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Introduction to Solitons

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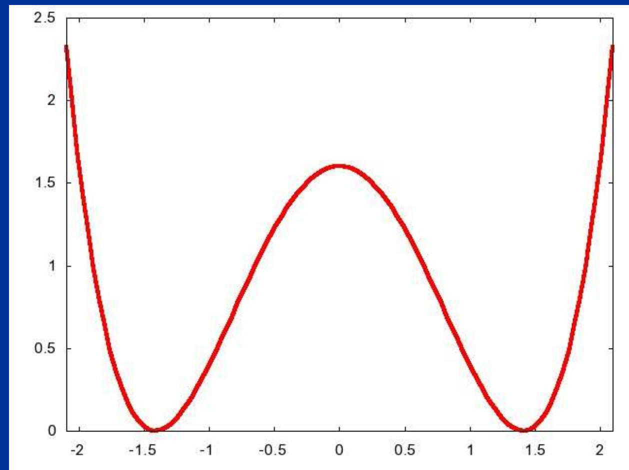
ϕ^4 model

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi); \qquad U(\phi) = \frac{\lambda}{4}\left(\phi^2 - a^2\right)^2$$



Field equation: $\partial_\mu\partial^\mu\phi + \frac{dU}{d\phi} = 0$

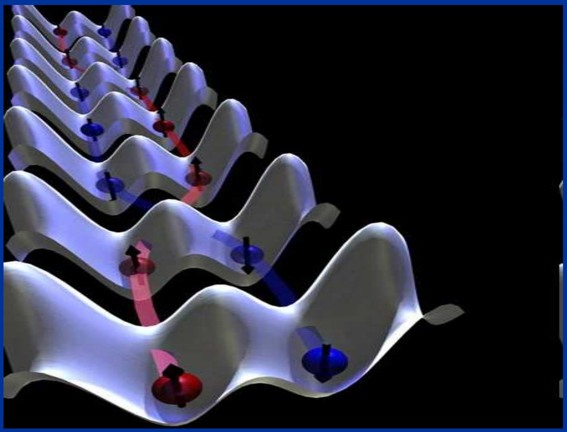
Potential energy: $V = \int\limits_{-\infty}^{\infty} dx \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + U(\phi)\right]$

Kinetic energy: $T = \frac{1}{2}\int\limits_{-\infty}^{\infty} dx \left(\frac{\partial\phi}{\partial t}\right)^2$

Vacuum: $\phi_0 = \pm a$ **Static configuration:** ***T=0***

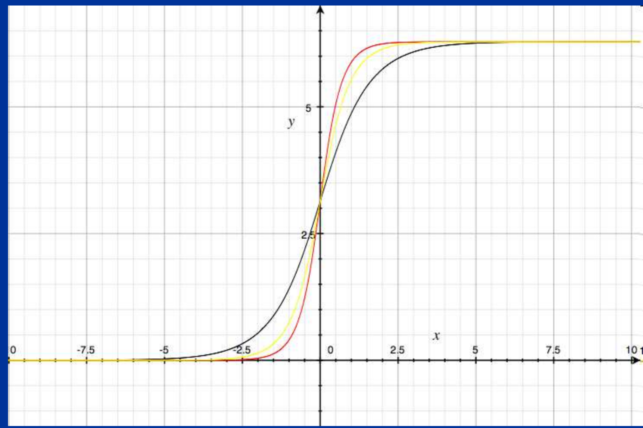
Energy bound: $E = V = \int\limits_{-\infty}^{\infty} dx \left[\frac{1}{\sqrt{2}}\phi' \pm \sqrt{U(\phi)}\right]^2 \mp \int\limits_{-\infty}^{\infty} dx \sqrt{2U(\phi)}\ \phi' \geq 0$

ϕ^4 model: Applications



- Phenomenological theory of second order phase transitions
- A model of the displacive phase transitions
- A model of uniaxial ferroelectrics
- A phenomenological theory of the non-perturbative transition in polyacetylene chain
- Condensed matter physics: solitary waves in shape-memory alloys
- Cosmology: model dynamics of the domain walls.
- Biophysics: soliton excitations in DNA double helices.
- Quantum field theory: a model example to investigate transition between perturbative and non-perturbative sectors of the theory.
- A model of quantum mechanical instanton transitions in double-well potential

Kink solution



Energy is minimal if $\frac{1}{\sqrt{2}}\phi' = \sqrt{U(\phi)}$

Kink solution: $\phi_{K\bar{K}} = \pm a \tanh\left(\frac{m(x-x_0)}{\sqrt{2}}\right)$

Mass of the kink: $M = \frac{2\sqrt{2}\lambda a^3}{3} = \frac{2\sqrt{2}m^3}{3\lambda}$

Mass of the scalar excitaion: $m = a\sqrt{\lambda}$

$$M = E = \int_{-\infty}^{\infty} dx \sqrt{2U(\phi)}\,\phi' = \sqrt{\frac{\lambda}{2}} \int_{\phi(-\infty)}^{\phi(\infty)} d\phi(a^2 - \phi^2) = \sqrt{\frac{\lambda}{2}} \left(a^2\phi - \frac{1}{3}\phi^3\right)\Big|_{\phi(-\infty)}^{\phi(\infty)}$$

Topological charge: $Q = \frac{1}{2a}\left[\phi(\infty) - \phi(-\infty)\right]$ - mapping (-a ; a) → $(-\infty\ ;\ \infty)$

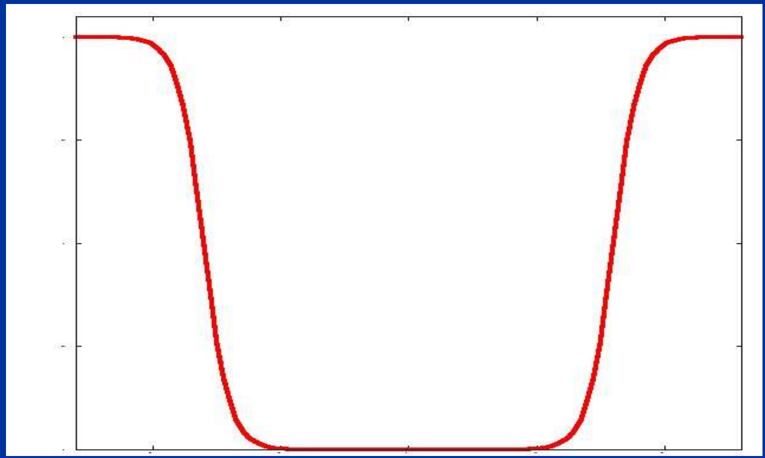
Topological current $J_\mu = \frac{1}{2a}\varepsilon_{\mu\nu}\partial^\nu\phi; \quad Q = \int_{-\infty}^{\infty} dx J_0 = \frac{1}{2a}\int_{-\infty}^{\infty} dx \frac{\partial\phi}{\partial x}$

Energy density: $\mathcal{E} = \frac{m^2a^2}{2}\mathrm{sech}^4\left(\frac{m(x-x_0)}{\sqrt{2}}\right)$

Interaction between the kinks

Kink-antikink pair (a=1, m = √2):

$$\phi(x) = 1 + \tanh(x-R) - \tanh(x+R)$$



Far away from the pair (somewhere at x ≈ 0)

$$\tanh(x-R) \approx -1 + 2e^{2(x-R)};$$
$$\tanh(x+R) \approx 1 - 2e^{-2(x+R)}$$

Interaction energy: $E_{int} \approx -16e^{-2L}, \quad L = 2R$

Kinks attracts each other with the force $F = \frac{dE_{int}}{dL} \approx 32e^{-2L}$

Linear oscillations on the static kink background: $\phi = \phi_K + \delta\phi$

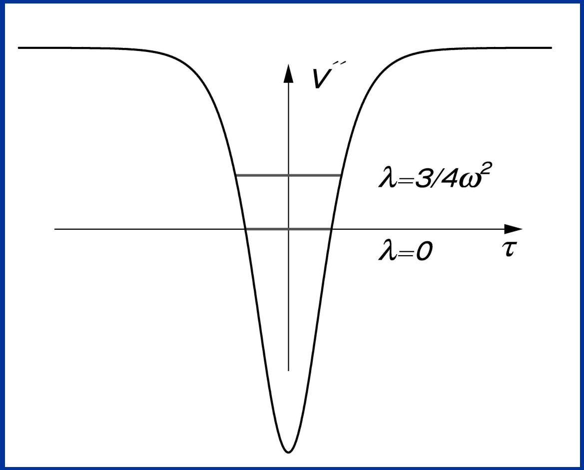
$$\ddot{\phi} - \phi'' - \lambda(a^2 - \phi^2)\phi = 0 \;\rightarrow\; \delta\ddot{\phi} - \delta\phi'' - m^2\left[1 - \frac{3}{2\cosh^2(mx/2)}\right]\delta\phi = 0$$

Modes of oscillations: ϕ^4 model

Modes on the ϕ^4 kink: $\delta\phi = \sum_{n=0}^{\infty} C_n(t)\eta_n(x)$

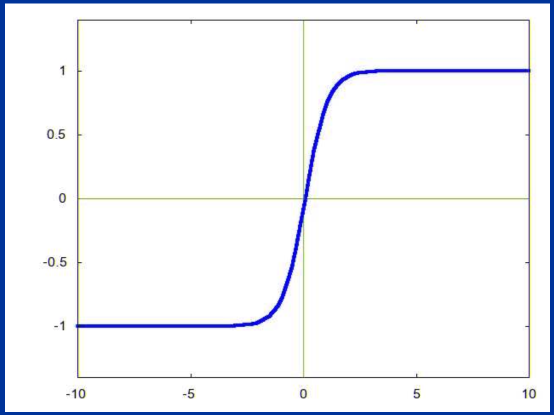
$$\frac{d^2\eta_n}{dx^2} + m^2\left[1 - 3\tanh\left(\frac{mx}{\sqrt{2}}\right)\right]\delta\eta_n = \omega^2\eta_n(x)$$

Peashle-Teller potential:



Zero (translational) mode: ω^2=0

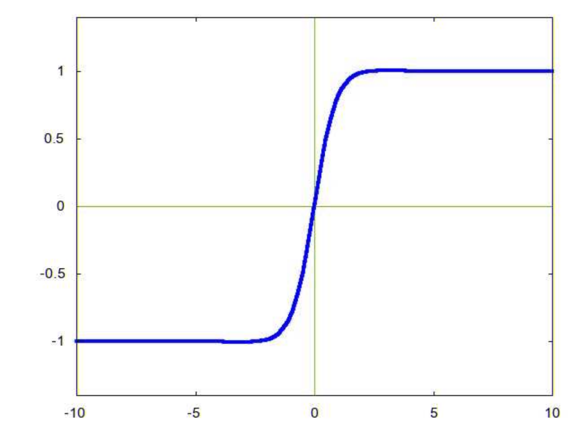
Internal mode: $\omega^2 = \frac{3m^2}{4}$



$$z = \frac{mx}{\sqrt{2}}$$

$$\eta_0 = \frac{2}{\sqrt{3}\cosh^2 z}$$

$$\eta_1 = \sqrt{\frac{2}{3}}\frac{\sinh z}{\cosh^2 z}$$



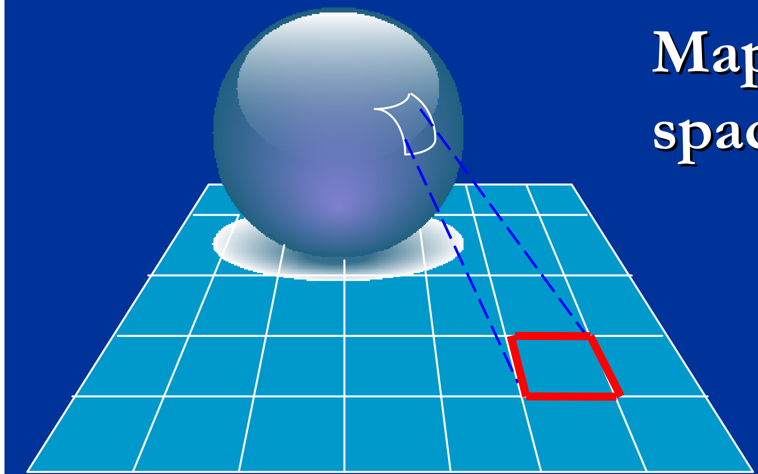
O(3) sigma-model

$L = \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a; \quad \phi^a \phi^a = 1$ $\qquad \phi^a = (\phi^1, \phi^2, \phi^3)$ $\qquad$ **d=2+1**

using the Lagrange multiplier $L = \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a + \lambda(1 - \phi^a \phi^a)$

Field equation: $\partial_\mu \partial^\mu \phi^a + (\partial_\mu \phi^b \partial^\mu \phi^b)\phi^a = 0$ $\qquad \lambda = \phi^a \partial_\mu \partial^\mu \phi^a$

Energy $E = \frac{1}{4}\int d^2x \, (\partial_n \phi^a \cdot \partial_n \phi^a)$ **remains finite if** $\phi \to (0,0,1)$ **as** $r \longrightarrow \infty$



Mapping from the (x,y) plane to the target space S²

$$(dx, dy) \to \left(\frac{\partial \vec{\phi}}{\partial x} dx, \frac{\partial \vec{\phi}}{\partial y} dy \right)$$

The boundary of the (x,y) plane is equivalent to a point, i.e. the plane is equivalent to the sphere S^2

Topological charge:
- mapping $S^2 \longrightarrow S^2$

$$Q = \frac{1}{4\pi}\int d^2x \, (\vec{\phi} \cdot \vec{\sigma}) = \frac{1}{4\pi}\int d^2x \, \vec{\phi} \left[\frac{\partial \vec{\phi}}{\partial x} \times \frac{\partial \vec{\phi}}{\partial y} \right]$$

O(3) sigma-model: Lump solitons

Problem: find solution in the sector with topological charge *Q=1*

SO(2) invariant ansatz: $\begin{cases} \phi^\alpha = n^\alpha \sin f(r), & \phi^3 = \cos f(r) \\ n^\alpha = (\cos\varphi; & \sin\varphi) \end{cases}$

Vacuum: $\phi^a = (0,0,-1)$ → $f(0) = 0; f(\infty) = \pi$

$$\partial_k \phi^\alpha = \frac{1}{r}\left(\delta^{k\alpha} - n^k n^\alpha\right)\sin f + n^k n^\alpha f' \cos f; \quad \partial_k \phi^3 = -n^k f' \sin f$$

Trick: do not solve field equation directly!

Consider $F_i^a = \partial_i \phi^a \pm \varepsilon_{abc}\varepsilon_{ij}\phi^b \partial_j \phi^c$ $\quad \frac{1}{2}F_i^a F_i^a = \partial_i \phi^a \partial_i \phi^a \mp \varepsilon_{abc}\varepsilon_{ij}\phi^a \partial_i \phi^b \partial_j \phi^c$

Because $\int d^2x\, F_i^a F_i^a \geq 0$

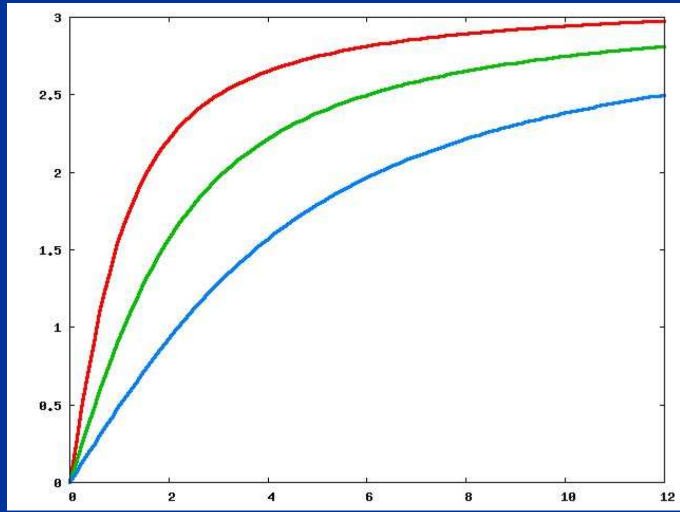
$$\int d^2x\, \partial_i \phi^a \partial_i \phi^a \geq \pm \int d^2x\, \varepsilon_{abc}\varepsilon_{ij}\phi^a \partial_i \phi^b \partial_j \phi^c$$

$$E \geq \pm 4\pi Q$$

$$\partial_i \phi^a \partial_i \phi^a = \varepsilon_{abc}\varepsilon_{ij}\phi^a \partial_i \phi^b \partial_j \phi^c$$

O(3) sigma-model: Lump solitons

$$f' = \frac{1}{r}\sin f \quad \longrightarrow \quad f = 2\arctan\frac{r}{r_0}$$



Solution:

$$\phi^\alpha = \frac{2x^\alpha r_0}{r_0^2 + r^2}; \quad \phi^3 = \frac{r_0^2 - r^2}{r_0^2 + r^2}$$

Topological charge:

$$Q = \frac{1}{4\pi}\int d^2x\ \varepsilon_{abc}\varepsilon_{ij}\phi^a\partial_i\phi^b\partial_j\phi^c = 1$$

Riemann sphere coordinates on the target space S^2:

$$R_1 = \frac{2\phi^1}{1-\phi^3}; \quad R_2 = \frac{2\phi^2}{1-\phi^3}; \quad R = R_1 + iR_2; \quad z = x + iy$$

$$E = \int d^2x\ \frac{|\partial_z R|^2 + |\partial_{\bar z} R|^2}{(1+|R|^2)^2}; \qquad Q = \frac{1}{\pi}\int d^2x\ \frac{|\partial_z R|^2 - |\partial_{\bar z} R|^2}{(1+|R|^2)^2}$$

Field equation becomes the Cauchy-Riemann equation:

$$\partial_{\bar z} R = 0$$

Skyrme family

- ***(2+1)-dim***: **Baby Skyrme model**

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{4}\left(\partial_\mu \phi \times \partial_\nu \phi\right)^2 - V(\phi)$$

$$\phi : S^2 \to S^2; \quad \phi_\infty = (0,0,1)$$

Standard choice: $V(\phi) = \mu^2(1-\phi_3)$

$$Q \in \mathbb{Z} = \pi_2(S^2)$$

$$Q = \frac{1}{4\pi}\int_{\mathbb{R}^2} \phi \cdot (\partial_1 \phi \times \partial_2 \phi) d^2x$$

- ***(3+1)-dim***: **Skyrme model**

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{4}\left(\partial_\mu \phi \times \partial_\nu \phi\right)^2 - V(\phi)$$

$$\phi : S^3 \to S^3; \quad \phi_\infty = (0,0,0,1)$$

$$R_\mu = \partial_\mu U U^\dagger; \quad U = \phi_0 \mathbb{I} + i\sigma^a \cdot \phi^a$$

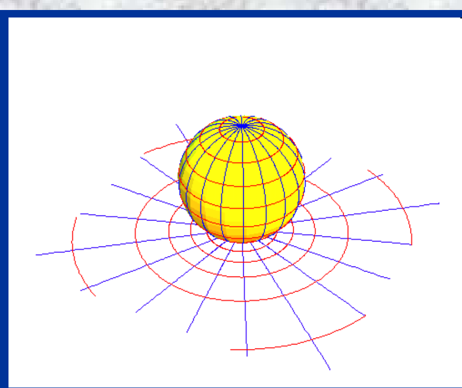
$$\mathcal{L} = -\ \mathrm{Tr}\left\{\frac{1}{2}(R_\mu R^\mu) + \frac{1}{16}([R_\mu, R_\nu][R^\mu, R^\nu]) + \mu^2(U - \mathbb{I})\right\}$$

$$Q \in \mathbb{Z} = \pi_3(S^3)$$

$$Q = \frac{1}{24\pi^2}\ \mathrm{Tr}\int_{\mathbb{R}^3} \varepsilon_{ijk} R_i R_j R_k d^3x$$

Baby Skyrme model

(Tchrakian, Zakrzewski, Leese (1990))



$$\phi = (\phi^1, \phi^2, \phi^3); \qquad \phi^a \cdot \phi^a = 1; \qquad \phi: \; S^2 \to S^2$$

$$Q = \frac{1}{4\pi} \int d^2x \; \varepsilon_{abc} \varepsilon_{ij} \phi^a \partial_i \phi^b \partial_j \phi^c = 1$$

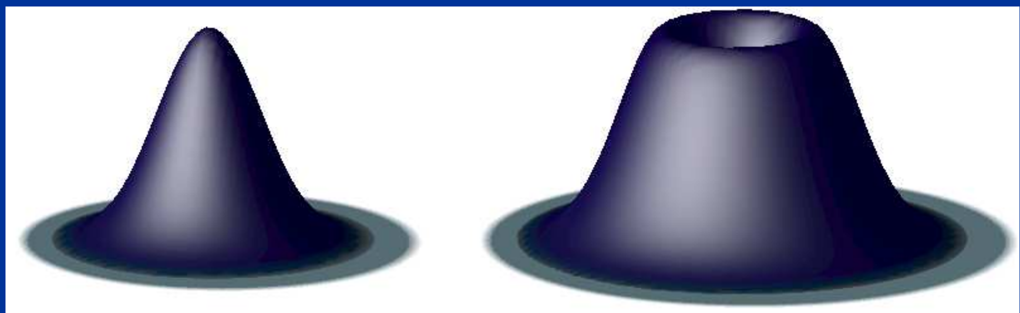
Derrick's scaling theorem: Skyrme term provides a scale but cannot stabilise the soliton: potential term is necessary

$$L = \frac{1}{4}(\partial_\mu \phi^a)^2 - \frac{\kappa}{8}\Big[(\partial_\mu \phi^a \partial_\mu \phi^a)^2 - (\partial_\mu \phi^a \partial_\nu \phi^a)(\partial^\mu \phi^a \partial^\nu \phi^a)\Big] + m^2(1 - \phi^3)$$

$E \geq \pm 4\pi Q$ equality is possible if $\kappa = 0$ and $m = 0$

Axially symmetric ansatz:

$$\phi^1 = \sin f(r) \cos(Q\varphi - \delta);$$
$$\phi^1 = \sin f(r) \sin(Q\varphi - \delta);$$
$$\phi^3 = \cos f(r)$$



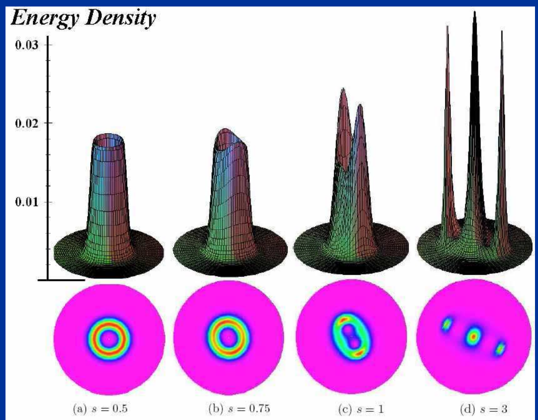
Q=1 Q=2

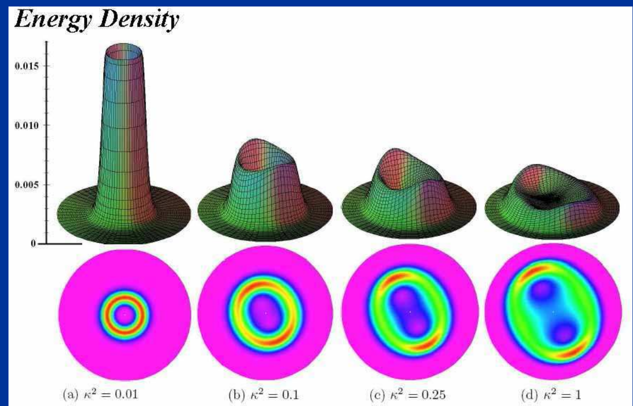
Baby Skyrme model

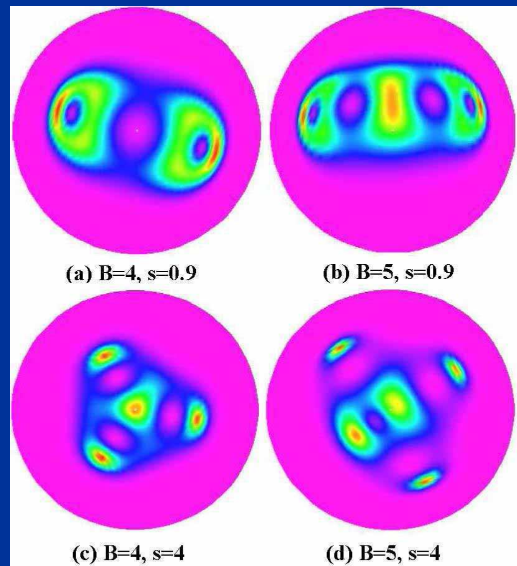
Potential of the baby Skyrme model: potential term U(ϕ) may be chosen almost arbitrarily, however must vanish at infinity for a given vacuum field value in order to ensure existance of the finite energy solutions: $\phi^a_{(0)} = (0,0,1)$

Several potential terms have been studied in great detail:

- "Old" model, with $U(\phi) = m^2(1-\phi_3)$
- Holomorphic model, with $U(\phi) = m^2(1-\phi_3)^4$
- "Double vacuum" model, with $U(\phi) = m^2(1-\phi_3^2)$





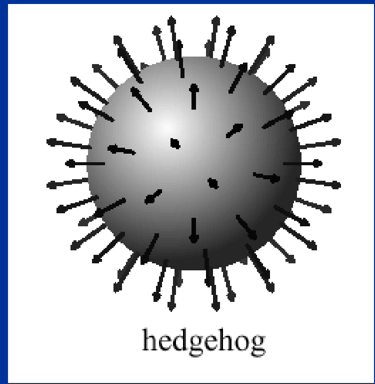


Karliner, Hen (2007)

Skyrme model: $\phi = (\phi^1, \phi^2, \phi^3, \phi^4); \qquad \phi^a \cdot \phi^a = 1; \qquad \phi: \ S^3 \to S^3$

$$L = -\int d^3x \left\{ \tfrac{1}{2}(\partial_\mu \phi^a)^2 - \tfrac{1}{4}[(\partial_\mu \phi^a \partial_\nu \phi^a)^2 - (\partial_\mu \phi^a)^4] + m^2 \ (1-\phi^3) \right\}$$

There is no self-dual skyrmions: E ⩾ Q



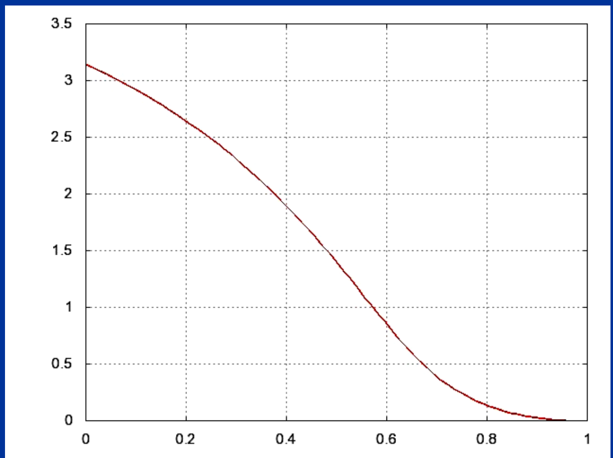
Spherically symmetric skyrmion ***(Q= 1)***

Hedgehog ansatz: $U = \phi^4 + \phi^a \cdot \sigma^a = \cos f(r) + i\hat{\mathbf{n}} \cdot \sigma \sin f(r)$

$U: \ S^3 \to S^3$ $\qquad \hat{\mathbf{n}} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

The topological charge

$$Q = \frac{1}{24\pi^2} \int d^3x \varepsilon_{ijk} \varepsilon_{abcd} \phi^a \partial_i \phi^b \partial_j \phi^c \partial_k \phi^d) = \frac{1}{\pi} \left[f(r) - \frac{\sin 2f(r)}{2} \right]_0^\infty$$



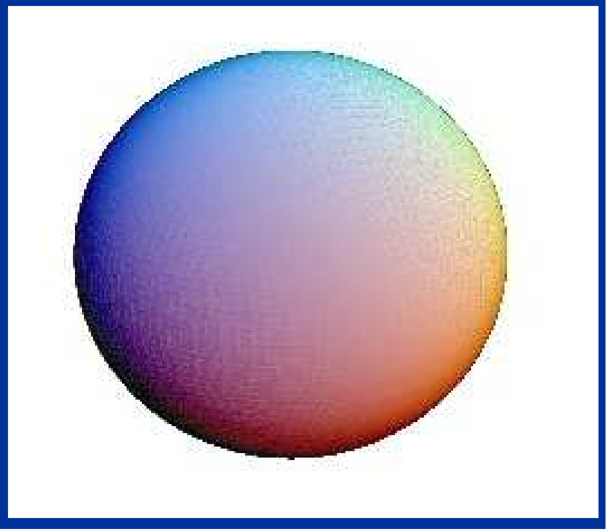
The boundary conditions

$$f(0) = \pi, f(\infty) = 0$$

correspond to Q=1

Good approximation:

$$f(r) = 4\arctan(e^r)$$



Low energy QCD ($ 1000000 Problem)

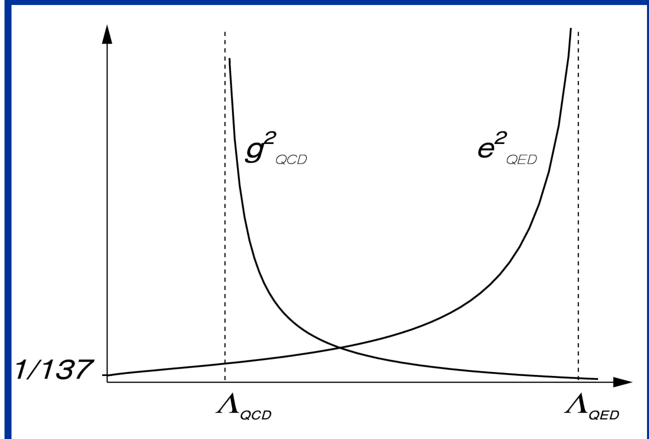
$\Lambda_{QSB} \simeq 1$ GeV

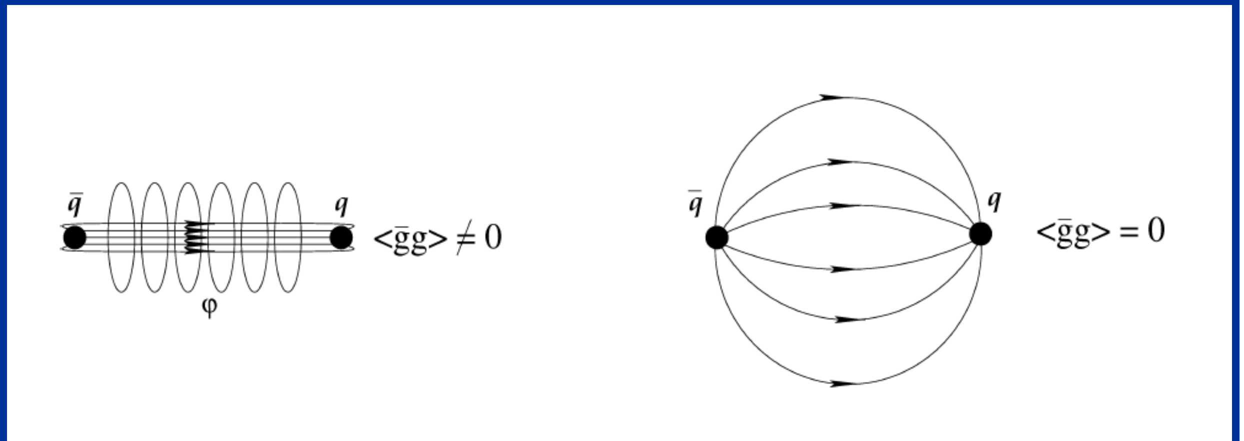
$\Lambda_{QCD} \simeq 180$ MeV

Perturbative QCD (Quarks & gluons) — *Low-energy effective theory* — *Hadrons*

Weak definition of the confinement: There are no color states in physical spectrum

Strong definition of the confinement: The quarks in hadrons are binded by a linear potential





Skyrmions and hadrons ("Top Gear approach")
by courtesy of Paul Sutcliffe

- **If you are a computer simulation freak:**

Nuclear physics from QCD, large scale lattice calculations

- **If you believe that $3 \simeq \infty$:**

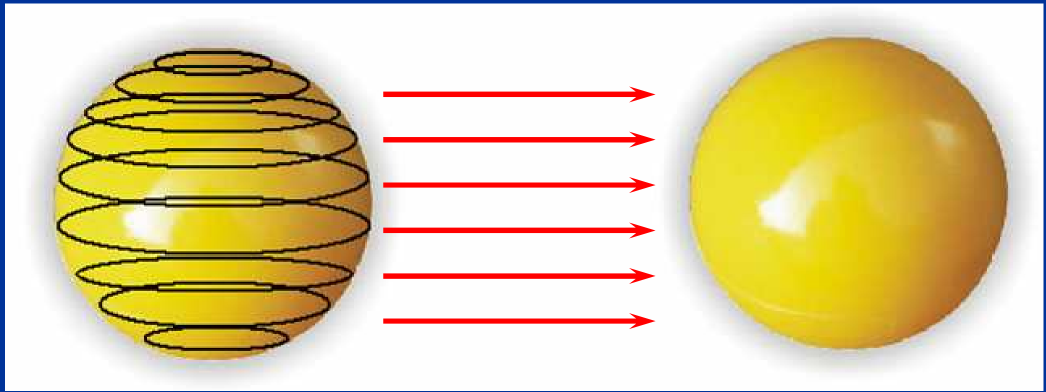
Skyrme model as low-energy QCD:

- **If you are a particle phenomenologist:**

Quark potential models, fitting experimental data

Rational map Skyrmions

The Skyrme field is effectively a map $U: S_3 \to SU(2) \sim S_3$



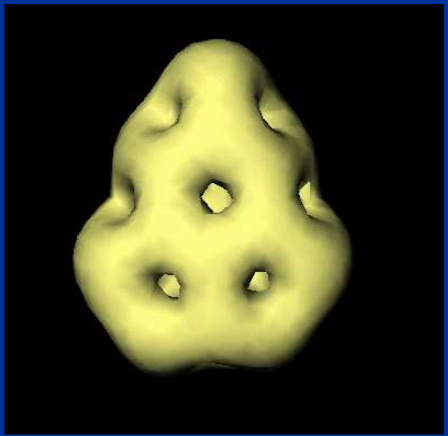
(N.S. Manton, C.Houghton & P.Sutcliffe, 1998)

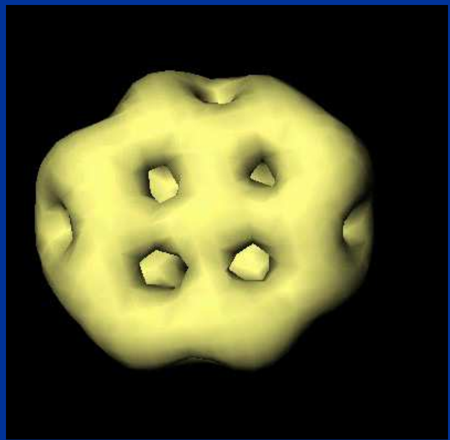
The idea of the rational map ansatz:

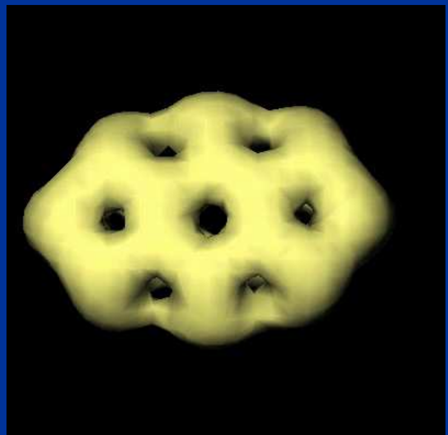
- Separate the radial and the angular dependence of the Skyrme field **as**

$$U = \exp\{i f(r) \hat{\mathbf{n}}_R \cdot \sigma\}$$

- *I*dentify spheres S_2 with concentric spheres in compactified R_3
- *I*dentify target space S_2 with spheres of latitude on S_3

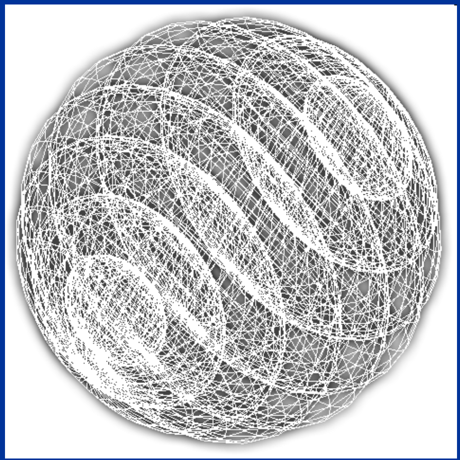






R. A. Battye
P. Sutcliffe
(2007)

Rational map Skyrmions



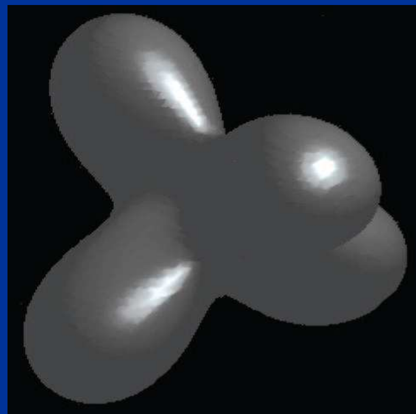
Stereographic projection: $z = \tan(\theta/2)e^{i\varphi}$

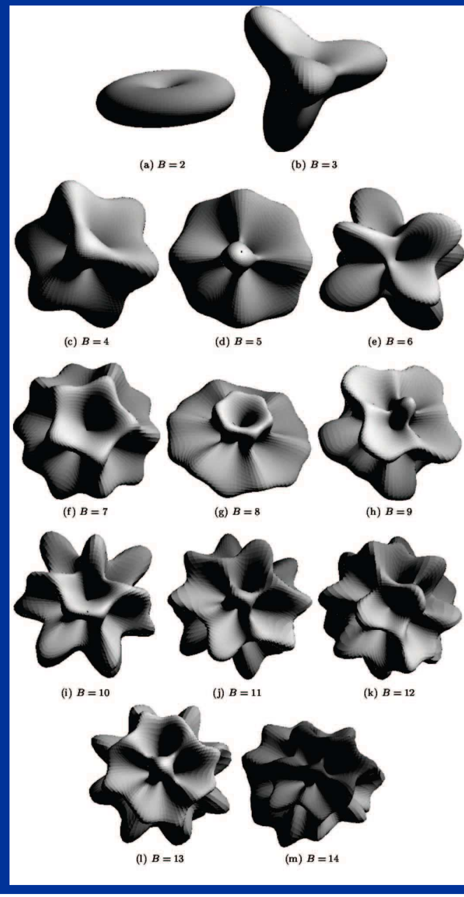
$$U = \exp\{if(r)\hat{\mathbf{n}}_R\cdot\sigma\} \qquad \hat{\mathbf{n}}_R:\ S^2 \to S^2$$

$$\hat{\mathbf{n}}_z = \frac{1}{1+|z|^2}(z+z^*,\ i(z^*-z),\ 1-|z|^2)$$

$$\hat{\mathbf{n}}_R = \frac{1}{1+|R|^2}(R+R^*,\ i(R^*-R),\ 1-|R|^2)$$

$$R = a(z)/b(z)$$

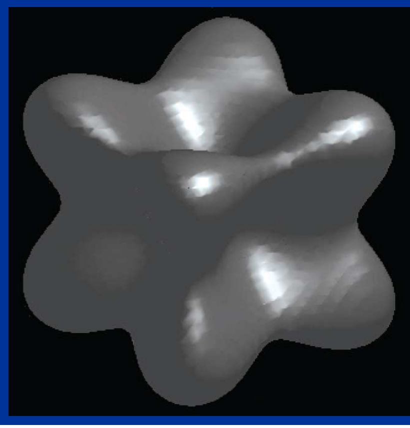




The holomorphic map of degree B:

B= 4: $R(z) = \dfrac{z^4+2i\sqrt{3}z^2+1}{z^4-2i\sqrt{3}z^2+1}$

(Octahedral Skyrmions)

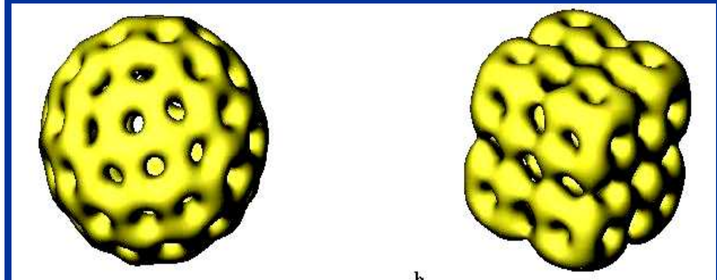


B= 7: $R(z) = \dfrac{z^7-7z^5-7z^2-1}{z^7+7z^5-7z^2+1}$

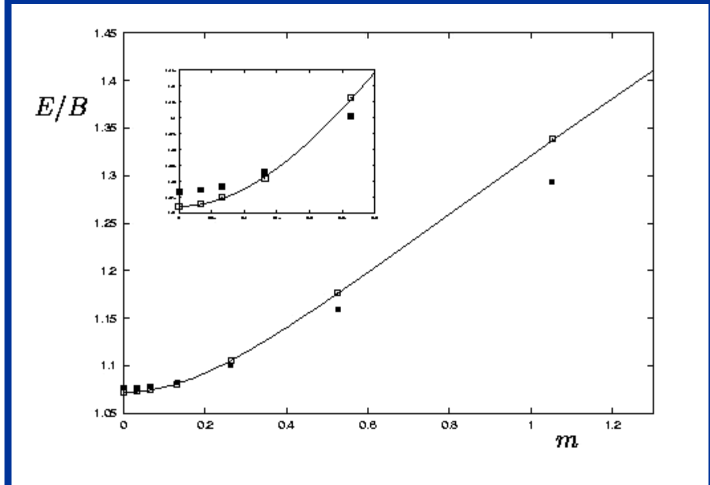
(Icosahedral Skyrmions)

Crystalline structure of nucleons

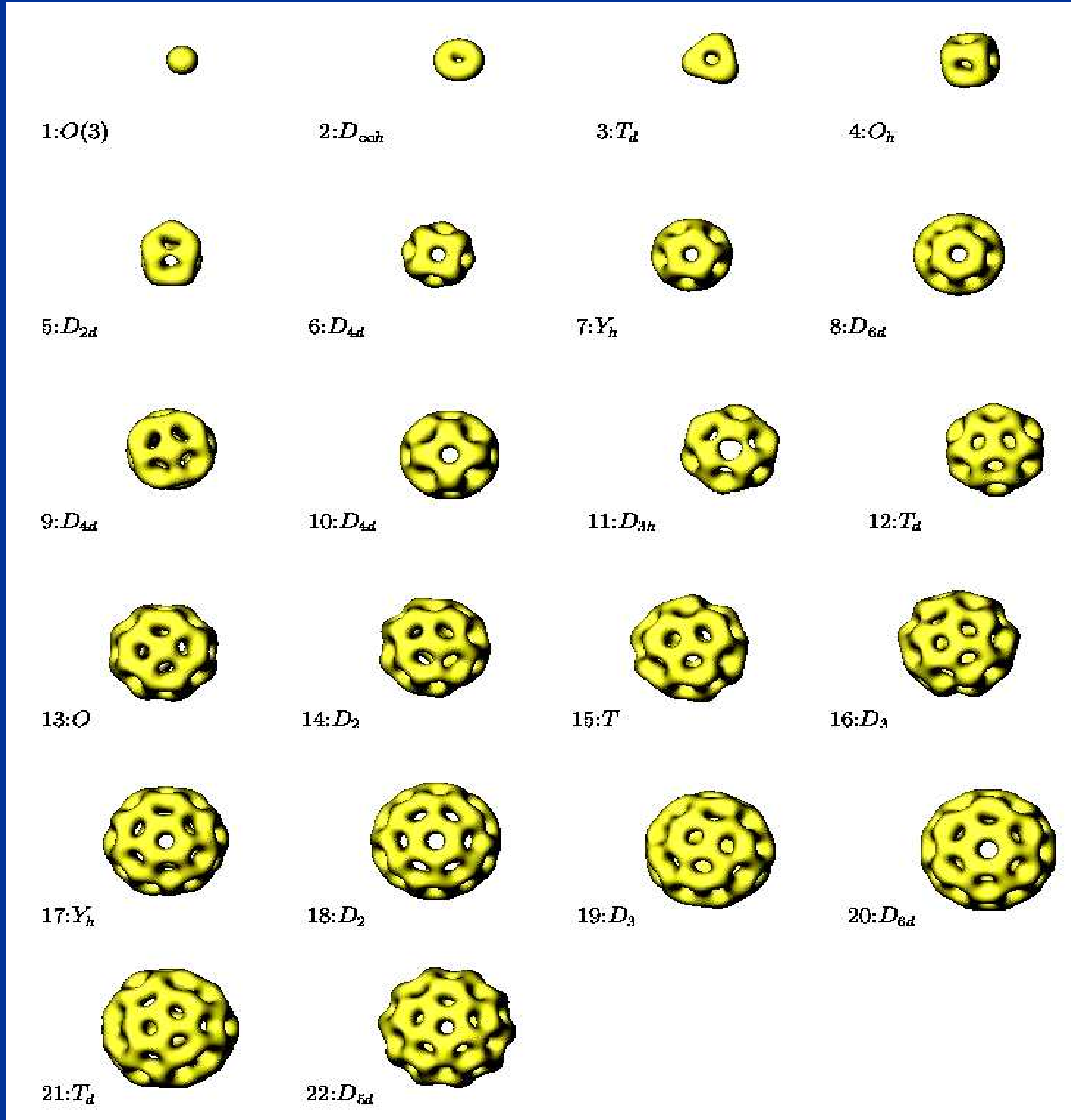
R. A. Battye, N. S. Manton, C.Houghton
and P. Sutcliffe (1996,2004)



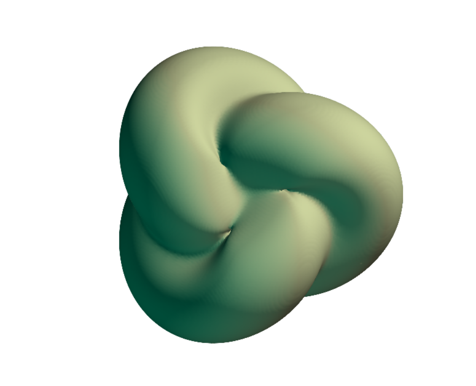
Shell vs. Crystal



Shell wins for m c 0.16



Faddev-Skyrme model



- Carrier field: *3D* unit vector field ϕ^a in R_3, locally smooth
- *3D* unit vectors can be fepresented by points on the sphere S_2
- **Hopf map:** ϕ**:** $S_3 \to S_2$, the homotopy class $\pi_3(S_2) = \mathbb{Z}$
- $\phi^a = (0, 0, 1)$ at infinity (in any direction)
- $\phi^a = (0, 0, -1)$ on the ring $x^2 + y^2 = 1$, $z = 0$ (vortex core).

The Hopf charge:

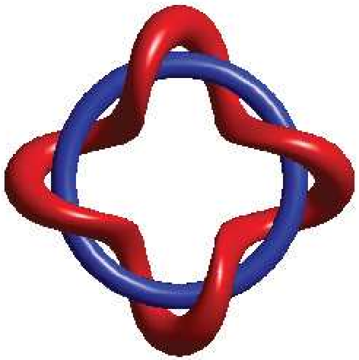
$$Q = \frac{1}{16\pi^2} \int d^3x \; \varepsilon_{ijk} \; A_i \; F_{jk}$$

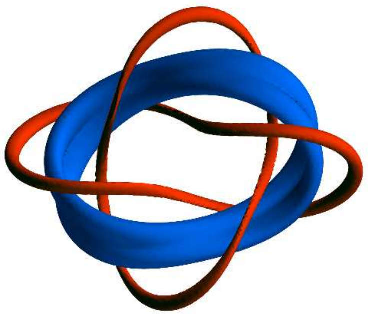
Given ϕ: $\to S_2$ define $F_{ij} = \epsilon_{abc} \, \phi^a \, \partial_i \, \phi^b \, \partial_j \, \phi^c$

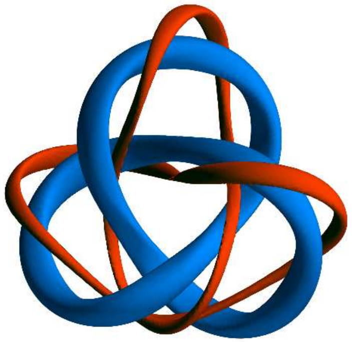
Given F_{ij} construct potential $F_{ij} = \partial_i A_j - \partial_j A_i$

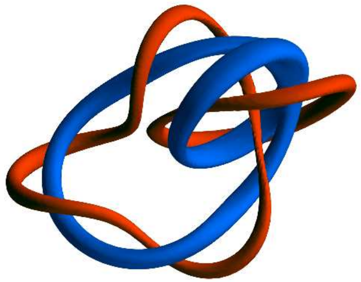
Faddeev-Skyrme Lagrangian:

$$L = \frac{1}{2}(\partial_i \phi^a)^2 - \frac{\kappa^2}{4} \left(\varepsilon_{abc} \; \phi^a \partial_i \phi^b \partial_j \phi^c\right)^2$$

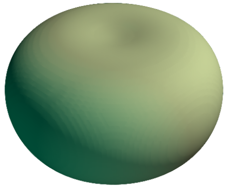




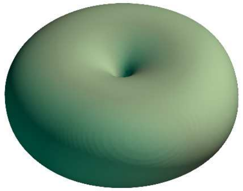




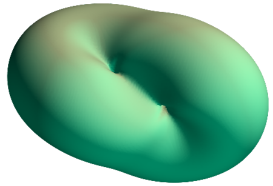
Solitons of the Faddeev-Skyrme model



Q=1 $1\mathcal{A}_{1,1}$

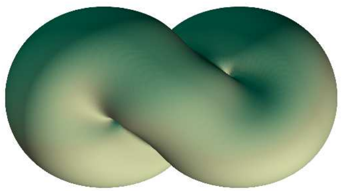


Q=2 $2\mathcal{A}_{2,1}$

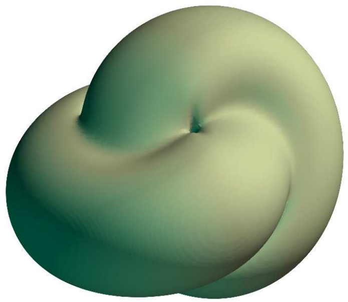


Q=3 $3\widetilde{\mathcal{A}}_{3,1}$

Q=4 $4\mathcal{A}_{2,2}$

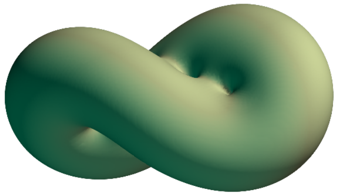


Q=4 $4\widetilde{\mathcal{A}}_{4,1}$

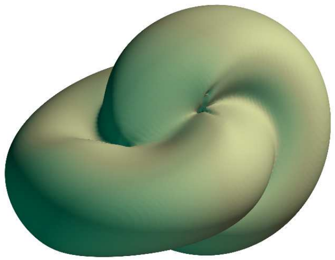


Q=4 $4\mathcal{L}_{1,1}^{1,1}$

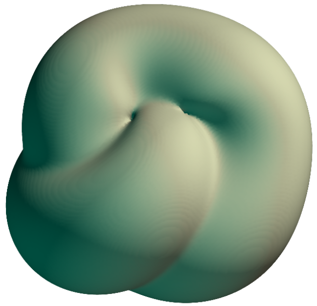
Solitons of the Faddeev-Skyrme model



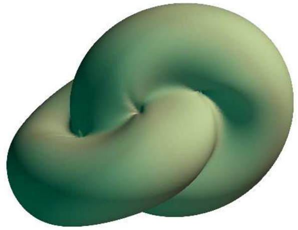
Q=5 $5\widetilde{\mathcal{A}}_{5,1}$



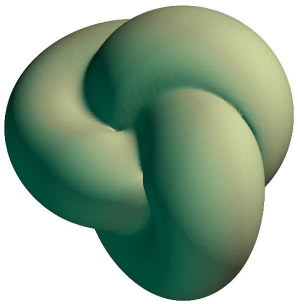
Q=5 $5\mathcal{L}^{1,2}_{1,1}$



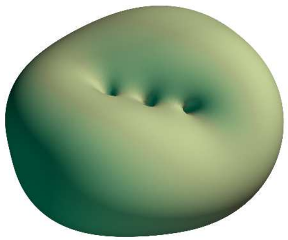
Q=6 $6\mathcal{L}^{2,2}_{1,1}$



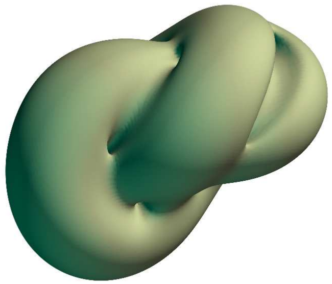
Q=6 $6\mathcal{L}^{3,1}_{1,1}$



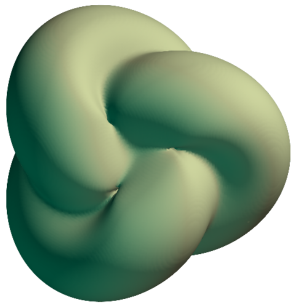
Q=7 $7\mathcal{K}_{3,2}$



Q=8 $8\widetilde{\mathcal{A}}_{4,2}$

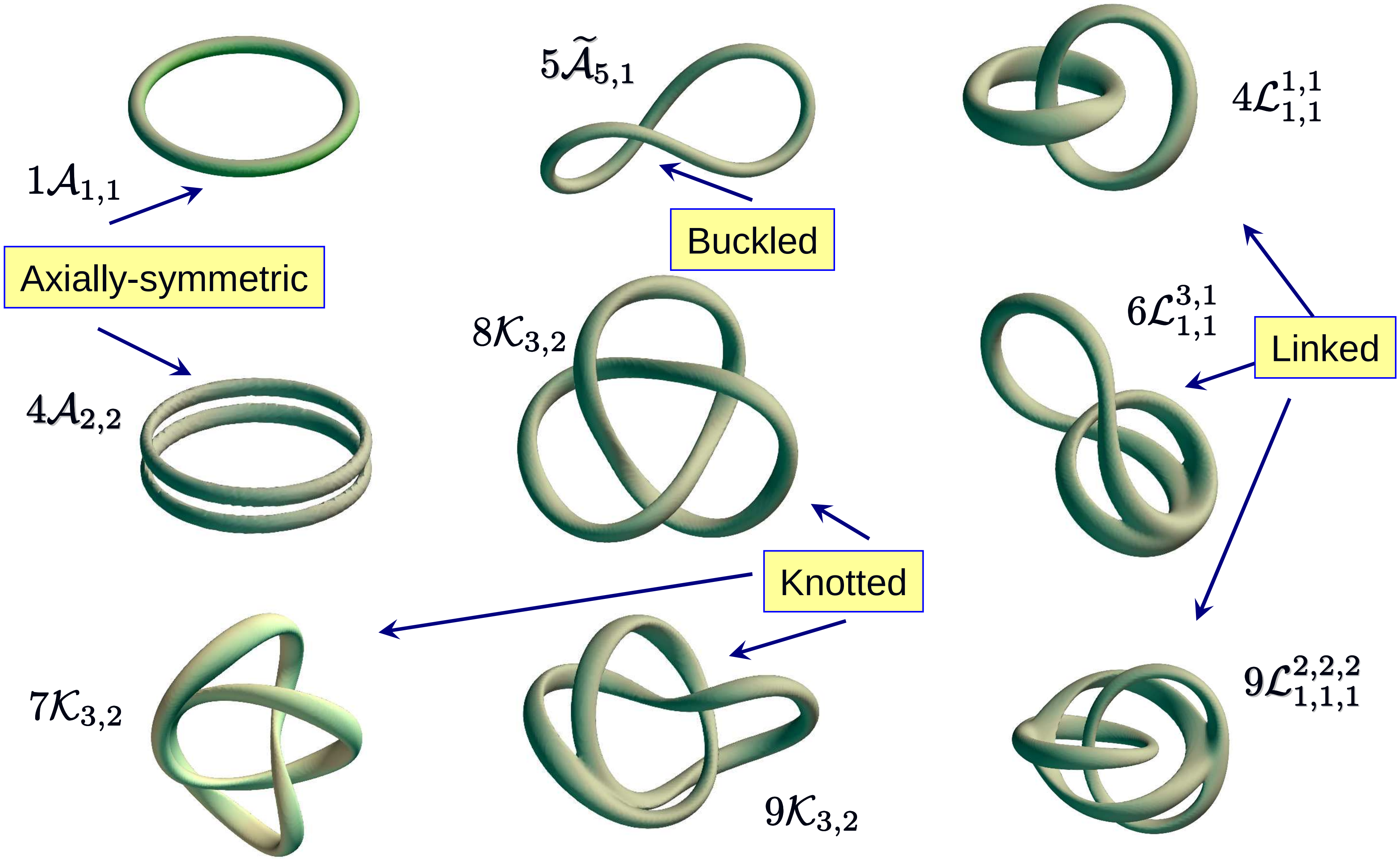


Q=8 $8\mathcal{L}^{3,3}_{1,1}$

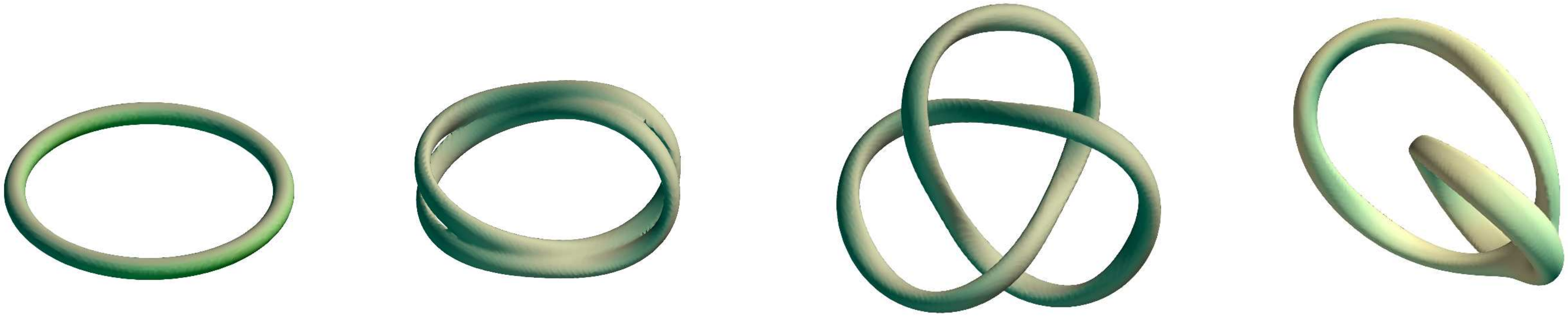


Q=8 $8\mathcal{K}_{3,2}$

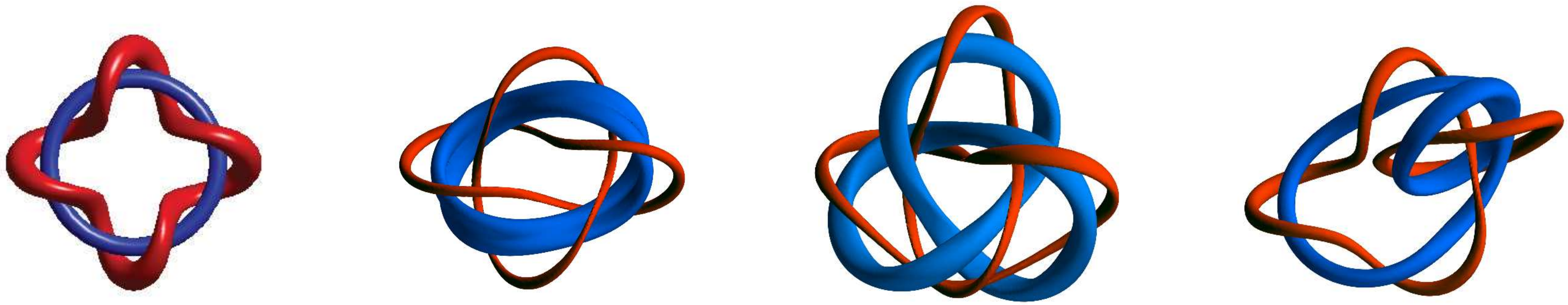
Buckled, linked and knotted hopfions



Position curves and linking numbers



$4\ \mathcal{A}_{4,1}$ $8\widetilde{\mathcal{A}}_{4,2}$ $8\mathcal{K}_{3,2}$ $6\mathcal{L}_{3,1}^{1,1}$



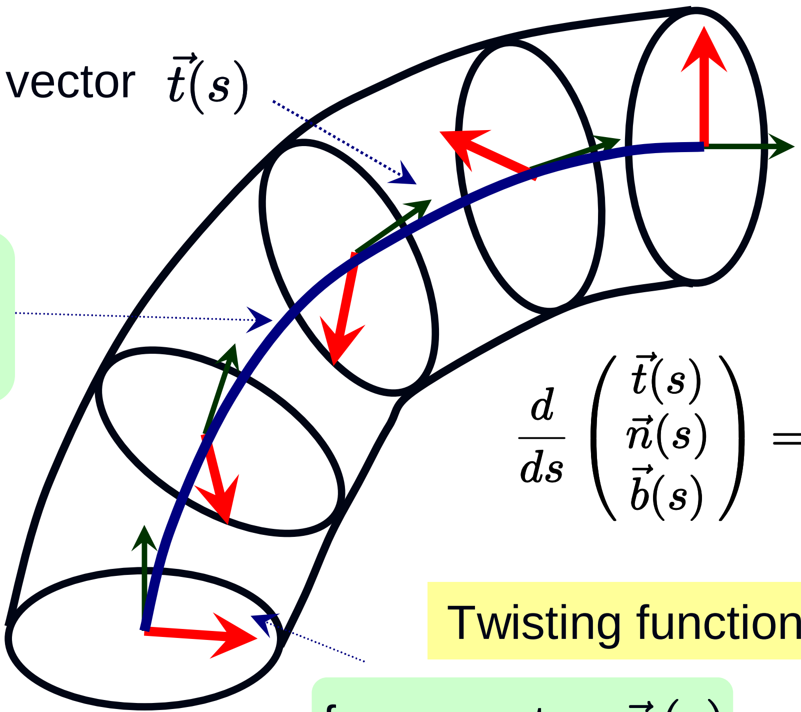
Elastic rod approximation

D. Harland, J.M. Speight and P.M. Sutcliffe
Phys. Rev.D83 (2011) 065008

Tubular coordinates:

- Arclength parameter $s \in [0, L]$
- Polar coordinates ρ, θ in the disk

- Torsion $\tau(s)$
- Curvature $\kappa(s)$



Frenet frame

$$\vec{m}(s) = \vec{n}(s)\sin\alpha(s) + \vec{b}(s)\cos\alpha(s)$$

$$\frac{d}{ds}\begin{pmatrix}\vec{t}(s)\\ \vec{n}(s)\\ \vec{b}(s)\end{pmatrix} = \begin{pmatrix}0 & \kappa(s) & 0\\ -\kappa(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{pmatrix}\begin{pmatrix}\vec{t}(s)\\ \vec{n}(s)\\ (s)\end{pmatrix}$$

Twisting function:

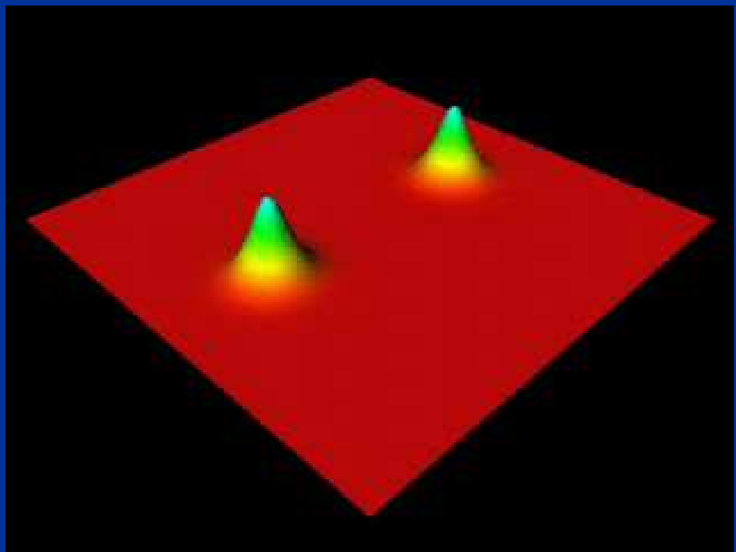
$$\alpha(s) = \vec{t} \cdot \vec{m}' \times \vec{m}$$

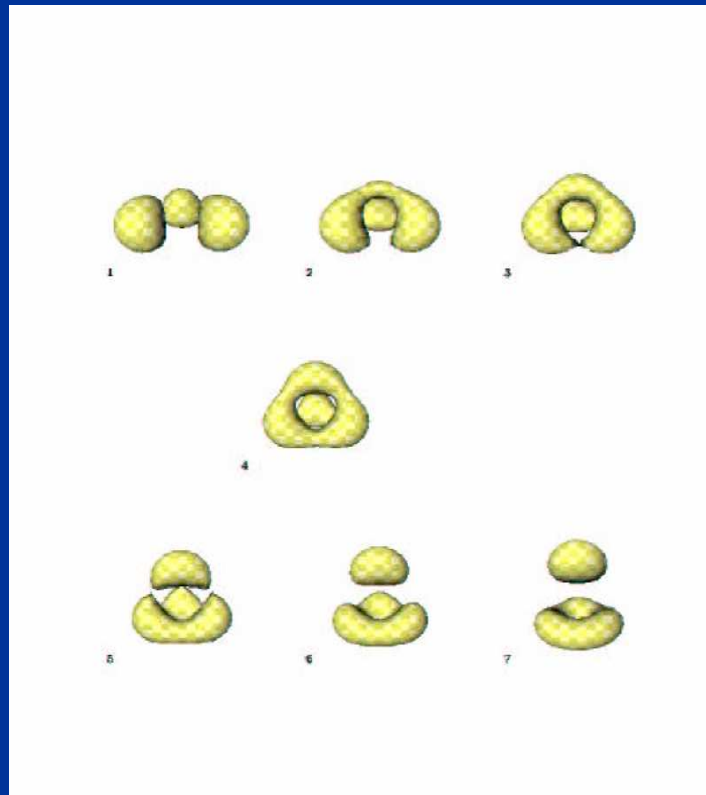
$$\alpha(L) = \alpha(0) + 2\pi N$$

Faddeev-Skyrme effective energy functional:

$$E = \int \left(A + B\kappa^2 + C(\alpha' - \tau)^2\right) ds$$

Dynamics of the solitons: moduli space





(Picture by courtesy of Paul Sutcliffe)