Explicit harmonic and spectral analysis on Bianchi I-VII groups

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Outline

INTRODUCTION

Introduction: Bianchi groups

All real 3-dim Lie groups - $Bi(N)/\Gamma$

> $Bi(N)$ - N=1,...,9. Bianchi I-IX groups.
> Γ - a discrete normal subgroup of $Bi(N)$

Some of Bianchi groups well known, others unknown.

$Bi(I)$	additive group $\mathbb{R}^3$	Abelian
$Bi(II)$	Heisenberg group $\mathbb{H}^{2+1}$	Nilpotent
$Bi(III)$	affine group $(ax+b)^{2+1}$	Solvable
$Bi(IV) - Bi(VII)$	Largely unknown	Solvable
$Bi(VIII)$	universal covering group $\widetilde{SL(2,\mathbb{R})}$	Simple
$Bi(IX)$	special unitary group $SU(2)$	Simple

Motivation: 3D exactly solvable systems

Quantum mechanics:

- Free non-relativistic particle on a 3D manifold G
- G equipped with arbitrary left invariant (anisotropic) Riemannian metric g
- Solvable in explicit terms

$$(-\Delta_g + m^2)\varphi(x) = \lambda\varphi(x), \quad x \in G.$$

Motivation: cosmology

Cosmology:

Homogeneous anisotropic spacetime models

Explicit mode decomposition of classical and quantum fields

$$(\Box_g + m^2)\Phi = (\partial_t^2 - \Delta_g + m^2)\Phi = 0.$$

HARMONIC ANALYSIS

Bianchi I-VII: Semidirect structure

Every Bianchi I-VII group is a semidirect product

$$G = \mathbb{R}^2 \rtimes_F \mathbb{R},$$

where homomorphism $F : \mathbb{R} \mapsto \mathrm{Aut}(\mathbb{R}^2)$ is $F(r) =$

I	II	III	IV
1^r	$\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$	$\begin{pmatrix} e^r & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} e^r & 0 \\ re^r & e^r \end{pmatrix}$
V	VI	VII	
$\begin{pmatrix} e^r & 0 \\ 0 & e^r \end{pmatrix}$	$\begin{pmatrix} e^r & 0 \\ 0 & e^{-qr} \end{pmatrix}$ $-1 < q \leq 1$	$e^{pr} \begin{pmatrix} \cos(r) & -\sin(r) \\ \sin(r) & \cos(r) \end{pmatrix}$ $p \geq 0$	

Bianchi I-VII: group properties

All exponential solvable, hence diffeomorphic to $\mathbb{R}^3$.

Faithful matrix representation:

$$\rho(x) = \begin{pmatrix} F(x_3) & \begin{matrix} x_1 \\ x_2 \end{matrix} \\ 0 & 1 \end{pmatrix}, \quad (x_1, x_2, x_3) = x \in G.$$

The modular function and the left Haar measure:

$$\Delta(x) = \det F(-x_3), \quad dg(x) = \mathfrak{h}\Delta(x)dx, \quad \forall\mathfrak{h} > 0.$$

Bianchi I-VII: Irreps

The null space $\mathcal{N}_F$ of the matrix $\partial_r \log F(r)$:

$$\mathcal{N}_F = \mathbb{R}\left\{\check{x} \in \mathbb{R}^2 : \quad \partial_r \log F(r)\check{x} = 0\right\}$$

The dual space $\hat{G}$, equivalence classes of unitary irreducible representations:

$$\hat{G} = \hat{G}_1 \cup \hat{G}_\infty, \quad \hat{G}_1 \simeq \mathcal{N}_F \times \mathbb{R}, \quad \hat{G}_\infty \simeq (\mathbb{R}^2 \setminus \mathcal{N}_F)/F^\top(\mathbb{R}).$$

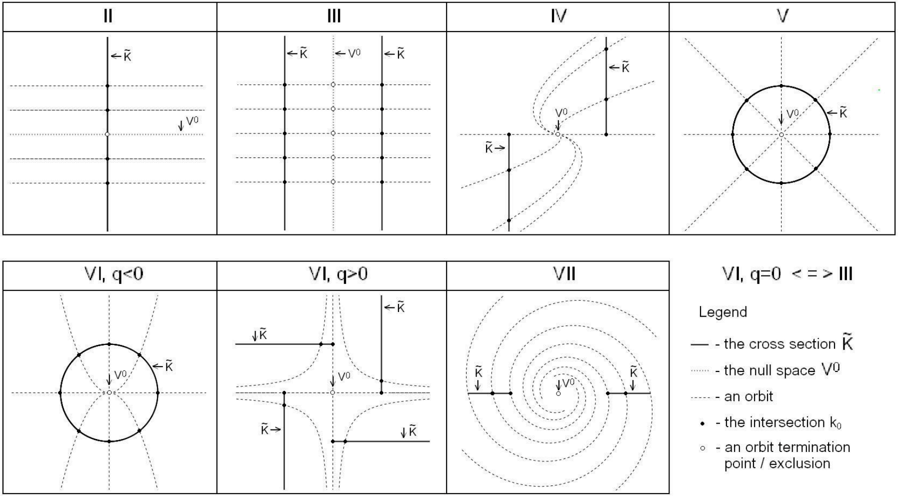
Explicitly,

$$\hat{G}_1 \ni p = (p_1, p_2, p_3), \quad U_p(x) = e^{i\langle p,x\rangle},$$

$$\hat{G}_\infty \ni \check{p} = (p_1, p_2), \quad U_{\check{p}}(x) = \mathrm{Ind}_{\mathbb{R}^2}^G e^{i\langle \check{p},\check{x}\rangle}, \quad \forall x = (\check{x}, x_3) \in G.$$

Bianchi II-VII: co-adjoint orbits

Co-adjoint orbits $F^{\top}(\mathbb{R})\mathbb{R}^2$:



Bianchi I-VII: The Plancherel measure

Embedding map $\check{k} : \mathfrak{K} \mapsto \mathbb{R}^2$

The cross sections and Plancherel measures:

Bianchi I:	$\mathfrak{K} = \mathbb{R}^3$	$d\nu(k) = dk$
Bianchi II:	$\mathfrak{K} = \mathbb{R}$	$d\nu(k) = \lvert k\rvert dk$
Bianchi III:	$\mathfrak{K} = \mathbb{R} \times \mathbb{Z}_2$	$d\nu(k) = dk$
Bianchi IV:	$\mathfrak{K} = \mathbb{R}_{+0} \times \mathbb{Z}_2$	$d\nu(p) = (1 + k_1)dk$
Bianchi V:	$\mathfrak{K} = \mathbb{R}/2\pi\mathbb{Z}$	$d\nu(k) = dk$
Bianchi VI, $q < 0$:	$\mathfrak{K} = \mathbb{R}/2\pi\mathbb{Z}$	$d\nu(k) = (1 - (1 + q)\sin^2(k))dk$
Bianchi VI, $q > 0$:	$\mathfrak{K} = \mathbb{R}_{+0} \times \mathbb{Z}_4$	$d\nu(k) = q^{k_2 \mod 2} dk$
Bianchi VII:	$\mathfrak{K} = [\, 1, e^{\pi p}\,) \times \mathbb{Z}_2$	$d\nu(k) = \lvert k_1\rvert dk$

SPECTRAL ANALYSIS

Bianchi I-VII: The spectral problem

Schrödinger equation on G:

- Left invariant (anisotropic) Riemannian metric g on G
- Metric Levi-Civita connection ∇
- Metric Laplace operator $\Delta_g = g^{ab}\nabla_a\nabla_b$

$$(-\Delta_g + m^2)\zeta(x) = \lambda\zeta(x), \quad x \in G.$$

Spectral problem:

- (G, g) - complete metric space (Milnor'76)
- Δ_g - essentially self-adjoint in $L^2(G)$ (Chernoff'73)

Bianchi I-VII: The spectrum

Left invariant metric g:

$$g_{ab}(x) = \mathbf{h}_{ij}\omega^i_a(x)\omega^j_b(x), \quad i,j = 1,...3, x \in G,$$

where

$\mathbf{h}_{ij}$ - any 3×3 symmetric positive definite matrix,
$\{\omega^i\}_{i=1}^3$ - basis of left invariant 1-forms.

Spectrum:

$$\mathrm{Spec}(-\Delta_g) = \mathrm{EssSpec}(-\Delta_g) = [\ \lambda^F_{min}, +\infty\ )$$

$$\lambda^F_{min} = \mathrm{Tr}(S_F^2) + \mathrm{Tr}(S_F)^2,$$

$$S_F = \frac{1}{2}\left(\partial_r \log F(r) + (\mathbf{h})^{2\times 2}\partial_r \log F(r)(\mathbf{h}^{-1})^{2\times 2}\right).$$

Bianchi I-VII: The spectral measure

Bianchi I:

Dual parameter $\mathbb{R}^3 \ni k = (k_x, k_y, k_z)$
Eigenvalue $\lambda(k) = k^\top \mathbf{h} k$
Spectral measure $d\mu(k) = d\nu(k) = dk$.

Bianchi II-VII:

Dual parameter $\mathfrak{K} \times \mathrm{Spec}(-\Delta_g) \times \mathbb{R} \times \mathbb{Z}_2 \ni \alpha = (k, \lambda, r, s)$
Eigenvalue $\lambda(\alpha) = \lambda$
Spectral measure $d\mu(\alpha) = d\nu(k) d\lambda dr d\#(s)$

Bianchi I-VII: Eigenfunctions

Bianchi I:

$$\zeta_k(x) = \frac{1}{\det \mathbf{h}} e^{i\langle k,x\rangle}$$

Bianchi II-VII:

$$\zeta_\alpha(x) = \frac{1}{\det \mathbf{h}} \det F(-r) e^{i\langle \check{k},\check{x}\rangle} P_{k,\lambda,s}(x_3 - r),$$

where

$$\ddot{P}_{k,\lambda,s}(z) + i\gamma(z;k,\mathbf{h})\dot{P}_{k,\lambda,s}(z) + \omega^2(z;k,\lambda,\mathbf{h})P_{k,\lambda,s}(z) = 0.$$

SUMMARY

Summary of results

Harmonic analysis:

- Group characteristics: structure, Haar measure, modular function
- Dual space: representations, Plancherel measure

Spectral analysis:

- Spectrum, spectral measure and eigenfunctions

Outlook

Open questions:

- Bianchi VIII-IX: harmonic analysis well known
- Bianchi VIII-IX: spectral analysis difficult
- Quotient groups $Bi(N)/\Gamma$: harmonic and spectral analysis
- Bianchi I-IX: vector-valued spectral analysis

Reference:

Zh.A., R. Verch, “Explicit harmonic and spectral analysis in Bianchi I-VII type cosmologies”, Class. Quant. Grav. (30)

THANK YOU