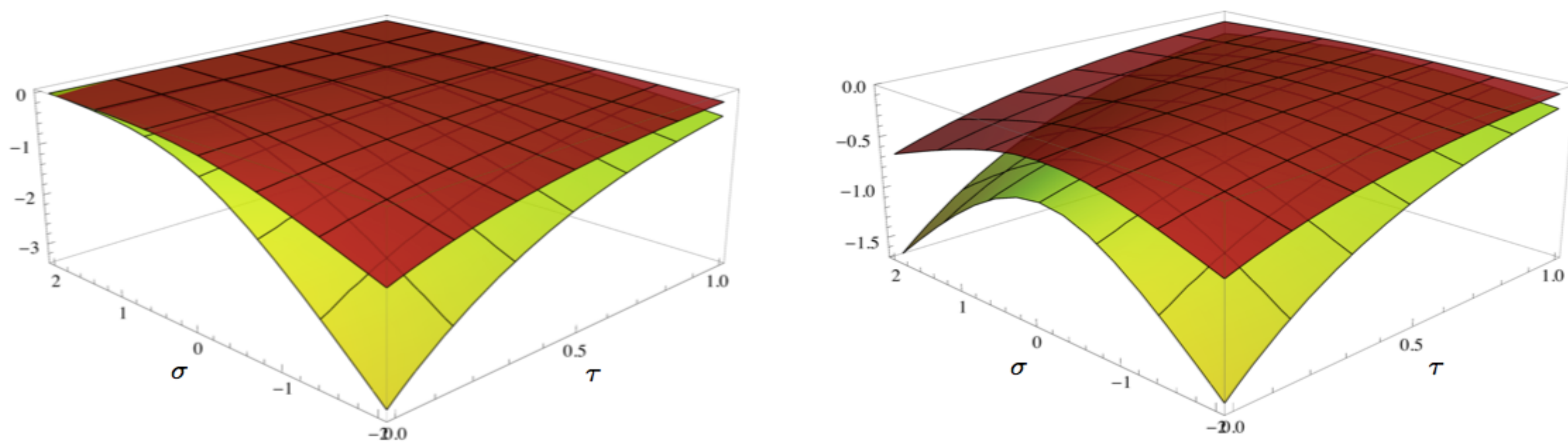
$$P(u|v) = 1 + \frac{2\pi}{\sqrt{\lambda}} K(\theta,\theta') + O(1/\lambda)$$

$$K(\theta,\theta') = \frac{i\cosh(2\theta)\cosh(2\theta')}{2\sinh(2\theta - 2\theta')}\left[\sqrt{2}\cosh\left(\theta - \theta' - i\frac{\pi}{4}\right) + 1\right]$$

$$\mathcal{W} = \exp\left[-\frac{\sqrt{\lambda}}{2\pi} YY_{cr}\right]$$

Exactly the same result as for the TBA solution of Y-system in minimal AdS5 surface problem! (YangYang functional)

OPE for Wilson loops. N=4 SYM finite coupling.



a) $\mathcal{W}_{\psi\bar{\psi}}$ **b)** $\mathcal{W}_{\psi\bar{\psi}} - \frac{1}{2}\Gamma_{\text{cusp}} e^{-2\tau}\sigma$

Figure 8: **a)** The two-fermion contribution $\mathcal{W}_{\psi\bar{\psi}}$ for $g = 1/2$ (top/red) and $g = 1$ (bottom/yellow). The double pole in the fermion measure at zero momentum leads to a contribution that grows

Open questions

- Additional P.T. tests – NNMHV, more loops and legs.
- Radius of convergence – finite tau.
- Strong coupling sub leading corrections.
- Numerical evaluations for finite coupling constant and tau.
- Resummation – can we obtain more simple representation of OPE series at finite coupling?
- Can OPE series can be actually *derived* from some string theory model ?

OPE for Wilson loops. N=4 SYM

$$P(u|v) \equiv P_{FF}(u|v)$$

$$P(u|v)^2 = \left[\frac{f(u,v)}{g^2(u-v)(u-v-i)}\right]^\eta \frac{S(u,v)}{S(u^\gamma,v)}$$

$$S(u,v) = \frac{\Gamma(\frac{1}{2}-iu)\Gamma(\frac{1}{2}+iv)\Gamma(iu-iv)}{\Gamma(\frac{1}{2}+iu)\Gamma(\frac{1}{2}-iv)\Gamma(iv-iu)} F(u,v)\,,$$

$$S(u^\gamma,v) = \frac{\pi g^2 \sinh\left(\pi(u-v)\right)}{(u-v+i)\cosh\left(\pi u\right)\cosh\left(\pi v\right)} G(u,v)\,.$$

OPE for Wilson loops. N=4 SYM

$$\log F = 2i \int_0^\infty \frac{dt}{t}(J_0(2gt)-1)\frac{e^{-t/2}(\sin(ut)-\sin(vt))}{e^t-1} - 2if_1 + 2if_2\,,$$

$$\log G = 2 \int_0^\infty \frac{dt}{t}(J_0(2gt)-1)\frac{e^{t/2}(\cos(ut)+\cos(vt))-J_0(2gt)-1}{e^t-1} + 2f_3 - 2f_4\,.$$

$$K_{ij} = 2j(-1)^{j(i+1)} \int_0^\infty \frac{dt}{t}\frac{J_i(2gt)J_j(2gt)}{e^t-1}$$

$$\mathbb{M} \equiv (1+K)^{-1} = 1 - K + K^2 - K^3 + \dots \qquad \mathbb{Q}_{ij} = \delta_{ij}(-1)^{i+1} i$$

$$f_1(u,v) = 2\,\tilde\kappa(u)\cdot\mathbb{Q}\cdot\mathbb{M}\cdot\kappa(v)\,, \qquad f_2(u,v) = 2\,\tilde\kappa(v)\cdot\mathbb{Q}\cdot\mathbb{M}\cdot\kappa(v)$$

$$f_3(u,v) = 2\,\tilde\kappa(u)\cdot\mathbb{Q}\cdot\mathbb{M}\cdot\tilde\kappa(v)\,, \qquad f_4(u,v) = 2\,\kappa(v)\cdot\mathbb{Q}\cdot\mathbb{M}\cdot\kappa(v)$$

OPE for Wilson loops. N=4 SYM

$$E(u) = 1 + 4g\left(\mathbb{Q}\cdot\mathbb{M}\cdot\kappa(u)\right)_1\,, \qquad p(u) = 2u - 4g\left(\mathbb{Q}\cdot\mathbb{M}\cdot\tilde{\kappa}(u)\right)_1$$