1. Verify that the Lax equation \( L_t = [M, L] \) for the Lax pair given by the matrix linear differential operators

\[
L = \begin{pmatrix} -i\partial_x & \psi^*(x) \\ -\psi(x) & i\partial_x \end{pmatrix},
\]

\[
M = \begin{pmatrix} -4i\lambda^3 + 2|\psi|^2 + \psi^*\psi_x - \psi^*\psi^*_x & -4i\lambda^2\psi^* - 2i\lambda\psi^*_x + i\psi^*_{xx} + 2i\psi^*\psi_x \\ -4i\lambda^2\psi + 2i\lambda\psi_x + i\psi_{xx} + 2i\psi^2\psi^* & 4i\lambda^3 - 2\lambda|\psi|^2 - \psi_x\psi^* + \psi^*_{x} \end{pmatrix},
\]

where the real spectral parameter \( \lambda = \text{const} \), is equivalent to the nonlinear Schrödinger equation

\[
i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0
\]

2. Using the Lax matrices of the Problem 1 solve the problem of the time evolution of the scattering data for the nonlinear Schrödinger equation. Show that the scattering data are reflectionless.

3. Hirota’s Method. Linearize the generalized sine-Gordon equation \( u_{xt} + m^2\sin u = 0 \), \( m^2 \) is a constant, by the transformation \( u = 2i\log f \). Using the substitution \( f = 1 + f^{(1)} + f^{(2)} + \ldots \) construct

(a) One-soliton (kink) solution

(b) Two-kink solution

4. Solve the problem of small excitations \( u(x, t) = u_0 + \phi \) around the kink solution \( u_0 = 4\arctan e^{x-x_0} \) of the sine-Gordon model given by the equation

\[
\phi_{tt} - \phi_{xx} + \phi\cos u_0 = 0
\]

Find the eigenfunctions of the continuum part of the spectrum. Show that the potential created by the soliton is reflectionless.

5. Analyse the effect of small perturbation of the \( \phi^4 \) kink solution given by the equation

\[
\phi_{tt} - \phi_{xx} - m^2\phi + \lambda\phi^3 + \varepsilon \frac{m^3}{\sqrt{\lambda}} = 0
\]

where \( m, \lambda \) are the constants of the non-perturbed solution \( \phi_0 = \frac{m}{\sqrt{\lambda}}\tanh \frac{m^2}{\sqrt{\lambda}} \) and \( \varepsilon \ll 1 \).