Onset of Chaos in the Transitional Region of Quantum Phase Transitions - Exceptional Points -

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As an illustration, the Lipkin model:

N Fermions occupying 2 degenerate levels, degeneracy at least *N*-fold.

Interaction lifts or lowers a Fermion pair

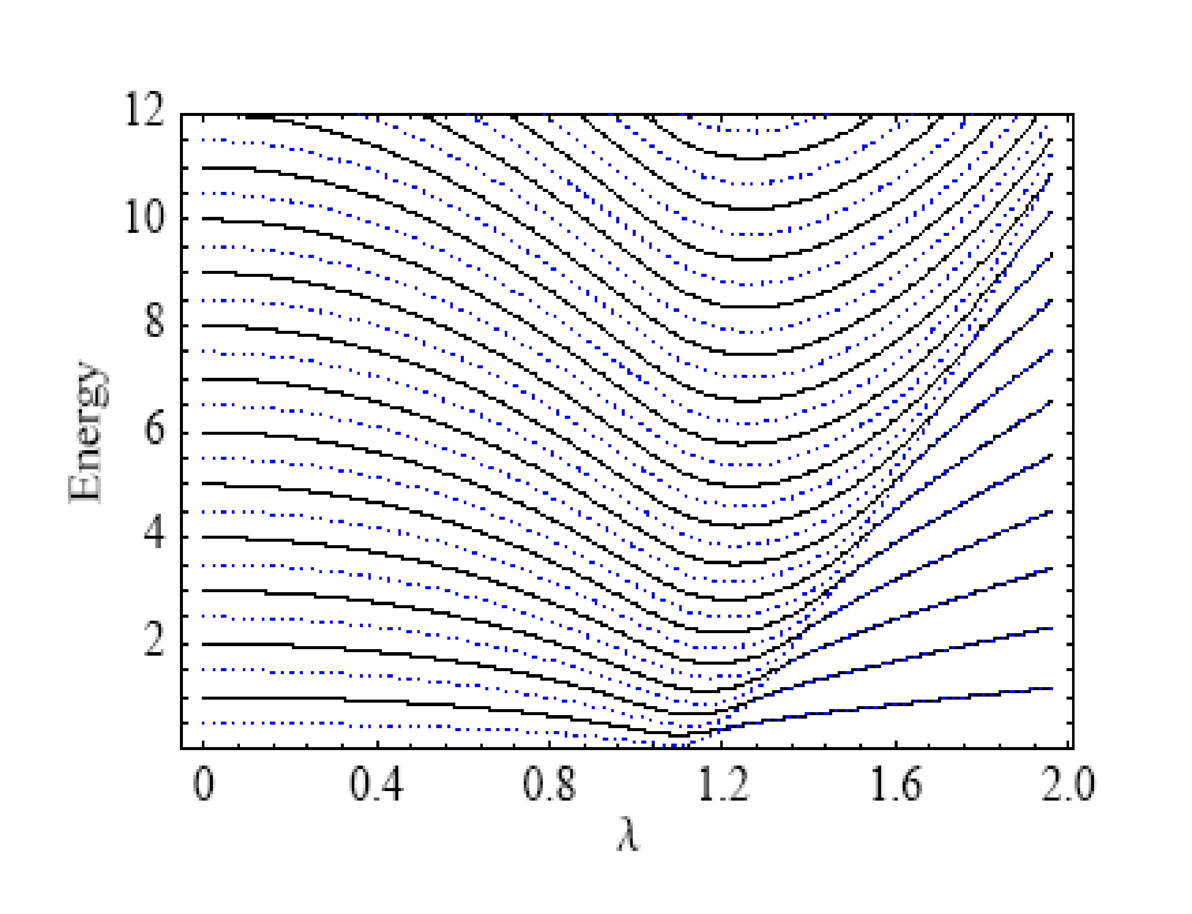
$$H = \sum a^{\dagger}_{k,m} a_{k,m} + \lambda \sum a^{\dagger}_{k,m} a^{\dagger}_{k',m'} a_{k',-m'} a_{k,-m}$$

as a consequence:

model is reducible into *even* or *odd N*

$$H = J_z + \frac{\lambda}{2N}(J_+^2 + J_-^2)$$

model shows phase transition at $\lambda = 1$ including ***symmetry breaking*** in that for $\lambda > 1$ a 'deformed' phase occurs where even and odd *N* become degenerate

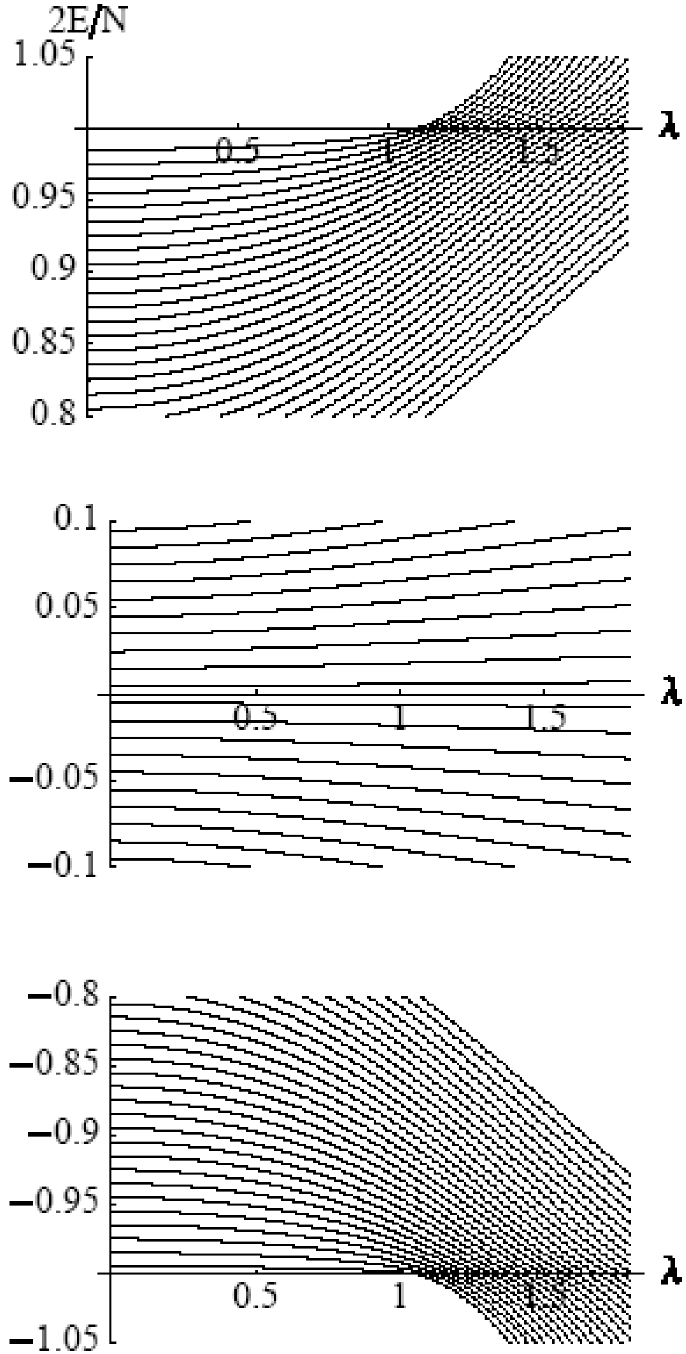


spectrum with respect to ground state

energy gap at the transition point,
for large but finite N

$$\Delta E \sim \frac{1}{N^{1/3}} \quad \text{for } \lambda = 1$$

$$\Delta E \sim \frac{\sqrt{\lambda^2 - 1}}{\log N} \quad \text{for } \lambda > 1$$

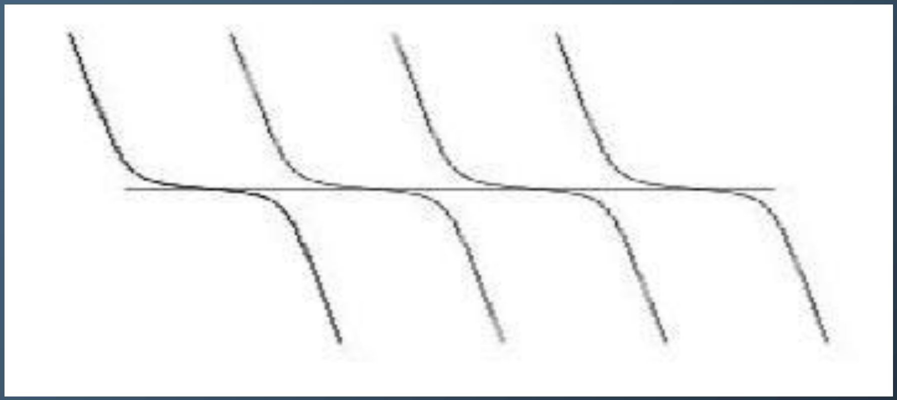


Spectrum as function of λ

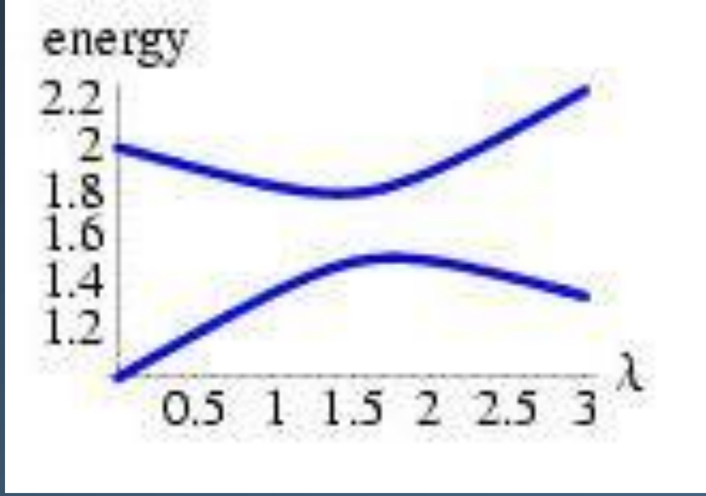
nothing interesting in middle, symmetry around E = 0

phase transition for all $\lambda > 1$ at *2E/N* = –1 (and *2E/N* = + *1*)

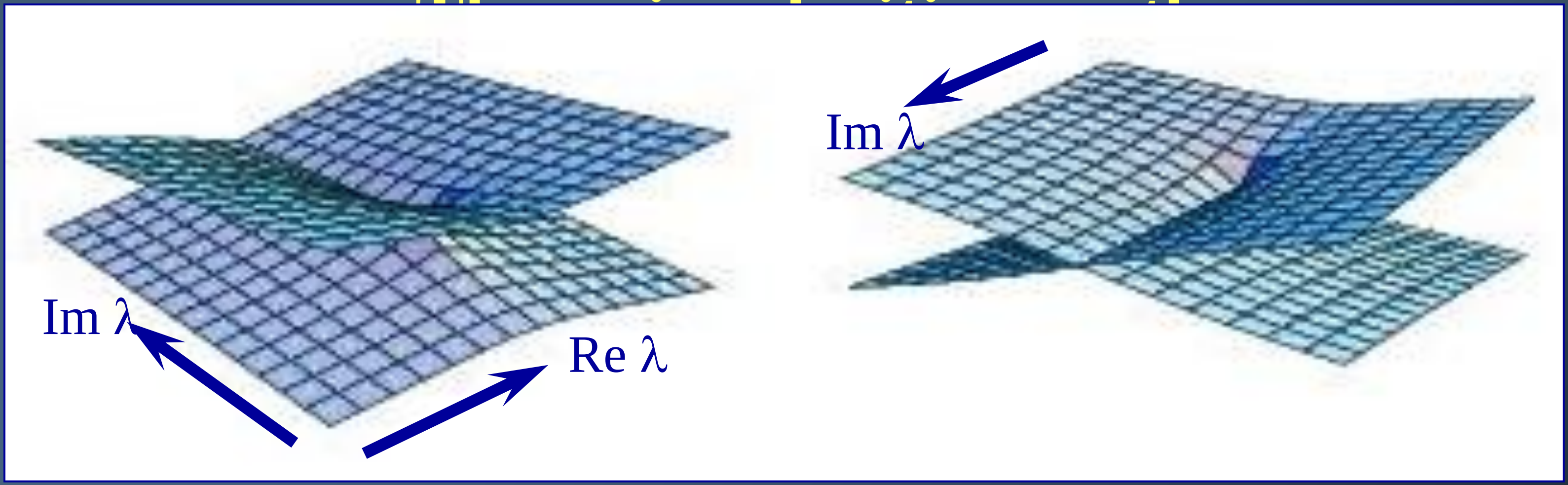
in fact, magnification along the line *2E/N* = –1 looks like



level repulsion – watch EP!

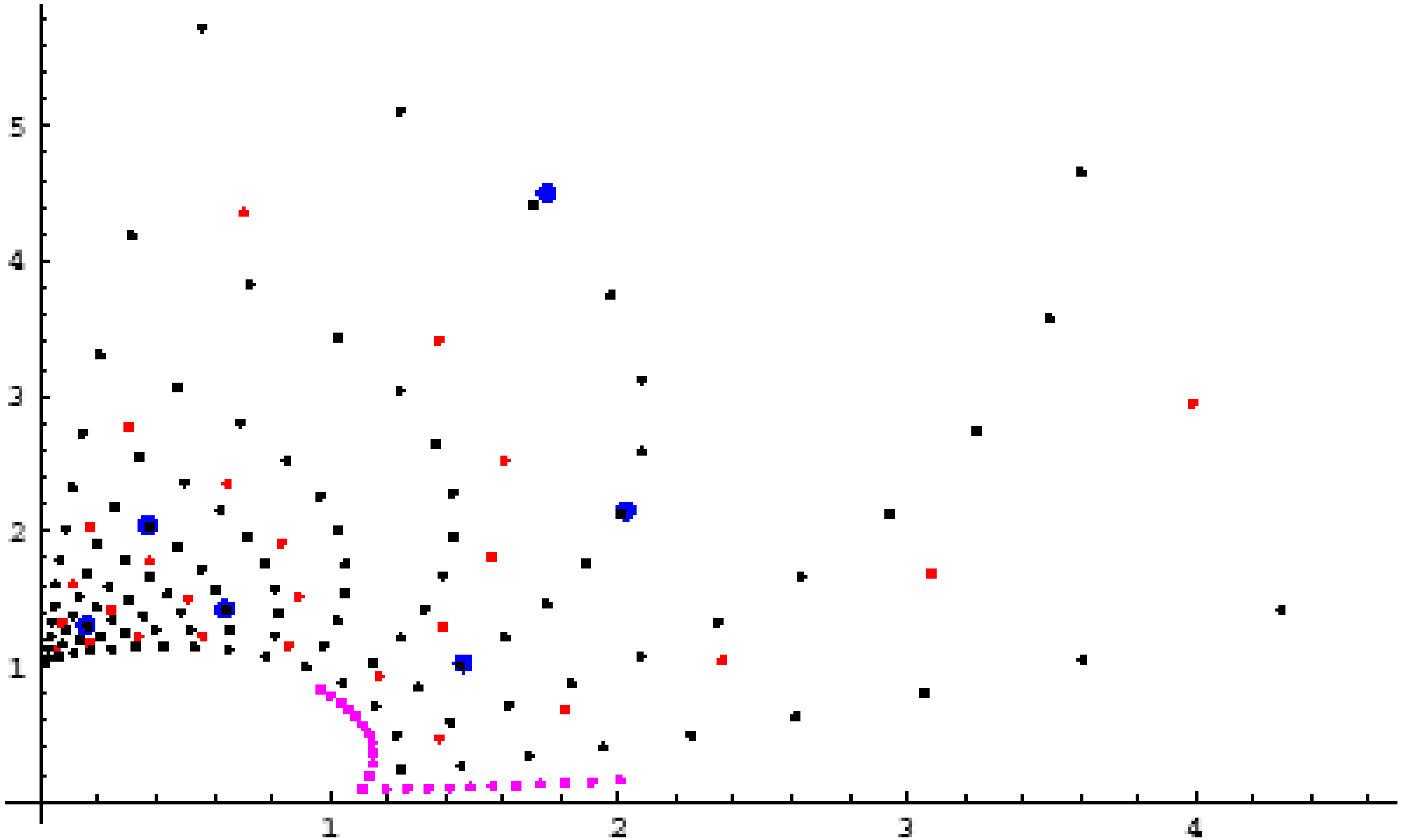


When there is level repulsion as a function of λ, the analytic continuation into the complex λ-plane yields a near square root singularity of energies and eigenfunctions: the two repelling levels are analytically connected.

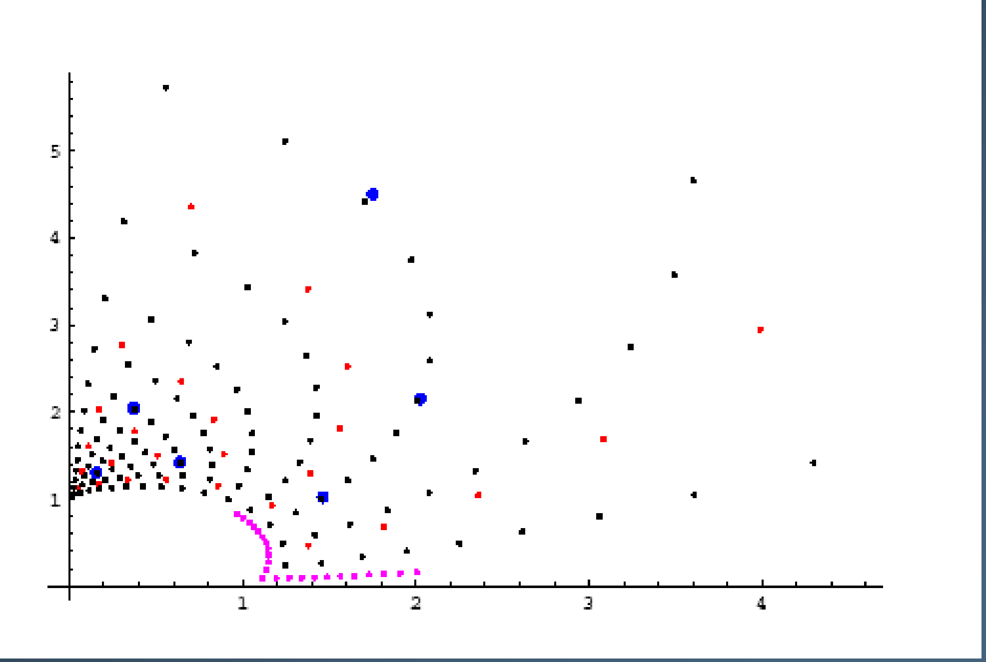


Exceptional Points are square root singularities where two levels *and* their eigenfunctions coalesce. They occur in the vicinity of level repulsions for complex values of the parameter which gives rise to level repulsion. For a finite N-dimensional problem all levels are analytically connected at the EPs; there are N(N-1) EPs.

EPs in complex λ - plane for various *N*



N=8 (blue), =16(red), =32(black), =96(pink)



The inner circle

$$|\lambda| < 1$$

remains free of singularities

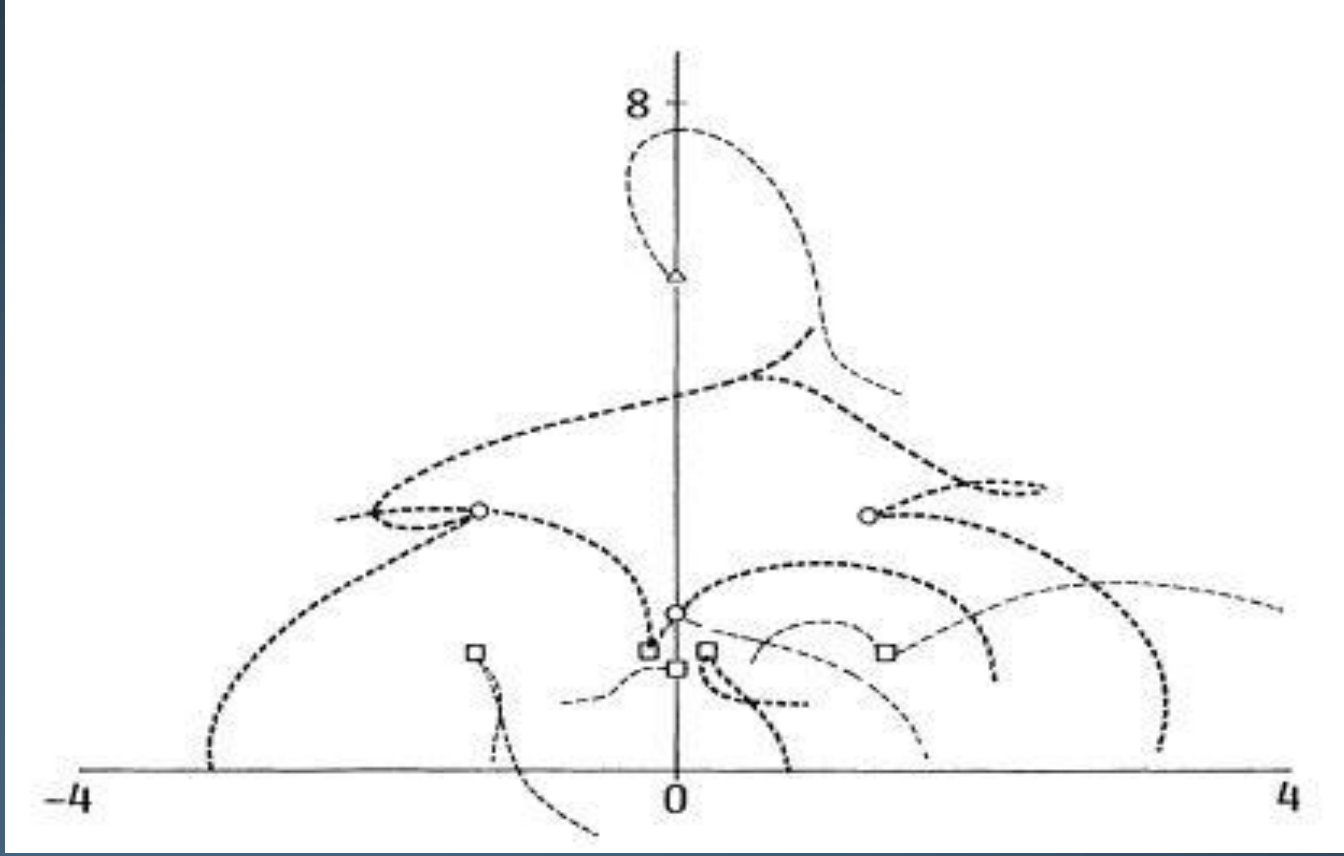
In contrast, for increasing *N*, the EPs accumulate along the real λ - axis for λ>1

But: the model yields them nicely ordered.
The slightest perturbation whirls them around.

The effect upon the **spectrum:**

Chaotic!

Trajectories of the EPs in the complex λ-plane for N=6 for the perturbation



$$J_+^2 + J_-^2 \rightarrow U^\dagger (J_+^2 + J_-^2) U$$

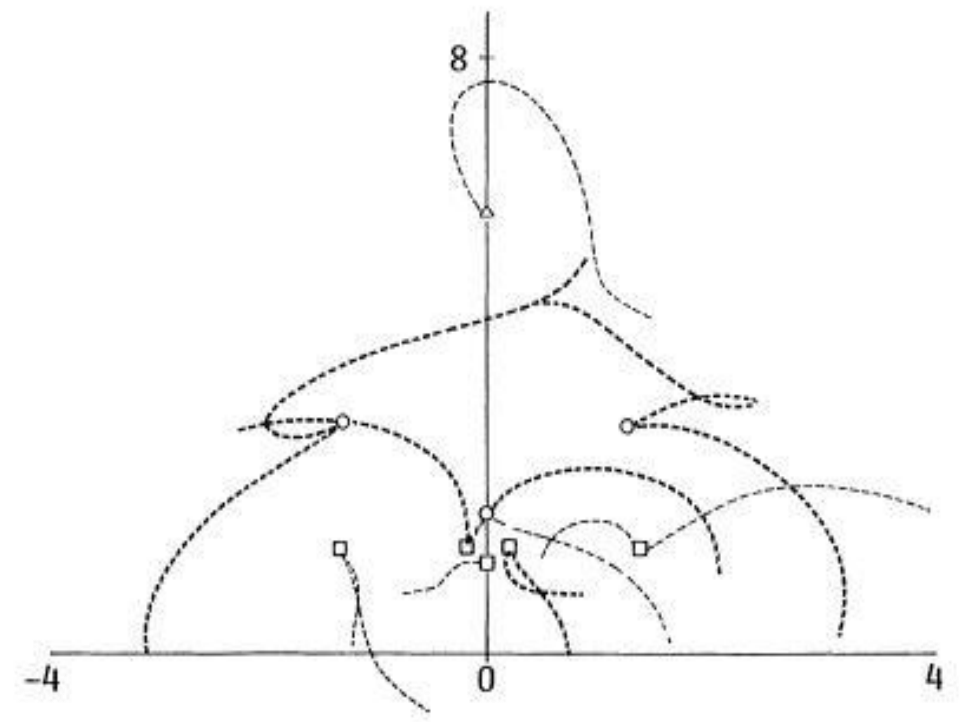
with U a random unitary matrix using random angles from the interval $\{0, \zeta_{max}\}$;

$\zeta_{max} \ll 1$, i.e. U is close to unity.

1.Note the following: The symmetry wrt the imaginary axis is destroyed.

2. Two trajectories emerge from each EP (except on the imaginary axis), i.e.the symmetry around E=0 is also destroyed.

3. A crossing of the real axis signals a genuine level crossing.

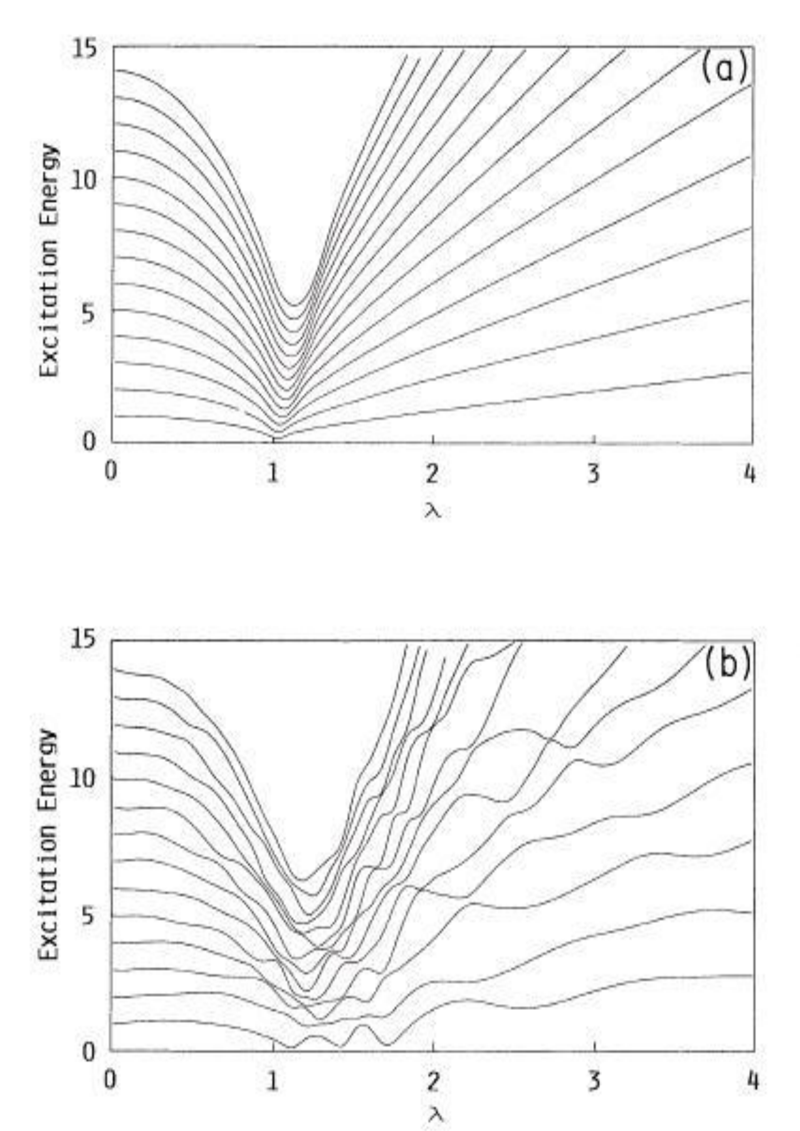


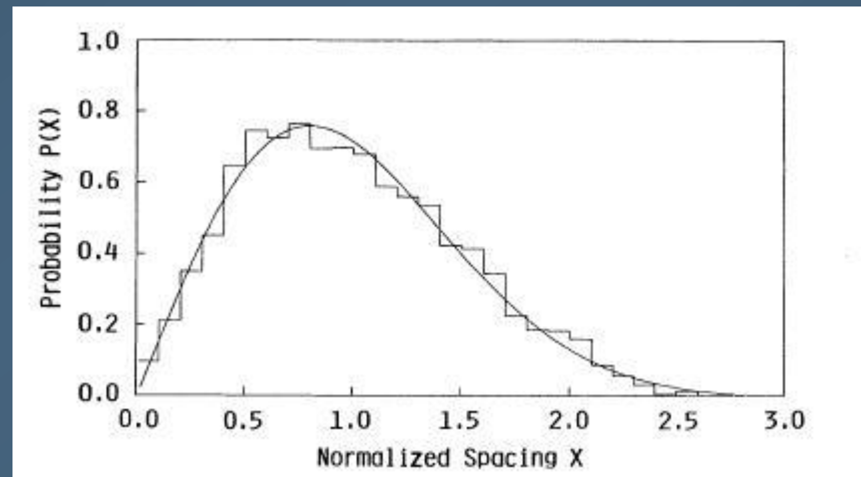
The larger N the smaller ζ_{max} can be chosen to produce typical signatures of chaotic behaviour of spectrum and eigenfcts.

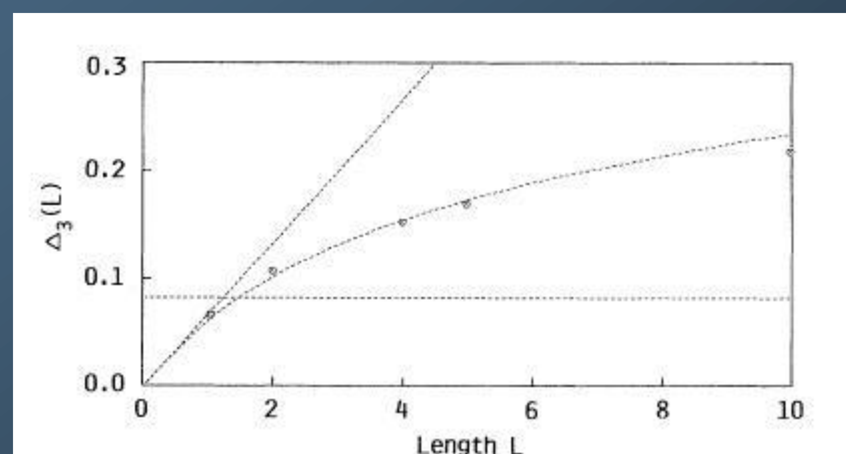
Further increase of ζ_{max} leaves the statistics at λ_{crit} unchanged.

While the level statistics is that of the harmonic oscillator for the unperturbed case, we now obtain the typical Wigner surmise for

$\lambda=\lambda_{crit}$







It is **significant** that this chaotic behaviour does not occur when λ is sufficiently distant from λ_{crit} This is best seen when we use a mean field approach: the ground state is given by

$$|HF\rangle = e^{2i\gamma J_y}|0\rangle$$

where $|0\rangle$ is the unperturbed ground state and

$$\cos 2\gamma = 1 \text{ for } \lambda < 1$$

$$\cos 2\gamma = \frac{1}{\lambda} \text{ for } \lambda > 1$$

and the energy is given by

$$E_{HF} = -\frac{N}{2}\cos 2\gamma - \frac{\lambda N}{4}(1-\cos^2 2\gamma)$$

The contribution – linear in the angles of *U* – from the perturbation

$$J_+^2 + J_-^2 \rightarrow U^\dagger (J_+^2 + J_-^2) U$$

can be obtained analytically and it turns out that only at λ around λ_{crit} there is an appreciable effect while outside the transitional region the perturbation leaves energy and state vectors virtually unchanged.

Summary:

1. Due to the absence of Exceptional Points the normal phase remains virtually unaffected under small perturbation; so does the deformed region.

2. Transitional regions are the most sensitive against small random perturbation owing to the high density of Exceptional Points.

3. Within the transitional region the pattern of the EPs looks like

and – for full chaos - the distribution of EPs becomes independent of the direction in the λ-plane, is centred around λ_{crit} and given by $1/|\lambda^2|$.

Comments:

1. The sensitivity and immediate onset of chaos is reminiscent of the classical situation at the crossing point of a separatrix.

2. These findings explain the inherent difficulties of many body calculations in transitional regions,
such as in nuclear physics

The End

thank you for your attention

Note! Workshop at Stellenbosch:

The Physics of Exceptional Points

at
Stellenbosch
2^{nd} Nov to 5^{th} Nov 2010

about ten speakers from overseas:
atomic physics, optics, scattering,…

Students are generously supported:
talk to me, or write to
dieter@physics.sun.ac.za

W.D. H and A.L.Sannino
Transitional Regions of Finite Fermi Systems and Quantum Chaos
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