CARL VON OSSIETZKY universität OLDENBURG

University of Oldenburg and BSU Minsk

Introduction to Solitons

Ya Shnir

Institute of Theoretical Physics and Astronomy
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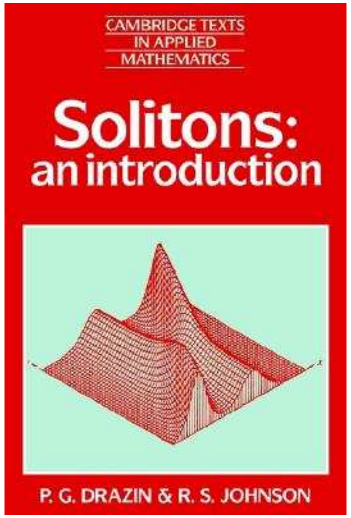
Outline

- Historical remarks
- Linear waves. D'Alembert's solution. Dispersion relations.
- The Burger's equation. The Korteweg and de Vries equation.
- The scattering and inverse scattering problems.
- The nonlinear Schrodinger equation, its soliton solutions.
- Hirota's method and Bäcklund transformations.
- Further integrable non-linear differential equations, the Lax formulation.
- The Fermi-Pasta-Ulam problem.
- Models for dislocations in crystals. The sine-Gordon equation.
- Kink soliton solutions in $\lambda\phi^4$ model
- Idea of topological classification
- Sigma-model, Baby Skyrmions, Skyrmions and Hopfions
- Magnetic monopoles

References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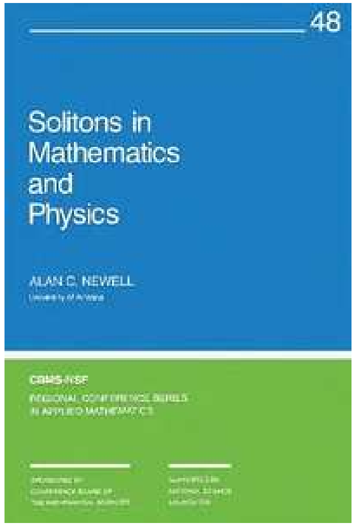
Solitons: an introduction
P. G. DRAZIN & R. S. JOHNSON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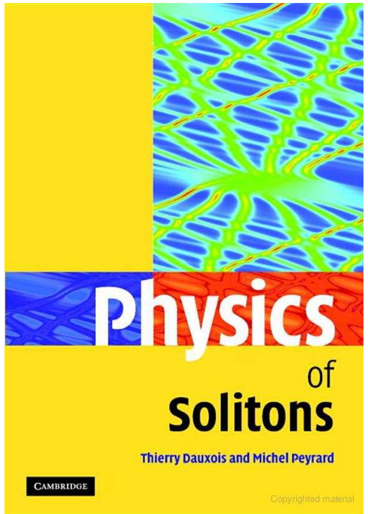
Solitons in Mathematics and Physics
ALAN C. NEWELL
CBMS-NSF


Physics of Solitons
Thierry Dauxois and Michel Peyrard
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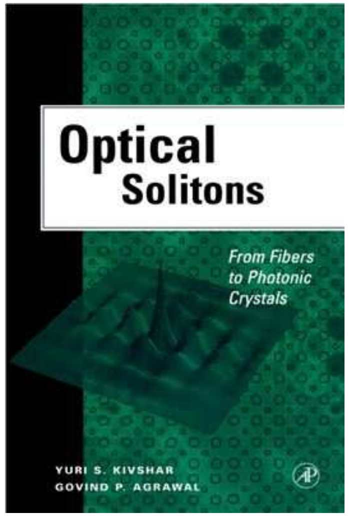
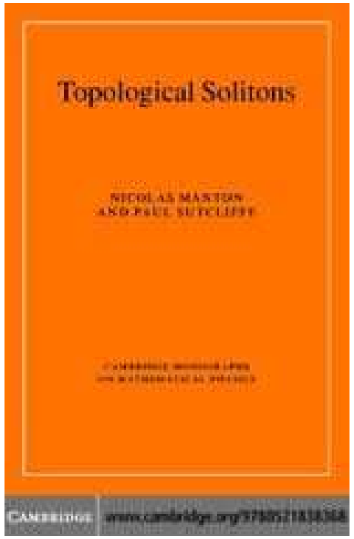

Optical Solitons
From Fibers to Photonic Crystals
YURI S. KIVSHAR
GOVIND P. AGRAWAL


Topological Solitons
NICOLAS MANTON AND PAUL SUTCLIFFE
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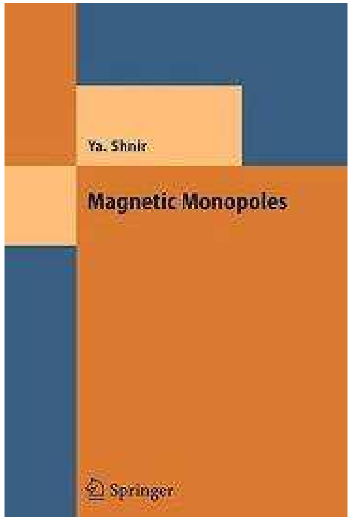
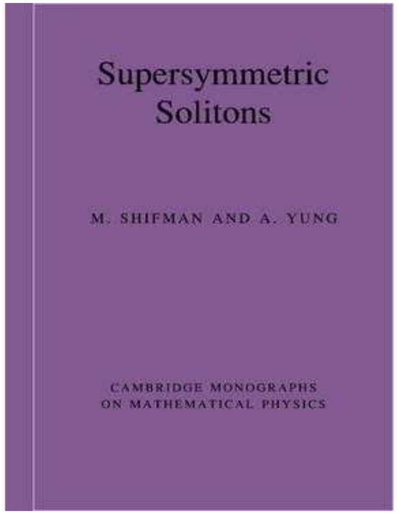
SOLITONS AND INSTANTONS
An Introduction to Solitons and Instantons in Quantum Field Theory


Ya. Shnir
Magnetic Monopoles
Springer


Supersymmetric Solitons
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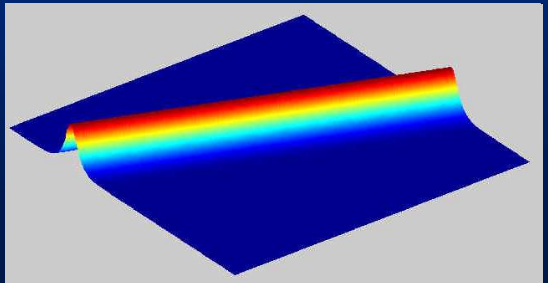
Solitons and lumps in non-linear physics

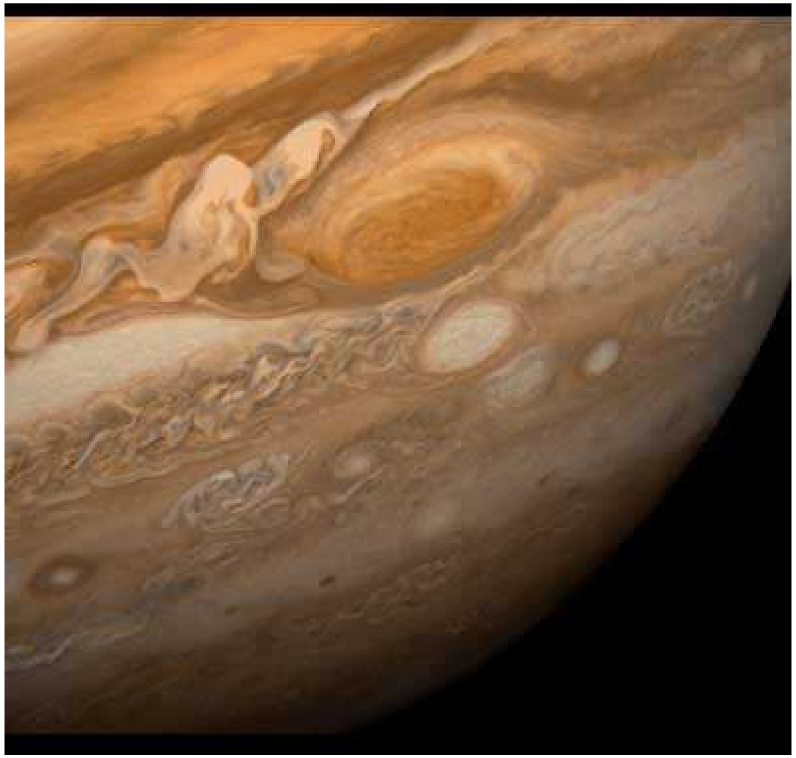
- Solitons, knots, vortons and sphalerons in the electroweak and strong interactions, caloron solutions in QCD, Q-balls, black holes, fullerenes and non-linear optics, etc...

<u>Soliton:</u> This is a solution of a nonlinear partial differential equation which represent a solitary travelling wave, which:

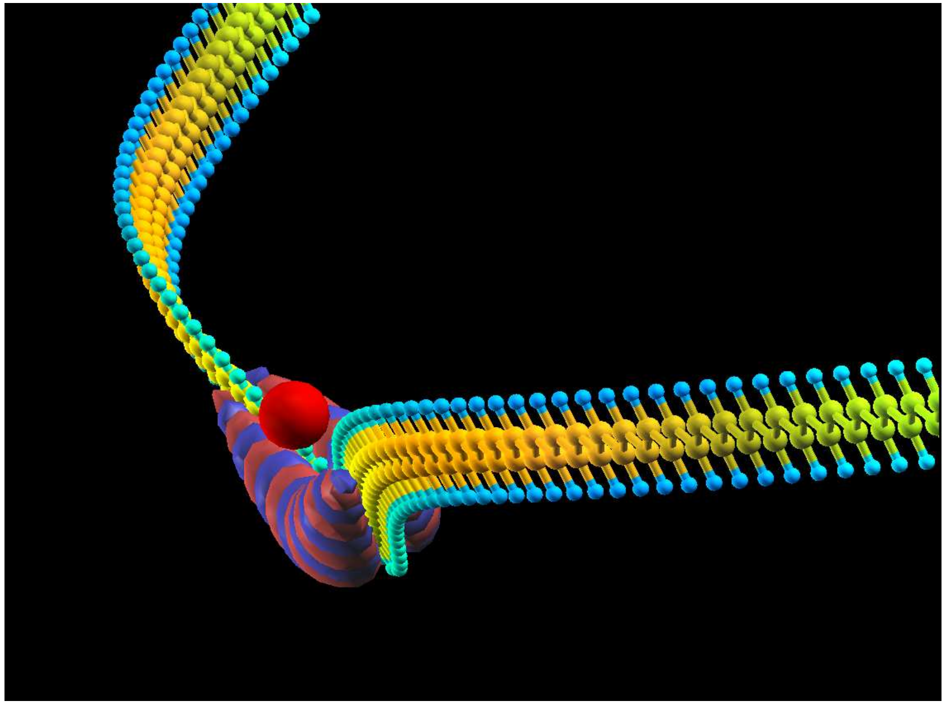
- **Has a permanent form;**
- **It is localised within a region;**
- **It does not obey the superposition principle;**
- **It does not disperse.**



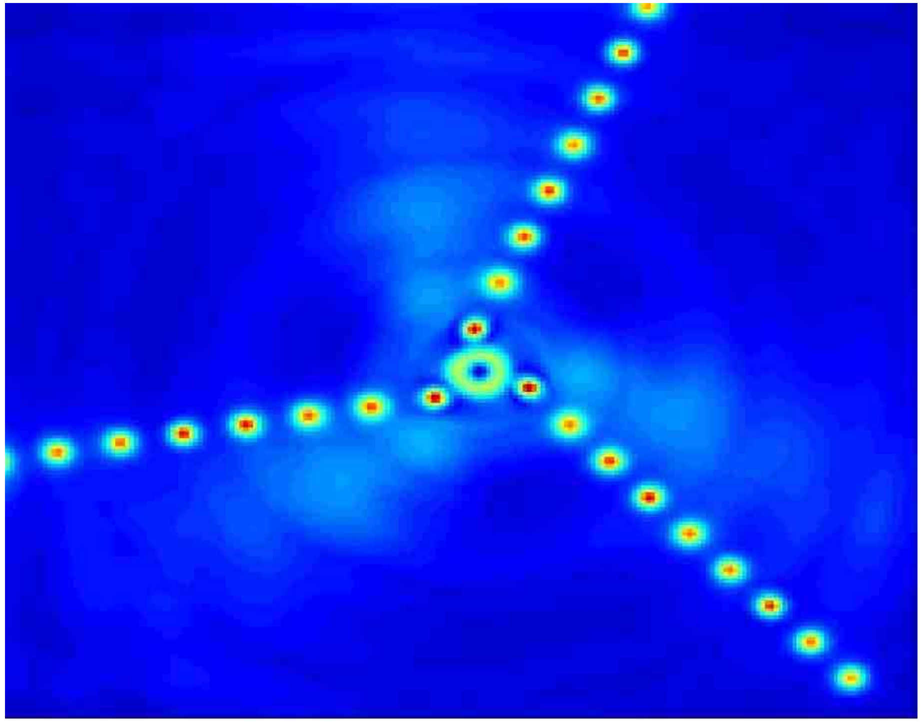
Optical fibres → NLSE
Josephson junctions → sine-Gordon model
Bose-Einstein condensate → Skyrme model
Superconductivity → Abrikosov-Nielsen-Olesen model

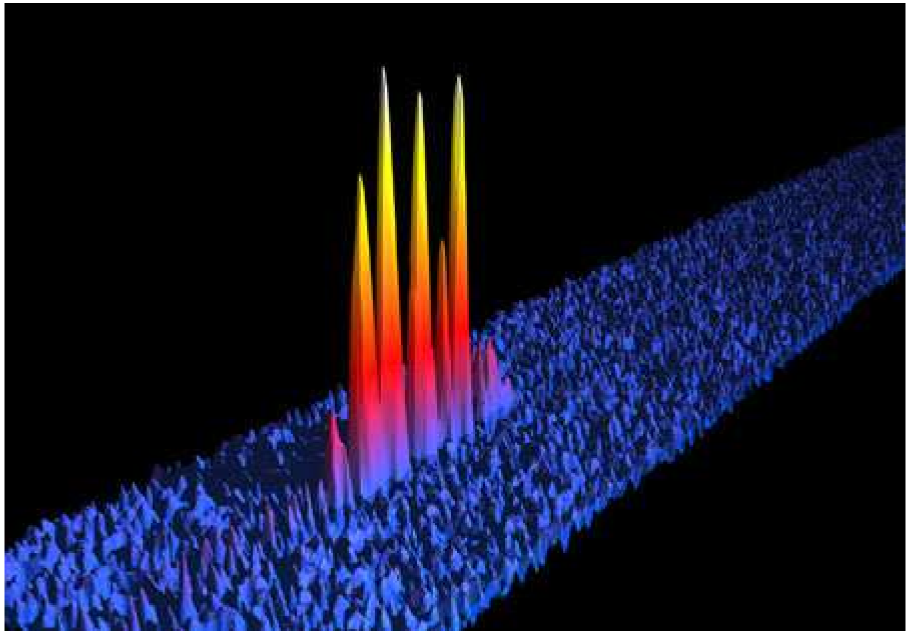


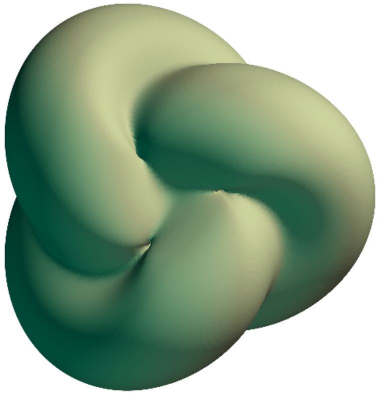
Jupiter's Great Red Spot – vortex soliton

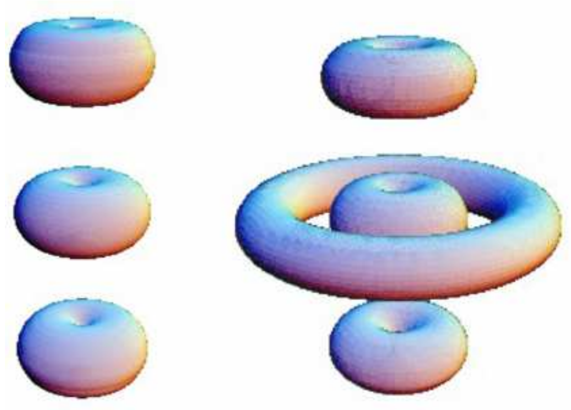


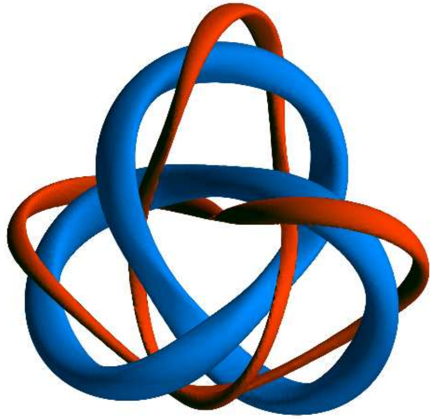
Soliton wavefunction of trans-polyacetylene doped by a counter ion – kink soliton

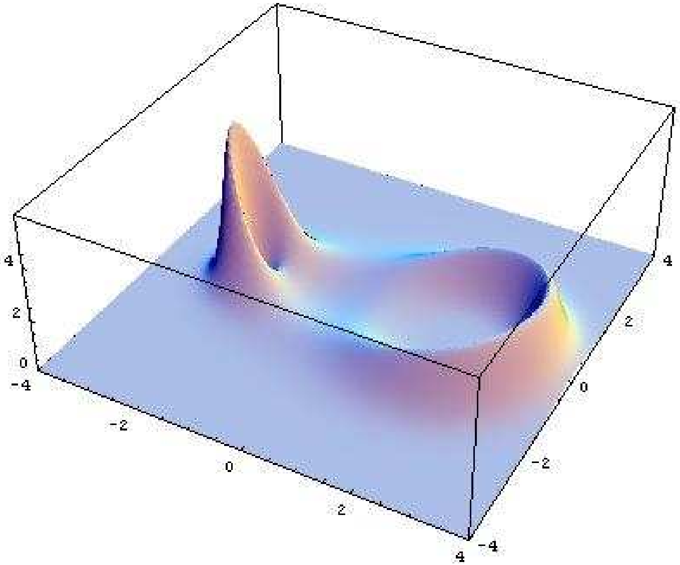









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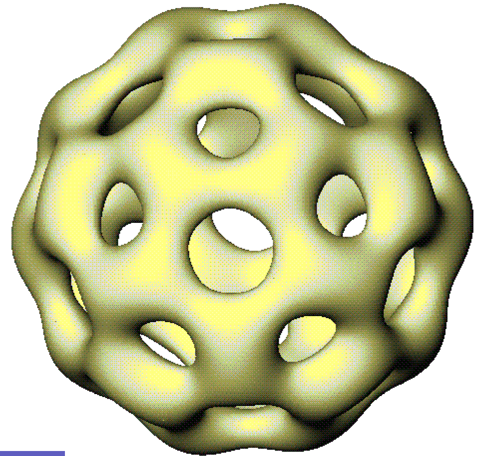
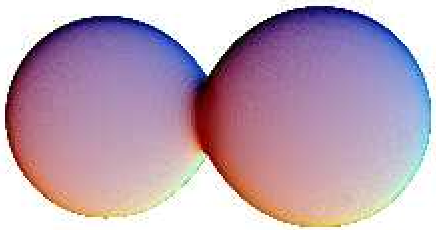
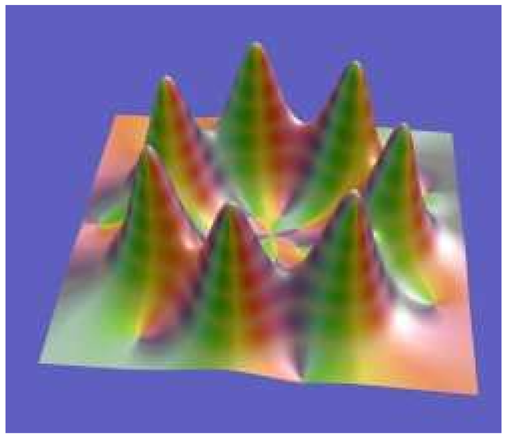
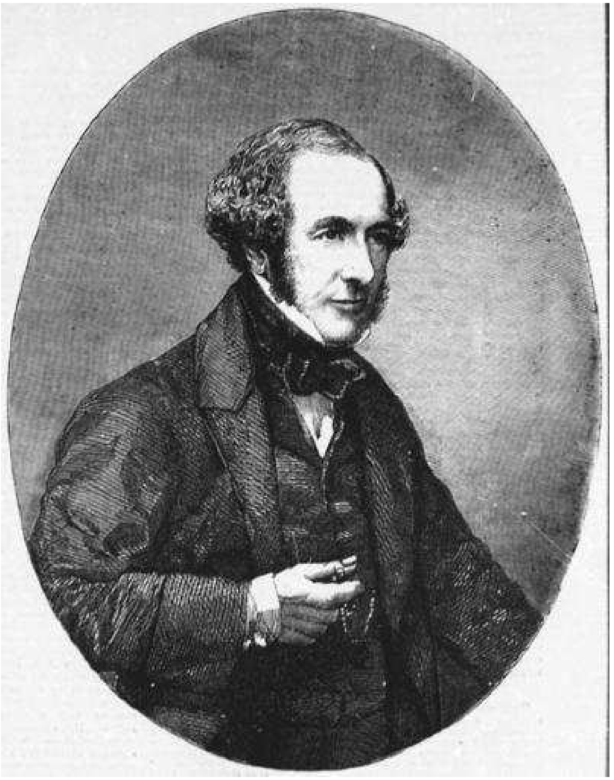






J S Russel observation of Solitary Waves

John Scott Russell (1808-1882) - engineer, naval architect and shipbuilder



S.S. Great Eastern
(1858)

Union Canal at Hermiston, Scotland

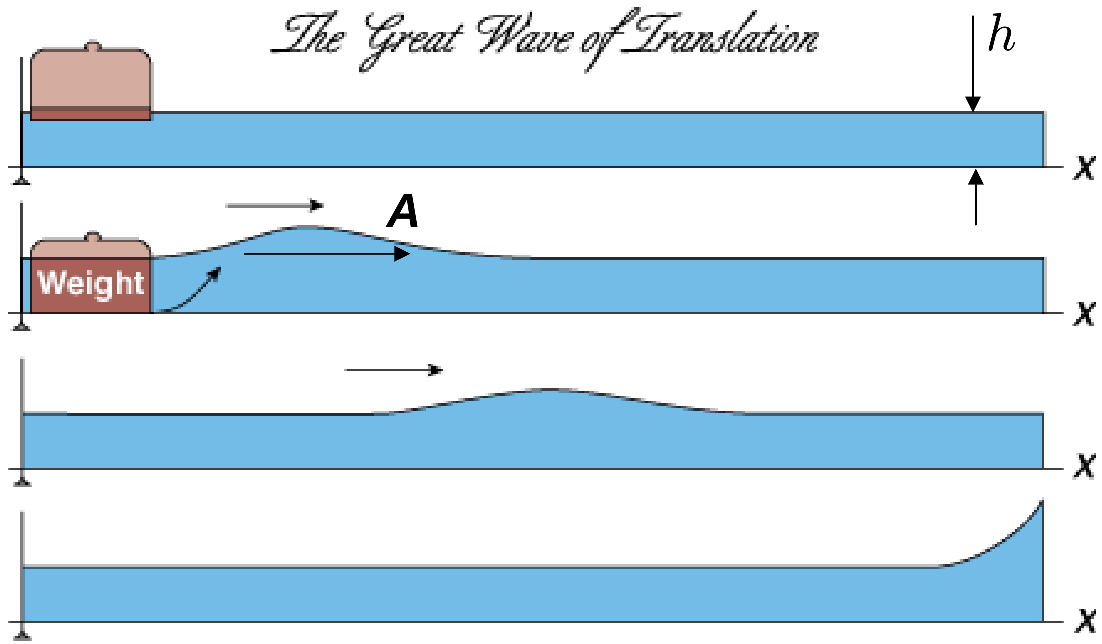
John Scott Russel Aqueduct

“I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed…”

This is impossible !!!

George Airy:

- Unconvinced of the Great Wave of Translation
- Consequence of linear wave theory?



Observation by J Scott Russel:

$$c^2 = g(h + A)$$

G. G. Stokes:

-Doubted that the solitary wave could propagate without change in form

Boussinesq (1871) and Rayleigh (1876):

- Discovered a correct nonlinear approximation theory

Linear wave equations

Simplest (second order) linear wave equation:

$$\boxed{\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0}$$

- There is no dissipation
- There is no dispersion

D'Alembert's solution $u(x,t) = f(x-ct) + \cancel{g(x+ct)}$ ***f, g are arbitrary functions***

Harmonic wave solution $u(x,t) = e^{i(kx-\omega t)}$ $\Rightarrow$ $k - \omega = 0$

(Dispersion relation)

rescaling ***t→ct:*** $\boxed{u_t + u_x = 0}$

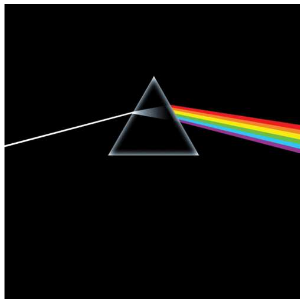
Less simple linear wave equation with dispersion:

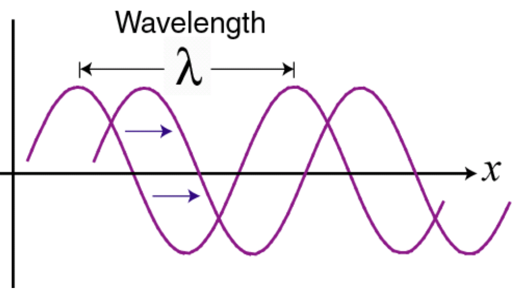
$$\boxed{u_t + u_x + u_{xxx} = 0}$$

$$k - k^3 - \omega = 0 \Rightarrow kx - \omega t = k[x - (1-k^2)t]$$

Phase velocity depends on the wave number:

$$v_p = \frac{\omega}{k} = 1 - k^2$$



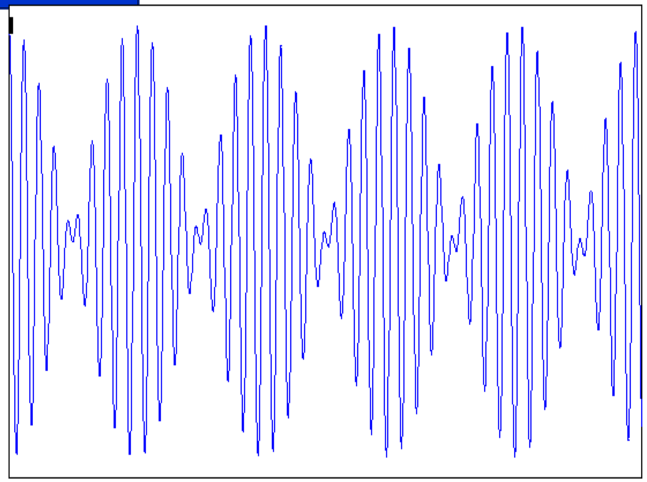


Dispersion and Dissipation

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} A(k) e^{i(kx-\omega t)} dk}$$

$$\omega(k) = k - k^3$$

group velocity: $v_g = \frac{d\omega}{dk} = 1 - 3k^2$

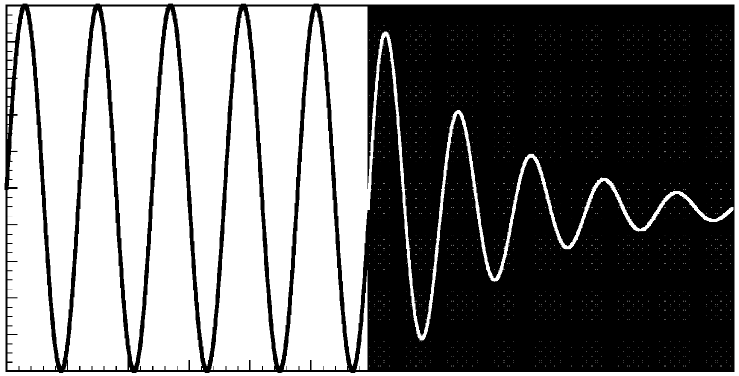


Question: If the dispersion finction $\omega(k)$ is always real ?

Less simple linear wave equation with dissipation:

$$\boxed{u_t + u_x - u_{xx} = 0} \Rightarrow k - \omega - ik^2 = 0 \Rightarrow u(x,t) = e^{ik(x-t)} \boxed{e^{-k^2 t}}$$

The wave decays exponentially!



Odd powers in spatial derivatives $\longrightarrow$ dispersion

Even powers in spatial derivatives $\longrightarrow$ dissipation

Non-linear wave equations

Simple Non-linear wave equation: $u_t + u_x + \underline{uu_x} = u_t + (1+u)u_x = 0$

Propagation velocity depends on the wave profile: $c = 1 + u$

General non-linear wave solution: $u(x,t) = f[x - (1+u)t]$

However:

- An explicit solution $u(x,t)$ can be not a single-valued function of x
- The solution becomes sharp at leading and trailing edges (shock wave formation)
- There is no superposition of the solutions

How about the dispersion/dissipation?

- Korteweg- de Vries equation (1895) (non-linear + dispersion)
- Burgers equation (1906) (non-linear + dissipation)

$$u_t + (1+u)u_x + u_{xxx} = 0$$

$$u_t + (1+u)u_x - u_{xx} = 0$$

KdV equation: Solution

Reparametrisation I:

$$1+u \to \alpha u, \quad t \to \beta t, \quad x \to \gamma x \quad \Rightarrow \quad u_t + \frac{\alpha\beta}{\gamma} u u_x + \frac{\beta}{\gamma^3} u_{xxx} = 0$$

Let us consider a particular case $\boxed{u_t + 6 u u_x + u_{xxx} = 0}$

Reparametrisation II: **We are looking for solutions of the type** $u(x,t) = u(x - vt)$

$$u \equiv u(\theta) \quad \text{where} \quad \theta = x - vt \quad \Rightarrow \quad \begin{cases} u_t = \dfrac{du}{d\theta}\dfrac{d\theta}{dt} = -vu' \\ u_x = \dfrac{du}{d\theta}\dfrac{d\theta}{dx} = u', \quad u_{xxx} = u''' \end{cases}$$

$$-vu' + 6uu' + u''' = \frac{d}{d\theta}\left(-vu + 3u^2 + u''\right) = 0$$

$$u'' + 3u^2 - vu + C_1 = 0 \quad \Big| \quad \times \text{ integrating factor } u'$$

$$\boxed{\frac{(u')^2}{2} + u^3 - \frac{v}{2}u^2 + C_1 u = C_2}$$

It looks like an equation of motion of a “particle”

in the “potential” $V_{eff} = u^3 - \dfrac{v}{2}u^2 + C_1 u$

KdV equation: Solitons

$$\frac{(u')^2}{2} + u^3 - \frac{v}{2}u^2 + C_1 u = C_2$$

Boundary conditions: $u = u' = 0, \quad \text{as} \quad \theta \to \pm\infty$

$$C_1 = C_2 = 0, \quad d\theta = \frac{du}{\sqrt{vu^2 - 2u^3}}$$

Separation of variables

$$\theta - \theta_0 = \frac{1}{\sqrt{v}} \int_{u_0}^{u} \frac{du}{u\sqrt{1 - \frac{2u}{v}}}$$

Soliton solution of the KdV equation:

$$\boxed{u(\theta) = \frac{v}{2}\operatorname{sech}^2\left(\frac{\sqrt{v}\theta}{2}\right)}$$

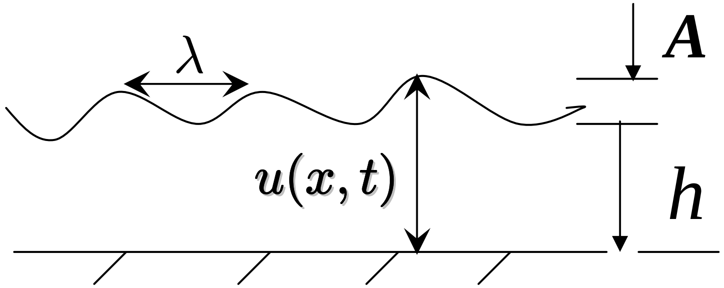
Note:

- $A = \dfrac{v}{2}$ -- amplitude is proportional to the velocity of propagation while the "width" is proportional to $\sqrt{v}$
  - taller solitary waves are thinner and move faster
- There is another solution: $u(\theta) = -\dfrac{v}{2}\operatorname{csch}^2\left(\dfrac{\sqrt{v}\theta}{2}\right)$ - singularity at $x = vt$
- More general solutions can be found for other choices of C_1 and C_2
- KdV equation has multisoliton solutions
- There is ***anti-soliton*** solution of the another KdV equation obtained by replacing ***u*** $\to$ -***u*** **:** $u_t - 6uu_x + u_{xxx} = 0$
- KdV equation is not Lorentz-invariant

KdV equation: shallow water waves

Assumptions:

- amplitude of the waves is small w.r.t. water depth, ***A/h < 1***
- Long waves on shallow water: ***h << λ***
- Nearly 1d motion
- Unrotated incompressible inviscid liquid



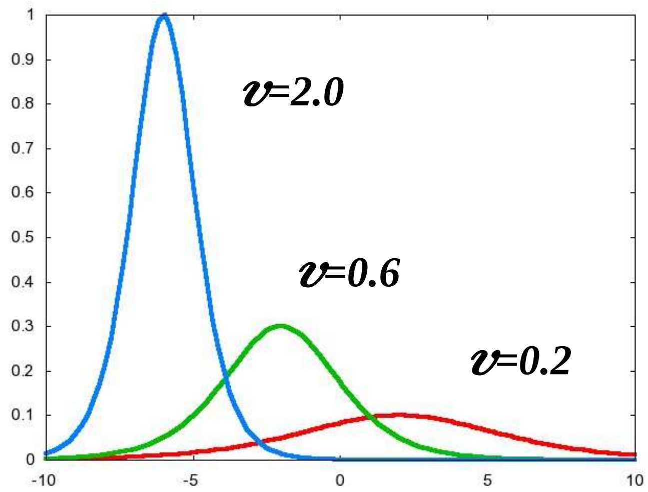
The Euler equation for such a system bounded by the rigid plane (bottom) and by a free surface from above:

$$\boxed{u_t = \frac{3}{2}\sqrt{\frac{g}{h}}\left(\frac{2}{3}\epsilon u_x + u u_x + \frac{1}{3}\sigma u_{xxx}\right)}$$

Ratio $\epsilon = A/\lambda$

surface tension

$$\sigma = \frac{1}{3}h^3 - \frac{Th}{g\rho}$$



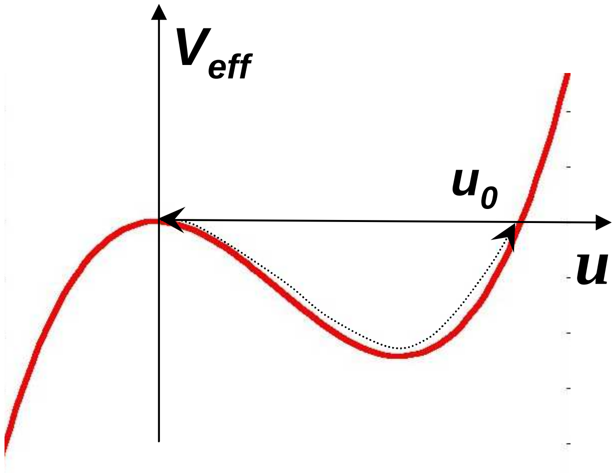
Note: If the depth $h \gg A$ the equation is reduced

$$u_t = c u_x$$

KdV equation: Boundary conditions and other solutions

total "energy"

$$\frac{(u')^2}{2} + V_{eff} = C_2; \qquad V_{eff} = u^3 - \frac{v}{2}u^2 + C_1 u$$



- **Solitary wave:** $C_1 = C_2 = 0$

The function ***V(u)*** has a local maximum at ***u=0***

The soliton solution corresponds to the 'motion' from $V(0)$ at $\theta \to -\infty$ to $V(u_1)$ at $\theta \to \infty$

- **Cnoidal wave:** a general solution, $C_1, C_2 \neq 0$

$$\left(\frac{du}{d\theta}\right)^2 = 2C_2 - 2u^3 + vu^2 - 2C_1 u = 2(u_1 - u)(u_2 - u)(u_3 - u)$$

Assume that $u_1 < u_2 < u_3$ $\Rightarrow$ $\frac{du}{d\theta} = \pm\sqrt{2(u_1 - u)(u_2 - u)(u_3 - u)}$

The cubic $P(u) = 2(u_1 - u)(u_2 - u)(u_3 - u) > 0$ for $u_2 < u < u_3$

Reparametrization:

$$\theta = \pm \int_{u_2}^{u} \frac{du}{\sqrt{P(u)}} = \pm\sqrt{\frac{2}{u_3 - u_1}} \int_0^{\phi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}}$$

$$k^2 = \frac{u_3 - u_2}{u_3 - u_1}; \qquad u = u_3 - (u_3 - u_2)\sin^2\varphi$$

KdV equation: Cnoidal waves and soliton lattice

Cnoidal wave solution:

$$u(\theta) = u_3 - (u_3 - u_2)\,\mathrm{sn}^2(\eta); \qquad \eta = \sqrt{\frac{u_3 - u_1}{2}}\theta$$

Jacobi elliptic function

What does it mean?

$$\xi(\phi, k) = \int_0^{\phi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}} \Rightarrow \begin{cases} \mathrm{sn}(\xi, k) = \sin\phi; \\ \mathrm{cn}(\xi, k) = \cos\phi \end{cases}$$

$$\mathrm{sn}^2(\xi, k) + \mathrm{cn}^2(\xi, k) = 1$$

For $\phi = 2\pi$

Complete elliptic integral

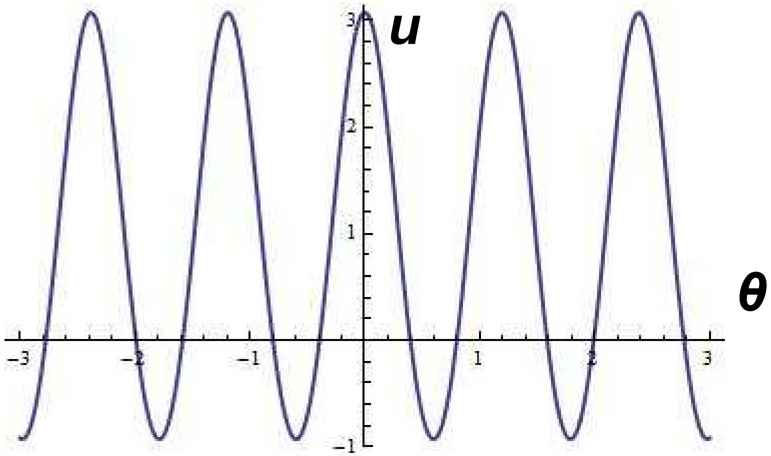
$$Z = \int_0^{2\pi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}} = 4\mathcal{K}(k)$$

For ***k=0*** $\begin{cases} \mathrm{sn}(\xi, 0) = \sin\xi; \\ \mathrm{cn}(\xi, 0) = \cos\xi \end{cases}$

Trigonometric limit

Soliton lattice:

$$\lambda = \frac{2\sqrt{2}\mathcal{K}(k)}{\sqrt{u_3 - u_1}}$$



Note: If the limit k=1 the soliton solution ***sech(ϕ)*** is recovered

Linear transport equation

Simplest ***non-linear PDE equation***
(dispersionless KdV equation)

$$\boxed{u_t + u u_x = 0}$$

- ***Poisson and Riemann (1820s)***

Definition: the solutions to the PDE are constant on the characteristic curves $x(t)$

The characteristic curves are the level sets of the characteristic variables

- **Example I** - ***Linear transport equation*** $u_t + c u_x = 0 \Rightarrow \frac{dx}{dt} = c = \ const$

The solutions are travelling waves $u \equiv u(\theta)$ where $\theta = x - ct$ **characteristic variable**

- **Example II** - ***Linear transport equation*** $u_t - x u_x = 0 \Rightarrow \frac{dx}{dt} = -x$

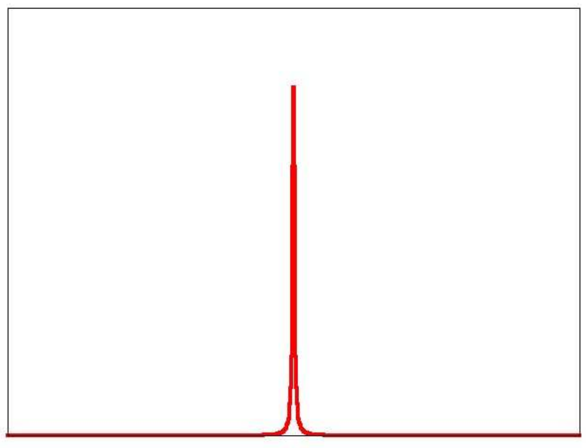
The characteristic variable is $\theta = x e^t$

The general solution: $u(x,t) = f(\theta)$

The initial data $u(0,x) = f(x)$, e.q. $u(0,x) = \dfrac{1}{1+x^2}$

Then

$$u(x,t) = 1/(1+(xe^t)^2) = \frac{e^{-2t}}{x^2 + e^{-2t}}$$



Non-linear transport equation

$$u_t + uu_x = 0$$

Definition: the solutions to the PDE are constant on the characteristic curves which are solutions to the autonomous ODE

$$\frac{dx}{dt} = u(x,t)$$

Note: $\frac{du}{dt} = u_x \frac{dx}{dt} + u_t = uu_x + u_t = 0 \Rightarrow \frac{dx}{dt} = u(x_0, 0) = \ const$

The characteristic curve must be a straight line: $x = x_0 + u(x_0, 0)t$

The characteristic variable is $\theta = x - ut$ The general solution is $u(x,t) = f(\theta)$

- **Example III** - ***Non-Linear transport equation has a solution*** $f(\theta) = \alpha\theta + \beta$

Then $u(x,t) = \alpha(x - ut) + \beta \Rightarrow u(x,t) = \frac{\alpha x + \beta}{1 + \alpha t}$

There is a problem:

- Straight lines may have a different slope, so they may cross…

- **First (trival) scenario:** all characteristic lines are parallel $\longrightarrow$ $u = \ const$
- **Second (less trival) scenario:** the function $f(\theta)$ increases monotonically, all characteristic lines never cross as ***t > 0*** $\longrightarrow$ *rarefaction* wave
- **Third (non-trival) scenario:** $f'(\theta) < 0$ **-** there are multiply-valued solutions

$\longrightarrow$ *shock wave* formation

Lecture 1: Summary

- **KdV equation:**

$$u_t + 6uu_x + u_{xxx} = 0$$

Non-linearity

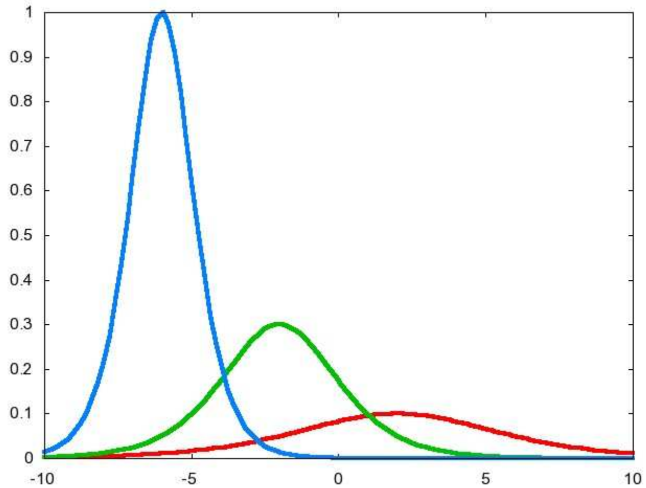
Dispersion

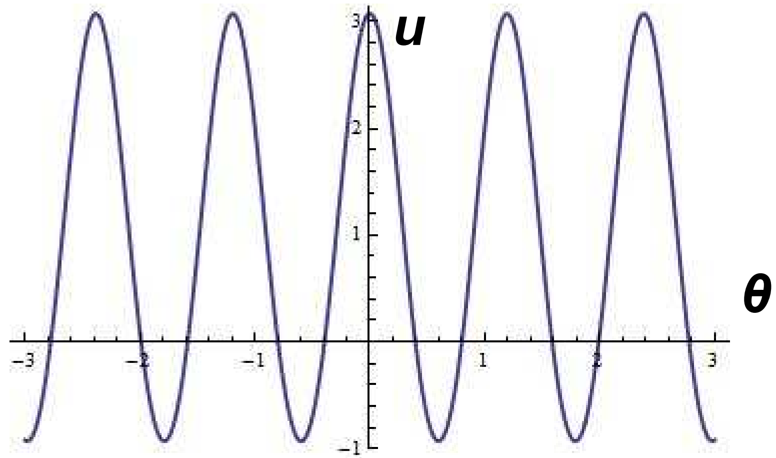
- **Soliton solution of the KdV equation:**

$$u(\theta) = \frac{v}{2}\mathrm{sech}^2\left(\frac{\sqrt{v}\theta}{2}\right)$$

- **Cnoidal wave solution of the KdV equation:**

$$u(\theta) = u_3 - (u_3 - u_2)\ \mathrm{sn}^2(\eta); \quad \eta = \sqrt{\frac{u_3 - u_1}{2}}\theta$$





Burgers equation: Solution

Simplest non-linear equation with viscous term (diffusion):

$$u_t + uu_x - \gamma u_{xx} = 0$$

- **Traveling wave solution:**

$$u = u(\theta), \quad \theta = x - vt$$

$$\frac{\partial u}{\partial x} = u'; \quad \frac{\partial^2 u}{\partial x^2} = u''; \quad \frac{\partial u}{\partial t} = -vu';$$

$$-vu' + uu' = \gamma u''$$

Separation of variables

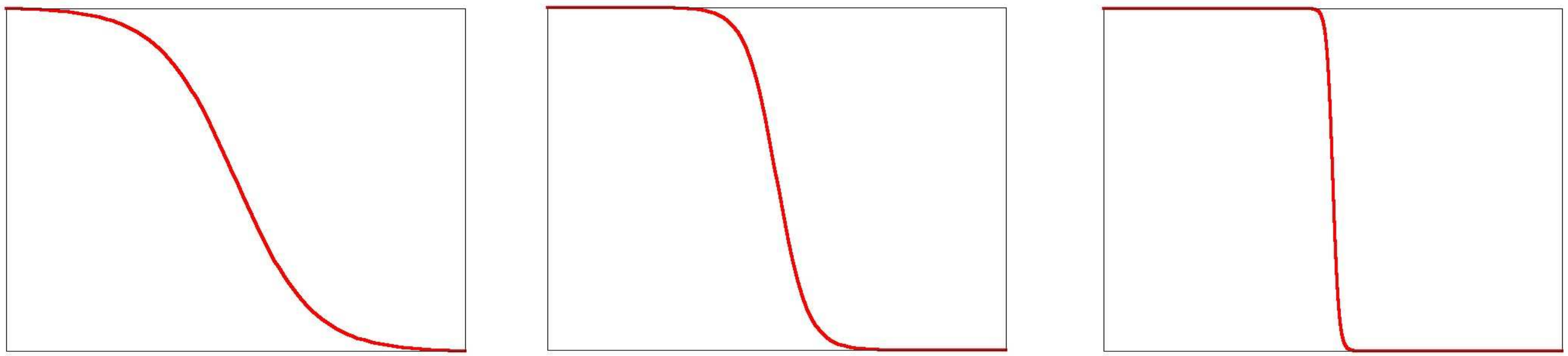
$$-vu + \frac{1}{2}u^2 - \gamma u' = 0$$

Boundary conditions: $u = u' = 0, \quad \text{as} \quad \theta \to \pm\infty$

$$\theta - \theta_0 = \int \frac{2\gamma du}{u^2 - 2vu}$$

$$\theta - \theta_0 = \frac{\gamma}{v} \ln \left| 1 - \frac{2v}{u} \right|$$

$$u(x,t) = \frac{2v}{1 - e^{\frac{v}{\gamma}(x - vt)}}$$



Burgers equation: the Hopf-Cole transformation

Remarkable observation: ***the non-linear Burgers equation can be converted Into the linear heat equation!***

J. Cole and E. Hopf (1950 - 1951)

$$\boxed{u_t - \gamma u_{xx} = 0} \longleftrightarrow \boxed{u_t + u u_x - \gamma u_{xx} = 0}$$

Reparametrisation:

$$u(x,t) = e^{\alpha\phi(x,t)} \leftrightarrows \phi(x,t) = \frac{1}{\alpha}\ln u(x,t)$$

$$u_t = \alpha\phi_t e^{\alpha\phi}; \quad u_x = \alpha\phi_x e^{\alpha\phi}; \quad u_{xx} = (\alpha\phi_{xx} + \alpha^2\phi_x^2)e^{\alpha\phi}$$

Potential Burgers equation: $\boxed{\phi_t = \gamma\phi_{xx} + \alpha\gamma\phi_x^2}$

Differentiation with respect to x: $\phi_{xt} = \gamma\phi_{xxx} + 2\alpha\gamma\phi_x\phi_{xx}$

Definition: the potential function is $u = \dfrac{\partial\phi}{\partial x}$ $\Rightarrow$ $\boxed{u_t = \gamma u_{xx} + 2\alpha\gamma u u_x}$

Any positive solution $v(x,t)$ to the linear heat equation solves the Burgers equation:

$$u(x,t) = \frac{\partial}{\partial x}\left(-2\gamma\ln v(x,t)\right) = -2\gamma\frac{v_x}{v}$$

KdV equation: Conservation laws

Definition: A conservation law is an equation of the form

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0$$

Conserved density

Flux

Note: the non-linear transport equation has the form of the conservation law:

$$u_t + uu_x = \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0$$

$$\frac{d}{dt}\int_{-\infty}^{\infty} T dx = \int_{-\infty}^{\infty} \frac{\partial X}{\partial x} dx = X\Big|_{-\infty}^{\infty} = 0$$

- **Example I** - ***KdV equation*** $u_t - 6uu_x + u_{xxx} = 0$

1 $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(u_{xx} - 3u^2\right) = 0 \Rightarrow \int_{-\infty}^{\infty} u dx = const = M$ **Conservation of mass**

2 $u_t - 6uu_x + u_{xxx} = 0 \,\Big|\, \times u \Rightarrow \frac{\partial}{\partial t}\left(\frac{u^2}{2}\right) + \frac{\partial}{\partial x}\left(uu_{xx} - 2u^3 - \frac{u_x^2}{2}\right) = 0$

$$\int_{-\infty}^{\infty} \frac{u^2}{2} dx = const = P$$

Conservation of momentum

Conservation of energy

3 $3u^2 \times \mathrm{KdV} + u_x \times \frac{\partial}{\partial x}\mathrm{KdV} \Rightarrow \int_{-\infty}^{\infty}\left(u^3 + \frac{u_x^2}{2}\right) dx = const = E$

$$\frac{\partial}{\partial t}\left(u^3 + \frac{u_x^2}{2}\right) + \frac{\partial}{\partial x}\left(-\frac{9}{4}u^4 + 3u^2 u_{xx} - 6uu_x^2 + u_x u_{xxx} - \frac{1}{2}u_{xx}^2\right) = 0$$

Apropos: KdV Lagrangian

$$u_t - 3(u^2)_x + u_{xxx} = 0 \quad \Longleftrightarrow \quad L = \frac{1}{2}\phi_x\phi_t - \phi_x^3 - \frac{1}{2}\phi_{xx}^2$$

Field equation: $\phi_{xt} - 3(\phi_x^2)_x + \phi_{xxxx} = 0$ $\quad \phi_x = u$

Symmetries of the KdV field theory

- **Translational invariance:**

Conservation of momentum

$$P = \int_{-\infty}^{\infty} \frac{\delta L}{\delta \phi_t}\phi_x dx = \frac{1}{2}\int_{-\infty}^{\infty} \phi_x^2 dx = \frac{1}{2}\int_{-\infty}^{\infty} u^2 dx$$

- **Time invariance:**

Conservation of energy

$$H = \int_{-\infty}^{\infty} \left(L - \frac{\partial L}{\partial \phi_t}\phi_t\right) dx = \int_{-\infty}^{\infty} \left(\frac{1}{2}\phi_{xx}^2 + \phi_x^3\right) dx = \int_{-\infty}^{\infty} \left(\frac{1}{2}u_x^2 + u^3\right) dx$$

- **Scale invariance:** $\phi \to \phi + \delta\phi$

Conservation of mass

$$M = \int_{-\infty}^{\infty} \frac{\partial L}{\partial \phi_t} dx = \frac{1}{2}\int_{-\infty}^{\infty} \phi_x dx = \frac{1}{2}\int_{-\infty}^{\infty} u dx$$

KdV equation: Gardner transform

$$T_1 \sim u; \quad T_2 \sim u^2; \quad T_3 \sim u^3 \ldots$$

$$T_4 = 5u^4 + 10uu_x + u_{xx}^2; \quad T_5 = 21u^5 + 105u^2u_x^2 + 21uu_{xx}^2 + u_{xxx}^2$$

Gardner transform: $\boxed{u = w + \epsilon w_x + A\epsilon^2 w^2}$ $\longrightarrow$ $u_t + uu_x + u_{xxx} = 0$

$$u_t = w_t + \epsilon w_{xt} + 2A\epsilon^2 w w_t = \left(1 + \epsilon\frac{\partial}{\partial x} + 2A\epsilon^2 w\right) w_t$$

$$uu_x + u_{xxx} = \left(1 + \epsilon\frac{\partial}{\partial x} + 2A\epsilon^2 w\right)\left(ww_x + w_{xxx} + A\epsilon^2 w^2 w_x\right) + \cancel{\epsilon^2(1+6A)w_x w_{xx}}$$

Trick: we can take A=-1/6

$$u_t + uu_x + u_{xxx} = \left(1 + \epsilon\frac{\partial}{\partial x} - \frac{\epsilon^2}{3}w\right)\left(w_t + ww_x + w_{xxx} - \frac{\epsilon^2}{6}w^2 w_x\right)$$

If w satisfies the ***Gardner equation***
$u = w + \epsilon w_x - \epsilon^2 w^2/6$ satisfies KdV

$$\boxed{w_t + ww_x + w_{xxx} - \frac{\epsilon^2}{6}w^2 w_x = 0}$$

Note: the Gardner equation can be written as conservation law:

$$\boxed{\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}\left(\frac{w^2}{2} - \frac{\epsilon^2}{18}w^3 + w_{xx}\right) = 0}$$

KdV equation: Conservation laws – How many?

$u = w + \epsilon w_x - \epsilon^2 w^2/6$ ⟹ Why don't we try the expansion $w = \sum_{n=0}^{\infty} \epsilon^n w_n$?

By comparing powers of ϵ

$$w_0 = u; \quad w_1 = -\frac{\partial w_0}{\partial x} = -u_x; \quad w_2 = -\frac{\partial w_1}{\partial x} + \frac{1}{6} w_0^2 = u_{xx} + \frac{1}{6} u^2 \ldots.$$

For n ≥3

$$w_n = -\frac{\partial w_{n-1}}{\partial x} + \frac{1}{3} u w_{n-2} + \frac{1}{6} \sum_{k=1}^{n-3} w_k w_{n-2-k}$$

Substituting the expansion of ω into the Gardner equation and collecting powers of ϵ

$$w = \sum_{n=0}^{\infty} \epsilon^n w_n \Longrightarrow \frac{\partial w}{\partial t} + \frac{\partial}{\partial x}\left(\frac{w^2}{2} - \frac{\epsilon^2}{18} w^3 + w_{xx}\right) = 0$$

$$\boxed{\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0;} \quad T(\epsilon) = w = \sum_{n=0}^{\infty} \epsilon^n T_n; \quad X(\epsilon) = \frac{w^2}{2} - \frac{\epsilon^2}{18} w^3 + w_{xx} = \sum_{n=0}^{\infty} \epsilon^n X_n$$

$$X_0 = \frac{w_0^2}{2} + w_{0,xx} = \frac{u^2}{2} + u_{xx}; \quad X_1 = w_0 w_1 + w_{1,xx} = -u u_x - u_{xxx}, \ldots$$

There is an infinite number of independent local conservation laws!!

KdV equation as a Hamiltonian System

Note: the KdV equation can be written as $u_t = \frac{\partial}{\partial x}\frac{\delta H}{\delta u}$

Here $$H = \int_{-\infty}^{\infty} T_3 dx = \int_{-\infty}^{\infty}(u^3 - \frac{1}{2}u_x^2)dx$$ is the "energy" integral and the variational derivative of the functional $H[u] = \int f(u, u_x; x)dx$ is

$$\frac{\delta H}{\delta u} = \frac{\partial f}{\partial u} - \frac{d}{dx}\left(\frac{\partial f}{\partial u_x}\right)$$

The **Poisson bracket**: $$\{H, G\} = \int_{-\infty}^{\infty} \frac{\delta H}{\delta u}\frac{\partial}{\partial x}\left(\frac{\delta G}{\delta u}\right)$$

Note: integration by parts yields $H = \int_{-\infty}^{\infty}(u^3 + uu_{xx})dx$ $\Rightarrow$ $\frac{\delta H}{\delta u} = 3u^2 + 2u_{xx}$

Question: How to link it to the traditional form of the finite-dimensional Hamiltonian system?

Fourier expansion: $u(x,t) = \sum u_k e^{ikx}$ $\Rightarrow$ $\{q_k = u_k/k, \quad p_k = u_{-k}, \quad \mathcal{H} = \frac{i}{2\pi}H\}$

We recover the usual Hamiltonian equations $$\frac{dq_k}{dt} = \frac{\partial \mathcal{H}}{\partial p_k}, \qquad \frac{dp_k}{dt} = -\frac{\partial \mathcal{H}}{\partial q_k}$$

with the Poisson bracket $$\{H, G\} = \frac{i}{2\pi}\sum_{k=-\infty}^{\infty} k\frac{\partial H}{\partial u_k}\frac{\partial G}{\partial u_{-k}}$$

Lecture 2: Summary

Existence of the infinite tower of conservation laws $\longrightarrow$ strong indication that we deal with a ***completely integrable system***

What does it mean?

$$u_t + 6uu_x + u_{xxx} = 0$$

- ***Gardner (1971) :*** The KdV equation represents an infinite-dimensional Hamiltonian system with an infinite number of integrals of motion in involution
- ***Gardner, Greene, Kruskal & Miura (1967):*** Inverse Scattering Transform (IST) method : the method to solve an initial-value problem for the KdV equation within a class of initial conditions.
- **Zakharov,Shabad and other (1971):** Inverse Scattering Transform for the nonlinear Schrödinger equation (NLS), the Sine-Gordon equation and many other completely integrable equations.

Note: The availability of the travelling wave (and, in particular, soliton) solutions for the KdV equation ***does not*** constitute its integrability. Practically the complete integrability means just the ability to integrate the KdV equation for a reasonably broad class of initial or boundary conditions.

KdV equation: Linearization?

Question: if the infinite number of conservation laws for KdV means that it is an analogue of a completely integrable Hamiltonian system?

Recall: the Burgers equation can be solved exactly through the Hopf-Cole transform:

$$\psi_t - \gamma\psi_{xx} = 0 \quad \Leftarrow \quad u = -2\gamma\frac{\psi_x}{\psi} \quad \Rightarrow \quad u_t + uu_x - \gamma u_{xx} = 0$$

How about KdV? $\quad u_t - 6uu_x + u_{xxx} = 0$

- **Step I:** Miura transform: $\boxed{u = v^2 + v_x}$

$$\left(\frac{\partial}{\partial x} + 2v\right)\underbrace{\left(v_t - 6v^2 v_x + v_{xxx}\right)} = 0$$

- **Step II:** Linearization of the modified KdV equation? Should we try

$$\boxed{u = \frac{\psi_{xx}}{\psi}} \qquad v_x = \frac{\psi_{xx}}{\psi} - \frac{\psi_x^2}{\psi^2} \quad \Leftarrow \quad v = \frac{\psi_x}{\psi} \quad \Rightarrow \quad v_t - 6v^2 v_x + v_{xxx} = 0$$

Galilean symmetry of KdV: $u \to u + E \quad \Rightarrow$ "Schroedinger" equation:

$$\boxed{\psi_{xx} - u\psi = E\psi}$$

Potential *Eigenvalue (mode)* *Eigenfunction*

This is like quantum mechanics!!

KdV equation: scattering problem

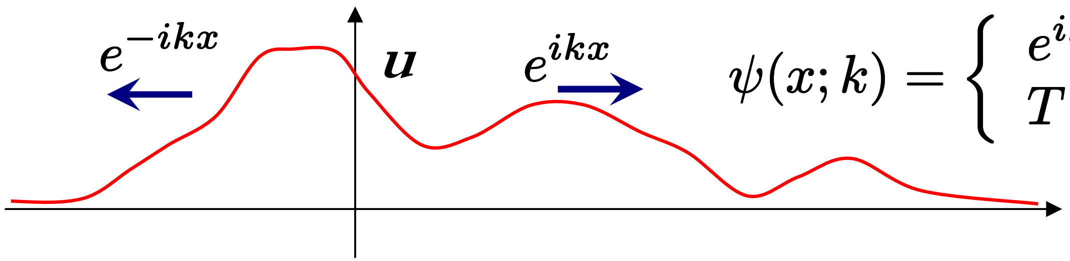
- **Scattering problems:** given a potential u, determine the spectrum $\{\psi, E\}$
- **Inverse scattering problem:** given a spectrum $\{\psi, E\}$, determine the potential

Assume $u(\pm\infty) = 0 \ \rightarrow \ |\psi|^2$ is integrable over $\mathbb{R}$ and it is normalizable

The discrete spectrum: $\psi_n(x) = c_n e^{-\kappa_n x}; \quad E = -\kappa_n^2 \ \text{ as } \ x \rightarrow \pm\infty$

The continuous spectrum: $E = k^2, \quad k \in \mathbb{R}$

Reflection coefficient



$$\psi(x;k) = \begin{cases} e^{ikx} + R(k)e^{-ikx} & \text{as } \ x \rightarrow -\infty \\ T(k)e^{ikx} & \text{as } \ x \rightarrow +\infty \end{cases}$$

Transmission coefficient

Question: What happened if $u = u(x,t)$ such that $u(x,t)$ solves KdV equation?

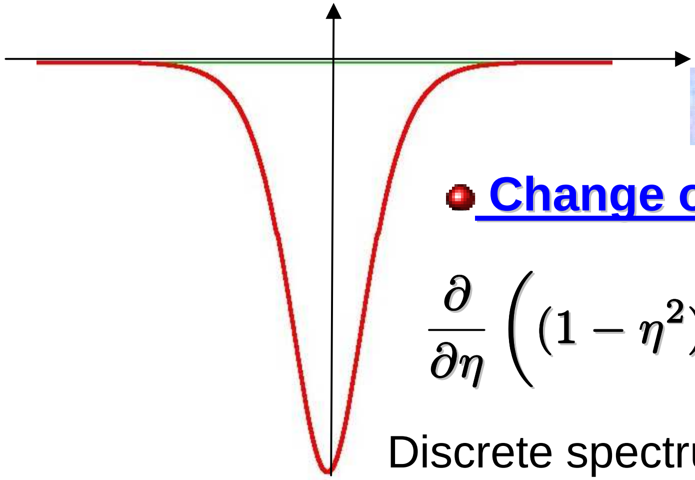
Naive answer: the eigenvalues ***E***, which in general depend on ***t*** through the parametric dependence in u, should change as t varies

?

- **Theorem I:** If $u(x,t)$ solves the KdV equation and it vanishes as $x \rightarrow \pm\infty$ the discrete eigenvalues of the Sturm-Liouville problem $\psi_{xx} + (\lambda - u)\psi = 0$ do not depend on ***t***

KdV equation: scattering problem

Recall: the soliton solution of the KdV equation is $u(x) = -\dfrac{u_0}{\cosh^2(x)}$



Sturm-Liouville equation: $\boxed{\psi_{xx} + \left(\lambda + \dfrac{u_0}{\cosh^2(x)}\right)\psi = 0}$

- **Change of variable:** $\eta = \tanh x$

$$\frac{\partial}{\partial \eta}\left((1-\eta^2)\frac{\partial \psi}{\partial \eta}\right) + \left(u_0 + \frac{\lambda}{1-\eta^2}\right)\psi = 0$$

Remark: **There are certain eigenvalues for those the potential is reflectionless**

Discrete spectrum: $u_0 = l(l+1); \quad \lambda = -k^2 < 0$ - Legendre polinomials

$$\psi = P_l^{(k)} = (-1)^k (1-\eta^2)^{k/2} \frac{d^k}{d\eta^k} P_l(\eta); \qquad P_l(\eta) = \frac{1}{2^l l!}\frac{d^l}{d\eta^l}(\eta^2 - 1)^l$$

Examples: $P_1^{(1)} = \sqrt{1-\eta^2} = -\dfrac{1}{\cosh x}; \qquad P_1^{(0)} = \eta = \tanh x$

Continuum: $\psi_k(x) = A\frac{2^{ik}}{(\cosh x)^{ik}} F_{2,1}(a,b,c;z); \qquad z = \frac{1+\eta}{2},$

$$a = \tfrac{1}{2} - ik + \sqrt{l(l+1) + \tfrac{1}{4}},\ b = \tfrac{1}{2} - ik - \sqrt{l(l+1) + \tfrac{1}{4}},\ c = 1 - ik$$

Asymptotically: $x \to -\infty$

$$\psi_k(x) \sim A\underbrace{\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}}_{=1} e^{-ikx} + A\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} e^{ikx}$$

Reflection coefficient

Linearized KdV and the Fourier transform

Consider linearized KdV equation: $u_t + u_{xxx} = 0; \quad x \in \mathbb{R}, \quad u(x,0) = u_0(x)$

- **Fourier transform:** $u(k) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty} u(x)e^{-ikx}dx; \quad u(x,t) = \int_{-\infty}^{+\infty} u(k,t)e^{ikx}dk$

$$u_x(x,t) = \frac{\partial}{\partial x}\int u(k,t)e^{ikx}dk = (ik)\int u(k,t)e^{ikx}dk \equiv \int u_x(k,t)e^{ikx}dk$$

$$u_x(k,t) = iku(k,t), \quad u_{xxx}(k,t) = -ik^3u(k,t)$$

Fourier transform of the linearized KdV: $\boxed{u_t(k,t) - ik^3u(k,t) = 0}$

$$u(k,t) = u_0(k)e^{-ik^3t} \Rightarrow u(x,t) = \int_{-\infty}^{+\infty} e^{-ik^3t}e^{-ikx}u_0(k)dk$$

$$\begin{tikzcd}
u(x,0) \arrow[r, "\mathcal{F}"] \arrow[d, "PDE"'] & u(k,0) \arrow[d, "Time\ evolution"] \\
u(x,t) & u(k,0)e^{-ik^3t} \arrow[l, "\mathcal{F}^{-1}"]
\end{tikzcd}$$

Scattering problem and the Fourier transform

Question: What is the analogue of the Fourier transform for KdV?

This is the Sturm-Liouville equation!

$$\psi_{xx} + (\lambda - u)\psi = 0$$

$$\psi(x;k) = \begin{cases} e^{ikx} + R(k)e^{-ikx} & \text{as} \quad x \to -\infty \\ T(k)e^{ikx} & \text{as} \quad x \to +\infty \end{cases}$$

$$\begin{tikzcd} u(x,0) \arrow[r, "S"] \arrow[d, "\text{Sturm- Liouville}"'] & \left\{ \begin{array}{c} R(k,0) \\ T(k,0) \end{array} \right\} \arrow[d, "\text{Time evolution}"] \\ u(x,t) & \left\{ \begin{array}{c} R(k,t) \\ T(k,t) \end{array} \right\} \arrow[l, "S^{-1}"] \end{tikzcd}$$

3 steps for solving the KdV equation:

- Given the initial condition $u(x,0)$ consider $-u$ as a potential in the Schrödinger equation and calculate the discrete spectrum $E = -\kappa^2$,the norming constant $c_n = c_n(0)$ and reflection coefficient $R(k) = R(k;0)$
- Introduce time dependence of these spectral data, the eigenvalues $E = -\kappa^2$ are fixed
- Carry out the procedure of the inverse scattering problem to recover $u(x,t)$

KdV equation: Lax Pair

Remark I: the KdV equation can be linearised via the spectral theory of the Schrödinger operator but not by means of an explicit change of variables

Remark II: the KdV equation $u_t + 6uu_x + u_{xxx} = 0$ can be viewed as a compatibility (integrability) condition for two linear differential equations for the same auxiliary function $\psi(x,t;\lambda)$

Spectral problem

$$\mathbf{L}\psi \equiv (-\partial^2_{xx} - u)\psi = \lambda\psi$$

Evolution problem

$$\psi_t = \mathbf{A}\psi \equiv (-4\partial^3_{xxx} - 6u\partial_x - 3u_x + C)\psi = (u_x + C)\psi + (4\lambda - 2u)\psi_x$$

λ is a complex parameter, $C(\lambda,t)$ depends on normalization of ψ

Compatibility condition +
Isospectral evolution:

$$\boxed{(\psi_{xx})_t = (\psi_t)_{xx} \quad + \quad \lambda_t = 0 \quad = \quad \text{KdV}}$$

Homework: Prove it!

The operators **L** and **A** are referred to as the **Lax pair**.

Remark III: the spectral equation **L** ψ is the Schrödinger equation we discussed!

Remark iV: the KdV equation is ***isospectral,*** i.e. $\lambda_t = 0$

Lax Pair

Remark V: the KdV equation can be represented in an operator form as

$$\boxed{\mathbf{L}_t = \mathbf{LA} - \mathbf{AL} \equiv [\mathbf{LA}]}$$

This operator representation provides a route for constructing the ***KdV hierarchy*** by appropriate choice of the operator **A**. Indeed, given the **L**-operator, the **A**-operator in the Lax pair is determined up to an operator commuting with **L**, which makes it possible to construct an infinite number of equations associated with the same spectral problem but having different evolution properties.

- **KdV hierarchy**

$$u_t + \frac{\partial}{\partial x}\mathbf{L}_{n+1}[u] = 0$$

$$\mathbf{L}_0[u] = \frac{1}{2}; \qquad \frac{\partial}{\partial x}\mathbf{L}_{n+1}[u] = \left(\frac{\partial^3}{\partial x^3} + 4u\frac{\partial}{\partial x} + 2\frac{\partial u}{\partial x}\right)\mathbf{L}_n[u]$$

---

1. $\mathbf{L}_1[u] = u; \Rightarrow u_t + u_x = 0$

2. $\mathbf{L}_2[u] = u_{xx} + 3u^2; \Rightarrow u_t + 6uu_x + u_{xxx} = 0$

3. $\mathbf{L}_3[u] = u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3;$

$$u_t + 10uu_{xxx} + 30u^2u_x + 20u_xu_{xx} + u_{xxxxx} = 0$$

Lax Pair operator formulation

Consider 2 linear equations: $\psi_x = L\psi; \quad \psi_t = A\psi$ $\qquad \psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}$

$$\begin{cases} \psi_{xt} = L_t\psi + L\psi_t; \\ \psi_{tx} = A_x\psi + A\psi_x. \end{cases} \qquad \underline{L_t\psi + LA\psi = A_x\psi + AL\psi;} \qquad \boxed{L_t - A_x = [A, L]}$$

Zero curvature condition

- **Let us take**

$$\begin{cases} L = i\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + i \begin{pmatrix} 0 & u \\ 1 & 0 \end{pmatrix}; \quad \lambda \in \mathbb{C} \\ A = -4\lambda^2 L - 2i\lambda \begin{pmatrix} -u & -iu_x \\ 0 & u \end{pmatrix} + \begin{pmatrix} u_x & iu_{xx} + 2iu^2 \\ 2iu & -u_x \end{pmatrix} \end{cases}$$

(an educated guess)

$$L_t = i \begin{pmatrix} 0 & u_t \\ 0 & 0 \end{pmatrix}; \qquad A_x = -4i\lambda^2 \begin{pmatrix} 0 & u_x \\ 0 & 0 \end{pmatrix} - 2i\lambda \begin{pmatrix} -u_x & -iu_{xx} \\ 0 & u_x \end{pmatrix} + \begin{pmatrix} u_{xx} & iu_{xxx} + 4iuu_x \\ 2iu_x & -u_{xx} \end{pmatrix}$$

Equalising the coefficients:

$$\mathrm{L}_t\Big|_{\lambda^0} = i \begin{pmatrix} 0 & \boxed{u_t} \\ 0 & 0 \end{pmatrix} = A_x + [A, L])\Big|_{\lambda^0} = i \begin{pmatrix} 0 & \boxed{u_{xxx} + 6uu_x} \\ 0 & 0 \end{pmatrix}$$

KdV equation!

The spectral equation: $\psi_x = L\psi \Longrightarrow \dfrac{\partial}{\partial x} \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} = \begin{pmatrix} i\lambda & iu \\ i & -i\lambda \end{pmatrix} \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}$

$$\begin{cases} \frac{\partial}{\partial x}\psi_{11} = i\lambda\psi_{11} + iu\psi_{21}; \\ \frac{\partial}{\partial x}\psi_{21} = i\psi_{11} - i\lambda\psi_{21} \end{cases} \Rightarrow \boxed{\left\{ \frac{\partial^2}{\partial x^2} + u + \lambda^2 \right\} \psi_{21} = 0}$$

Spectral problem

KdV equation: Direct scattering problem

Scattering data: $\{\kappa_n, c_n(0), r(k,0), t(k,0)\}$

- Discrete spectrum $\psi_n(x,0) \sim c_n(0)e^{-\kappa_n x}$ as $x \to \pm\infty$
- Continuous spectrum $\psi(x;k) = \begin{cases} e^{ikx} + R(k)e^{-ikx} & \text{as } x \to -\infty \\ T(k)e^{ikx} & \text{as } x \to +\infty \end{cases}$

Substitution into the second Lax equation yields for spectral data of continuum:

$$\psi_t = \mathbf{A}\psi \equiv (-4\partial^3_{xxx} - 6u\partial_x - 3u_x + c)\psi \implies c(\lambda,t) = 4ik^3; \quad \frac{dR}{dt} = 8ik^3 R; \quad \frac{dT}{dt} = 0$$

Hence $$\boxed{R(k,t) = R(k,0)e^{8ik^3 t}; \quad T(k,t) = T(k,0)}$$

Substitution into the second Lax equation yields for spectral data of the discrete spectrum:

$$c = c_n = 4\kappa_n^3 \Longrightarrow \frac{dc_n}{dt} = 4c_n\kappa_n^3 \Longrightarrow \boxed{c_n(t) = c_n(0)e^{4\kappa_n^3 t}}$$

Remark: the bound state problem can be viewed as an analytic continuation of the scattering problem defined on the real *k*-axis, to the upper half of the complex *k*-plane. Then the discrete points of the spectrum are found as *simple* poles $k = i\kappa$ of the reflection cofficient $R(k)$

KdV equation: Inverse scattering problem

It is well known from 1950s that the potential of the Schrödinger equation can be completely recovered from the scattering data – ***Gelfand-Levitan-Marchenko equation***

We define the function of the scattering data

Discrete spectrum data

Continuum data

$$F(x,t) = \sum_{n=1}^{N} c_n^2 e^{-\kappa_n x} + \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k,t) e^{ikx} dk$$

Then the potential $u(x,t)$ can be restored from the equation

$$\boxed{u(x,t) = 2\frac{\partial}{\partial x} K(x,x,t)}$$

where function $K(x,y,t)$ can be found from the linear integral-differential GLM equation

$$K(x,y) + F(x+y) + \int_{x}^{\infty} K(x,z) F(y+z) dz = 0$$

Note: at each step of solving of the KdV equation we consider a linear problem

Lecture 3: Summary

$$\psi(x;k) = \begin{cases} e^{ikx} + R(k)e^{-ikx} & \text{as} \quad x \to -\infty \\ T(k)e^{ikx} & \text{as} \quad x \to +\infty \end{cases}$$

$$\begin{tikzcd}
u(x,0) \arrow[r, "S"] \arrow[d, "\text{Sturm- Liouville}"'] & \left\{ \begin{matrix} R(k,0) \\ T(k,0) \end{matrix} \right\} \arrow[d, "\text{Time evolution}"] \\
u(x,t) & \left\{ \begin{matrix} R(k,t) \\ T(k,t) \end{matrix} \right\} \arrow[l, "S^{-1}"]
\end{tikzcd}$$

3 steps to solve the KdV equation:

- Given the initial condition $u(x,0)$ consider $-u$ as a potential in the Schrödinger equation and calculate the discrete spectrum $E = -\kappa^2$,the norming constant $c_n = c_n(0)$ and reflection coefficient $R(k) = R(k;0)$ (scattering data)
- Introduce time dependence of these spectral data, the eigenvalues $E = -\kappa^2$ are fixed
- Carry out the procedure of the inverse scattering problem to recover $u(x,t)$ making use of the GLM equation:

$$K(x,y) + F(x+y) + \int_x^\infty K(x,z)F(y+z)dz = 0$$

$$F(x,t) = \sum_{n=1}^{N} c_n^2 e^{-\kappa_n x} + \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k,t)e^{ikx}dk \qquad u(x,t) = 2\frac{\partial}{\partial x}K(x,x,t)$$