Cosmological billiards and Bruhat order

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As one knows (see, e.g. [**?**]), the term *cosmological billiard* is used to describe a remarkable limit behavior of an important dynamical system, the *1-dimensional* Σ*-model* on homogenous spaces. We shall briefly recall main definitions from this sphere.

Let $P = G/K$, where G is semisimple a Lie group and $\pi : G \to P$ the natural projection. We shall assume, that K is maximal compact subgroup of G. Then P is homeomorphic to $\mathbb{R}^n$ for a suitable n. Let $\mathfrak{k} \subseteq \mathfrak{g}$ be the Lie algebras of K and G, $\mathfrak{p}$ a complement of $\mathfrak{k}$ so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$; one can regard $\mathfrak{p}$ as the tangent space of P in any given point, since P is topologically trivial. Here we assume that $\mathfrak{k} \oplus \mathfrak{p}$ is a *Cartan decomposition* of $\mathfrak{g}$, in particular that the Killing form is positive-definite on $\mathfrak{p}$ and negative-definite on $\mathfrak{k}$. Thus we can use Killing form to induce a metric h on P.

For every map $\sigma : \Sigma \to P$ $(\dim\Sigma = 1)$ consider the formula

$$L_\sigma(x) = \mathrm{proj}_{\mathfrak{p}}^{\perp}(g^{-1}\dot{g}),$$

where $x \in \Sigma$, $g : P \to G$ is a local section of π around x and $\mathrm{proj}_{\mathfrak{p}}^{\perp}$ denotes the orthogonal projection.

One can show, that $L = L_\sigma$ does not depend of the choice of g and determines a section $L_\sigma : \Sigma \to TP = P \times \mathfrak{p}$, depending on σ. Consider the Lagrangian action:

$$\mathcal{S}(\sigma) = \int_\Sigma h(L_\sigma(x), L_\sigma(x)) d\mu. \tag{1}$$

Here integration is taken over a compact subset (or we assume additional conditions on the behavior of $\sigma(x)$ in infinity). We shall restrict our attention to the case, when $\Sigma = \mathbb{R}^1$, i.e. $\sigma : \mathbb{R}^1 \to G$, so that $d\mu = dt$.

In the latter case variation of the action $\mathcal{S}$ with respect to $\sigma(t)$ yields the following equation:

$$\frac{d}{dt} L_\sigma = -[M_\sigma, L_\sigma], \tag{2}$$

where $M_\sigma(x) = \mathrm{proj}^\perp_\mathfrak{k}(g^{-1}\dot{g})$. One can express this latter element as the function of L_σ; this can be done either in terms of the suitable positive roots system of $\mathfrak{g}$, or in terms of the anti-symmetrization of the matrix, representing L_σ. Also observe, that $[M_\sigma, L_\sigma] \in \mathfrak{p}$ due to the definition of Cartan pairs.

It is not difficult to see, that the equation (2) is the well-known *Toda equation*. For instance, when $G = SL(n,\mathbb{R})$, we can choose $K = SO(n,\mathbb{R})$; then $G/K \cong B^+(n,\mathbb{R}) \cong Sym_0(n,\mathbb{R})$; here $B^+(n,\mathbb{R})$ is the group of upper triangular matrices and $Sym_0(n,\mathbb{R})$ is the space of symmetric matrices with trace equal to 0 (the first isomorphism follows from the QR-decomposition and the second is clear from linear algebra). Then we choose Cartan pair as $\mathfrak{p} = Sym_0(n,\mathbb{R}),\ \mathfrak{k} = \mathfrak{so}(n,\mathbb{R})$ so that $L = L_\sigma$ and $M = M_\sigma$ are given by

$$L = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix},\ M = \begin{pmatrix} 0 & a_{12} & \dots & a_{1n} \\ -a_{12} & 0 & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ -a_{1n} & -a_{2n} & \dots & 0 \end{pmatrix},$$

i.e. the equation (2) obtains the usual form of the Adler-Kostant-Symes scheme. Similar equations can be written in other cases, i.e. for other semisimple groups G.

One can take the description, given above, as the definition of Toda system for arbitrary semisimple Lie group, below we shall repeat main steps of it. Properties of this system are rather well-studied today. The matrix L (more generally, the $\mathfrak{p}$-valued function L) is called *Lax matrix*, and equation (2) is called *Lax equation* of the Toda system.

First of all, one can show, that this system is an integrable Hamiltonian system, where the Poisson structure on P comes from its identification with the dual space of a Lie algebra. The corresponding Hamiltonian is equal to the integration term in (1). This system is integrable, i.e. it possesses many independent commuting first integrals.

In particular, in the case $G = SL(n, \mathbb{R})$, P is identified with $(\mathfrak{b}^+(n, \mathbb{R}))^*$. In this case, which is called *the full symmetric Toda flow*, eigenvalues of the Lax matrix L are invariants of the system and one can even give explicit formulas that will solve this equation for given initial data. This system also has many other important invariants; some of them will be important for our work.

Using explicit solutions (see previous slide), one can show that any solution of this integrable system tends to diagonal matrix, with given eigenvalues, when $t \to \pm\infty$ (in fact, see previous slide, eigenvalues don't depend on t). Moreover, if eigenvalues are different,

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n,$$

then almost all trajectories go from the matrix $\lambda_{-\infty} = diag(\lambda_1, \lambda_2, \dots, \lambda_n)$ as $t \to -\infty$ to the matrix $\Lambda_{+\infty} = diag(\lambda_n, \lambda_{n-1}, ;\lambda_1)$, as $t \to +\infty$. Remaining trajectories also connect two diagonal matrices $\hat{\Lambda}_{\pm\infty}$ (in fact, the set of singular points of this system coincides with diagonal matr, so that the number of inversions in the sequence of eigenvalues at $t \to +\infty$ is greater than the corresponding number at $t \to -\infty$, as it was observed in the paper [**?**]. A more accurate analysis in the neighborhood of diagonal matrices shows, that the order of eigenvalues is not changed instantaneously, but in a series of swaps, as the trajectory approaches singular points. It is this behavior, that earned the name of *cosmological billiard*.

In our previous paper we showed that the order, in which trajectories of Toda flow on $SL(n,\mathbb{R})$ connect diagonal matrices, is not quite arbitrary and is not completely determined by the number of swaps, or inversions in the sequence of eigenvalues. It turned out that the behavior of trajectories of this equation is determined by the *Bruhat order* on the group S_n (the Weyl group of $SL(n,\mathbb{R})$), if all the eigenvalues $\{\lambda_j\}$ of L are distinct. In fact, we proved the following result:

Theorem

Two diagonal matrices are connected by a trajectory of the full symmetric Toda flow, if and only if the corresponding permutations of eigenvalues are comparable in the sense of (strong) Bruhat order in S_n.

This can be rephrased as follows: *the phase portrait of the full symmetric Toda system coincides with the Bruhat diagram of the permutation group S_n.*

It turns out, that similar statements hold in many other cases too. Namely, in this talk I will explain the following two results:

- *Let* $G = SL(n, \mathbb{R})$*, but not all eigenvalues are distinct, then the trajectories of the system still connect diagonal matrices, so that two such matrices are connected iff the corresponding multiset permutations are compareble in the sense of Bruhat order;*
- *Let* G *be any simple group of rank* 2*; then the phase portrait of the Toda flow coincides with Bruhat diagram of the corresponding Weyl group.*

It is natural to conjecture, that the same statement is true always, namely that *the phase portrait of the trajectories of Toda system, associated with any semisimple Lie group* G*, coincides with the Bruhat diagram of the corresponding Weyl group.* However, at present we cannot prove this in full generality.

In the rest of this talk I will recall basic definitions, necessary for the understanding of these results, and give a brief outline of the proofs. We shall begin with the definition of Weyl group of a semisimple Lie group G.

Let $T \subseteq G$ be the maximal commutative subgroup of G (it is homeomorphic to a multidimensional torus). It can be regarded as the exponent of the maximal commutative Lie subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, called *Cartan subalgebra*. As one knows, all such subgroups/subalgebras are conjugate by the action of G, so their dimension does not depend on the choices made; this dimension is called *rank* of the group G.

Let $N_G(T) \subseteq G$ be the normalizer subgroup of T in G, i.e. the maximal subgroup of G, which contains T as a normal subgroup. Then the quotient group

$$W_G = N_G(T)/T$$

is called *Weyl group of G*. As one sees, this group does not depend on the choices we have made.

It follows from the definition, that Weyl group W_G acts on the maximal torus T: this action is induced by conjugations in $N_G(T)$; alternatively, one can consider the induced adjoint action on Cartan subalgebra. This representation is exact, so one can identify W_G with the corresponding group of linear transformations of $\mathfrak{t}$.

Under this assumption one can show, that Weyl group of any semisimple Lie group is generated by a finite number of reflections with respect to hyperplanes in a multidimensional Euclidean space $\mathfrak{t}$. This gives a very special set of generators t_i, such that $t_i^2 = 1$, where t_i is a reflection. Another way to formulate it is to say that W_G has structure of a finite Coxeter group.

Every Coxeter reflection group has a partial order on it. First of all, one can define *length* $l(g)$ of an element $g \in W$

$$l(g) = \min\{n \mid g = t_{i_1} \dots t_{i_n}\}.$$

Thus, we can order the elements by comparing their lengths. This gives (rather rude) partial order.

A more subtle partial order is the so called *strong Bruhat order*, which can be defined as the closure (in the sense of partial orders) of the following elementary relation:

$$u \prec v \Leftrightarrow v = tu, \text{ for some elementary reflection } t.$$

An important particular case of this order is Bruhat order on *symmetric group* S_n, which is equal to the Weyl group of $SL(n, \mathbb{C})$. In this case the role of reflections is played by transpositions of two elements and the length of a permutation w is equal to the number of inversions in the corresponding sequence $(w(1), w(2), \ldots, w(n))$.

A convenient way to represent this order is by drawing a graph, which is called the Hasse diagram of the order, or simply *Bruhat graph*. It is an oriented graph, whose vertices are elements of W_G, and any two vertices u and v are connected by an edge, iff the elementary relation $u \prec v$ (see above) holds.

There is an important relation between Bruhat order on S_n and the geometry of full flag space

$$Fl(n,\mathbb{R}) = SL(n,\mathbb{R})/B^+(n,\mathbb{R})$$

and of the group $SL(n,\mathbb{R})$ too; alternatively one can regard $Fl(n,\mathbb{R})$ as the space of all flags of vector subspaces $V_1 \subset V_2 \subset \cdots \subset V_{n-1}$ in $\mathbb{R}^n$.

To explain this relation, recall that there is a cell decomposition of $Fl(n,\mathbb{R})$, called *Schubert cell decomposition*. One can define it either with the help of the $SL(n,\mathbb{R})$ action on the flag space, or in terms of the geometry of spaces V_i. Cells X_w of this decomposition correspond bijectively to the elements w of S_n. Then one can show, that

$$u < v \Leftrightarrow X_u \subset \overline{X}_v,$$

where $\overline{X}$ denotes the topological closure of the cell. Moreover, one can define *dual Schubert cells* Y_v in $Fl(n,\mathbb{R})$; then

$$u < v \Leftrightarrow X_u \bigcap Y_v \neq \varnothing.$$

One can use Bruhat order on S_n in order to define similar order on multiset permutations. Recall, that multiset a is a "set with repeating elements", i.e. one can regard it as a collection of n elements $a_1, \dots, a_n$, where we assume that

$$a_1 = a_2 = \cdots = a_{k_1}, a_{k_1+1} = a_{k_1+2} = \cdots = a_{k_1+k_2}, \dots,$$
$$a_{k_1+\cdots+k_{m-1}+1} = a_{k_1+\cdots+k_{m-1}+2} = \cdots = a_{k_1+\cdots+k_{m-1}+k_m},$$

in particular $n = k_1 + k_2 + \cdots + k_m$.

Then any permutation $w \in S_n$ can be applied to the set $(a_1, \dots, a_n)$, sending it to the string

$$w(a) = (a_{w(1)}, a_{w(2)}, \dots, a_{w(n)}).$$

Clearly, many permutations w will yield the same string $w(a)$. We shall call all such strings "permutations with repetitions". The action of S_n on multiset a has an evident stabilizer $S_{k_1} \times S_{k_2} \times \cdots \times S_{k_m}$. Thus, for every permutation with repetition, there are exactly $k_1! \dots k_m!$ different permutations in S_n, that represent it, so that there are exactly $\dfrac{n!}{k_1!k_2!\dots k_m!}$ permutations with repetitions of the multiset a.

We shall denote the set of permutations of a multiset a by $S(a)$. One can use the considerations of previous slide to pull the Bruhat order from S_n to $S(a)$. Namely, for any two permutations $x, y \in S(a)$ we consider their preimages $s(x), s(y)$ in S_n (it is clear, that they do not intersect). Then we shall say that

$$x \prec y \Leftrightarrow \exists\, u \in s(x),\ v \in s(y):\ u \prec v.$$

Here on the right hand we use the Bruhat order in S_n.

One can find a geometrical interpretation of this partial order too. To this end we consider the partial flag space $Fl(k_1, \dots, k_m, \mathbb{R})$; it can be obtained from $Fl(n, \mathbb{R})$ by removing certain subspaces V_i from the list; there exists a natural projection

$$Fl(n, \mathbb{R}) \to Fl(k_1, \dots, k_m, \mathbb{R}),$$

given by forgetting the “extra” spaces. One can use this projection to pull Schubert cell decomposition to the partial flag space. This time the cells will be enumerated by the elements of $S(a)$. Then we shall have a similar relation between the order on $S(a)$ and intersections of the cells.

Let us now briefly recall the main results from the theory of Toda flows; we begin with the full symmetric Toda lattice. Recall that eigenvalues of the Lax matrix L are first integrals of the flow. Let us fix them. Then in dimension n the system induces a gradient flow on the orthogonal group $SO(n,\mathbb{R})$, which we here shall call *the Toda flow on* $SO(n,\mathbb{R})$, or just *Toda system* on $SO(n,\mathbb{R})$. This flow is determined by the vector field

$$M(\Psi) = \left((\Psi\Lambda\Psi^{-1})_+ - (\Psi\Lambda\Psi^{-1})_-\right)\Psi,$$

where Λ is the diagonal matrix of eigenvalues of the symmetric Lax matrix L and $\Psi \in SO(n,\mathbb{R})$. To obtain it observe that every symmetric matrix with distinct eigenvalues can be represented in the form $L = \Psi\Lambda\Psi^T$, and this representation is locally unique.

One can show, that vector field $M(\Psi)$ is equal to the gradient of a function F with respect to an invariant metric on $SO(n,\mathbb{R})$: for a fixed eigenvalues matrix Λ one can take

$$F(\Psi) = Tr(\Psi\Lambda\Psi^T N), \text{ where } N = diag(0, 1, \dots, n-1).$$

The $SO(n,\mathbb{R})$-invariant Riemannian structure that we use here is determined by its values on $\mathfrak{so}_n$, where it is given by the formula

$$\langle A, B\rangle_J = -Tr(AJ^{-1}(B)),$$

for any antisymmetric matrices A and B and a linear isomorphism $J : \mathfrak{so}_n \to \mathfrak{so}_n$. Then

$$M(\Psi) = grad_{\langle,\rangle_J} F,$$

It turns out that the last equation does not depend on whether the eigenvalues are all distinct, or not. Same formulas as above induce a vector field, a function and a Riemannian structure on the full flag space

$$Fl_n(\mathbb{R}) = SO(n;\mathbb{R})/T_n^+,$$

so that we obtain a gradient field M on $Fl_n(\mathbb{R})$. It can be shown (see Chernyakov, Sharygin, Sorin, CMP, 2014 and F. De Mari, M. Pedroni, J.G.A.,19 that the function F is Morse function both on $SO(n,\mathbb{R})$ and on the flag space $Fl_n(\mathbb{R})$.

In a similar way, one can modify the constructions of the previous slide to construct gradient flows on the partial flag spaces in case of a generic set of eigenvalues. So let us suppose that there are coinciding eigenvalues of Λ; more accurately, let there be $m < n$ distinct eigenvalues of Λ, with multiplicities $k_1, \dots, k_m$. In this case the vector field $M(\Psi)$ on $SO(n;\mathbb{R})$ is invariant with respect to $O(k_1,\mathbb{R}) \times \cdots \times O(k_m,\mathbb{R})$, i.e.

$$M(\Psi g) = M(\Psi)g$$

for all $g \in SO(n,\mathbb{R}) \bigcap (O(k_1,\mathbb{R}) \times \cdots \times O(k_m,\mathbb{R}))$.

This group action induces the projection $Fl(n,\mathbb{R}) \to Fl(k_1, \dots, k_m, \mathbb{R})$, mentioned earlier. Using the group action we can pull $M(\Psi)$ down along this projection and obtain a vector field $\widetilde{M}$ on the partial flag space. It turns out that all the naturality conditions hold (in particular the function F and the Riemannian structure $\langle , \rangle_J$ are invariant with respect tot the group action). Thus the field $\widetilde{M}$ is equal to the gradient of a function $\widetilde{F}$ ("image of F" with respect to projection) with respect to the induced Riemannian structure. Moreover, one can show that the function $\widetilde{F}$ is a Morse function on $Fl(k_1, \dots, k_m, \mathbb{R})$.

Another important property of the Toda system is that it admits a wide collection of invariant manifolds in orthogonal groups and in flag manifolds. An important large collection of such invariant sets is given by so-called *minor surfaces*: they are the null-sets of the functions $M_I^+(g),\ M_J^-$ (where $I,\ J$ are poly-index sets $I = (1 \le i_1 < i_2 < \cdots < i_p \le n),\ J = (1 \le j_1 < j_2 < \cdots < j_q \le n)$ and $g \in SO(n,\mathbb{R})$), given by the formulas

$$M_I^+(g) = \det(g^I_{1,2,\dots,p}),\ M_J^-(g) = \det(g^J_{n-q+1,\dots,n-1,n}),$$

where $g^I_{1,2,\dots,p}$) and $g^J_{n-q+1,\dots,n-1,n}$ are the sub matrices of g, spanned by the first p (resp. last q) rows and the columns, given by I (resp by J).

Another important family of invariant submanifolds of Toda flow on $SO(n,\mathbb{R})$ (or on the corresponding flag space) consists of Schubert varieties (or their images in $SO(n,\mathbb{R})$). Of course, one should add all the intersections of all minor surfaces and other invariant surfaces to this list.

Finally, one can embed Toda system on any classical group into the Toda system of $SL(n, \mathbb{R})$. Namely, if we embed the group G into a suitable $SL(n, \mathbb{R})$ so that the Cartan subalgebra is mapped into the diagonal matrices and root vectors correspond to the upper or lower triangular matrices; then the Toda equation on G is given by the usual matrix equation

$$\dot{L} = -[L, M].$$

One can show, that the system, obtained in this way is indeed Toda system on G as it has been introduced earlier. It is also possible to use this approach to show that the Toda system on any Lie group G can also be related to a gradient flow on the maximal compact subgroup of G. In fact, one can describe this gradient flow in terms of the Lie algebra structure of G (in terms of root vectors etc.), without the use of the embedding, we use here. However, this embedding is important, since it allows one use invariant surfaces from $SO(n, \mathbb{R})$ on the image of G to obtain invariant manifolds of the generalized Toda system.

Now the outline of the proofs is as follows:

- **Step 1** Show that invariant surfaces in $Fl(n, \mathbb{R})$, given by unions of Schubert cells or dual Schubert cells with common projection into $Fl(k_1, \dots, k_m, \mathbb{R})$, intersect if and only if the corresponding permutations with repetitions are comparebale with respect to the Bruhat order.
- **Step 2** Show that the stable/unstable manifolds of Toda flow in $Fl(k_1, \dots, k_m, \mathbb{R})$ coincide with the images of Schubert cells or dual Schubert cells (in neighborhoods of singular points).

These two steps prove the first statement. And the second one follows from the following

- **Step 3** Embed the remaining two rank 2 groups $Sp(4, \mathbb{R})$ and G_2 into suitable $SL(n, \mathbb{R})$ as explained in previous slide and use minor surfaces to describe the phase portrait of the Morse system.