# Course 345: INTRODUCTION TO SOLITONS

## Problem Set 4

**Date Issued: April 14, 2008**
**Date due: April 28, 2008**

1. Verify that the Lax equation $L_t = [M, L]$ for the Lax pair given by the matrix linear differential operators
$$L = \begin{pmatrix} -i\partial_x & \psi^*(x) \\ -\psi(x) & i\partial_x \end{pmatrix},$$
$$M = \begin{pmatrix} -4i\lambda^3 + 2\lambda|\psi|^2 + \psi^*\psi_x - \psi\psi_x^* & -4i\lambda^2\psi^* - 2i\lambda\psi_x^* + i\psi_{xx}^* + 2i\psi^{*2}\psi \\ -4i\lambda^2\psi + 2i\lambda\psi_x + i\psi_{xx} + 2i\psi^2\psi^* & 4i\lambda^3 - 2\lambda|\psi|^2 - \psi_x\psi^* + \psi\psi_x^* \end{pmatrix},$$
where the real spectral parameter $\lambda =$ const, is equivalent to the nonlinear Schrödinger equation
$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0$$

2. Using the Lax matrices of the Problem 1 solve the problem of the time evolution of the scattering data for the nonlinear Schrödinger equation. Show that the scattering data are reflectionless.

3. Hirota's Method. Linearize the generalized sine-Gordon equation $u_{xt} + m^2 \sin u = 0$, $m^2$ is a constant, by the transformation $u = 2i \log \frac{f^*}{f}$. Using the substitution $f = 1 + \lambda f^{(1)} + \lambda^2 f^{(2)} + \ldots$ construct
   (a) One-soliton (kink) solution
   (b) Two-kink solution

4. Solve the problem of small excitations $u(x,t) = u_0 + \phi$ around the kink solution $u_0 = 4 \arctan e^{x - x_0}$ of the sine-Gordon model given by the equation
$$\phi_{tt} - \phi_{xx} + \phi \cos u_0 = 0$$
Find the eigenfunctions of the continuum part of the spectrum. Show that the potential created by the soliton is reflectionless.

5. Analyse the effect of small perturbation of the $\phi^4$ kink solution given by the equation
$$\phi_{tt} - \phi_{xx} - m^2\phi + \lambda\phi^3 + \varepsilon\frac{m^3}{\sqrt{\lambda}} = 0$$
where $m, \lambda$ are the constants of the non-perturbed solution $\phi_0 = \frac{m}{\sqrt{\lambda}} \tanh \frac{mx}{\sqrt{2}}$ and $\varepsilon \ll 1$.