Higher-dimension invariants of $\mathcal{N}=(1, 1) 6D$ SYM from Harmonic Superspace

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Maximally extended gauge theories (with 16 supersymmetries) are under intensive study for the last few years:

\[ \mathcal{N} = 4, \, 4D \quad \Rightarrow \quad \mathcal{N} = (1, 1), \, 6D \quad \Rightarrow \quad \mathcal{N} = (1, 0), \, 10D. \]

- \( \mathcal{N} = 4, \, 4D \) SYM is UV finite and perhaps completely integrable.
- \( \mathcal{N} = (1, 1), \, 6D \) SYM is not renormalizable by formal power counting (the coupling constant is dimensionful) but is expected to also possess various unique properties.
- In particular, it respects the so called “dual superconformal symmetry” like its 4D counterpart.
- It provides the effective theory descriptions of some particular low energy sectors of string theory, such as D5-brane dynamics.
- \( \mathcal{N} = (1, 1) \) SYM is anomaly free (Frampton, Kephart, 1983, et al), as distinct from \( \mathcal{N} = (1, 0) \) SYM.
- \( \mathcal{N} = (1, 1) \) and \( \mathcal{N} = (1, 0) \) SYM are analogs of \( \mathcal{N} = 8 \) supergravity (also formally non-renormalizable).
The full effective action of D5-brane is expected to be of non-abelian Born-Infeld type, generalizing the $\mathcal{N} = (1, 1)$ SYM action (Tseytlin, 1997).

The newest perturbative explicit calculations in $\mathcal{N} = (1, 1)$ SYM show a lot of cancelations of the UV divergencies which cannot be predicted in advance.

The theory is UV-finite up to 2 loops, while at 3 loops only a single-trace counterterm of canonical dim 10 is required. The allowed double-trace counterterms do not appear (Bern et al, 2010, 2011; Berkovits et al, 2009; Bjornsson et al, 2011, 2012).

New non-renormalization theorems? The maximally supersymmetric off-shell formulations are needed!

Maximum that one can achieve - off-shell $\mathcal{N} = (1, 0)$ SUSY. The most natural off-shell formulation of $\mathcal{N} = (1, 0)$ SYM - in harmonic $\mathcal{N} = (1, 0), 6D$ superspace (Howe et al, 1985; Zupnik, 1986) as a generalization of the $\mathcal{N} = 2, 4D$ HSS (Galperin et al, 1984).
[\mathcal{N} = (1, 0) \text{ SYM} + 6D \text{ hypermultiplet}] = [\mathcal{N} = (1, 1) \text{ SYM}], with the second hidden on-shell \mathcal{N} = (0, 1) \text{ SUSY}.

How to construct higher-dimension \( \mathcal{N} = (1, 1) \) invariants in terms of \( \mathcal{N} = (1, 0) \) superfields?

The “brute-force” method: To start with the appropriate dimension \( \mathcal{N} = (1, 0) \) SYM invariant and then to complete it to \( \mathcal{N} = (1, 1) \) invariant by adding the proper hypermultiplet terms. Very cumbersome technically and actually works only for the lowest-order invariants.

The situation is simplified by the fact that for finding all admissible superfield counterterms it is enough to stay on the mass shell.

One of the main results of our work with Bossard and Smilga is developing of the new approach to constructing higher-dimension \( \mathcal{N} = (1, 1) \) invariants based on the concept of the on-shell \( \mathcal{N} = (1, 1) \) harmonic superspace with the double set of the harmonic variables \( u_i^\pm, u_A^\pm, i = 1, 2; A = 1, 2 \) (Bossard, Howe & Stelle, 2009) and solving the \( \mathcal{N} = (1, 1) \) SYM constraints in terms of \( \mathcal{N} = (1, 0) \) superfields.

The \( d = 8 \) and \( d = 10 \) invariants were explicitly constructed and an essential difference between the single- and double-trace \( d = 10 \) invariants was established.
6D superspaces

- The standard $\mathcal{N} = (1, 0), 6D$ superspace:
  \[ z = (x^M, \theta^a_i), \quad M = 0, \ldots, 5, \ a = 1, \ldots, 4, \ i = 1, 2, \]
  with Grassmann pseudoreal $\theta^a_i$.

- The harmonic $\mathcal{N} = (1, 0), 6D$ superspace:
  \[ Z := (z, u) = (x^M, \theta^a_i, u^{\pm i}), \quad u^-_i = (u^+_i)^*, \ u^+_i u^-_i = 1, \ u^{\pm i} \in SU(2)_R/U(1). \]

- The analytic $\mathcal{N} = (1, 0), 6D$ superspace:
  \[ \zeta := (x^M_{(an)}, \theta^a i, u^{\pm i}) \subset Z, \quad x^M_{(an)} = x^M + \frac{i}{2} \theta^a_i \gamma_{ab} \theta^b_i u^+ k u^- l, \quad \theta^{\pm a} = \theta^a_i u^{\pm i}. \]

- Basic differential operators in the analytic basis:
  \[ D^+_a = \partial_{-a}, \quad D^-_a = -\partial_{+a} - 2i \theta^{-b} \partial_{ab}, \]
  \[ D^0 = u^+_i \frac{\partial}{\partial u^+_i} - u^-_i \frac{\partial}{\partial u^-_i} + \theta^{+a} \partial_{+a} - \theta^{-a} \partial_{-a} \]
  \[ D^{++} = \partial^{++} + i \theta^a_i \theta^{-b} \partial_{ab} + \theta^a_i \partial_{-a}, \quad D^{--} = \partial^{--} + i \theta^{-a} \theta^{--} \partial_{ab} + \theta^{-a} \partial_{+a}, \]
  where $\partial_{\pm a} \theta^{\pm b} = \delta^b_a$ and $\partial^{++} = u^+_i \frac{\partial}{\partial u^-_i}$, $\partial^{--} = u^-_i \frac{\partial}{\partial u^+_i}$. 
Basic superfields

- **Analytic gauge** $\mathcal{N} = (1, 0)$ SYM connection:
  \[ \nabla^{++} = D^{++} + V^{++}, \quad \delta V^{++} = -\nabla^{++} \Lambda, \quad \Lambda = \Lambda(\zeta). \]

- **Second harmonic (non-analytic) connection**:
  \[ \nabla^{--} = D^{--} + V^{--}, \quad \delta V^{--} = -\nabla^{--} \Lambda. \]

- **Related by the harmonic flatness condition**
  \[ [\nabla^{++}, \nabla^{--}] = D^0 \Rightarrow D^{++} V^{--} - D^{--} V^{++} + [V^{++}, V^{--}] = 0 \]
  \[ \Rightarrow V^{--} = V^{--}(V^{++}, u^\pm). \]

- **Wess-Zumino gauge**:
  \[ V^{++} = \theta^+ a \theta^+ b A_{ab} + 2(\theta^+)^3 a \lambda^{-a} - 3(\theta^+)^4 D^{--}. \]
  Here $A_{ab}$ is the gauge field, $\lambda^{-a} = \lambda^a i u^-_i$ is the gaugino and $D^{--} = D^{ik} u^-_i u^-_k$, where $D^{ik} = D^{ki}$, are the auxiliary fields.
Covariant derivatives

\[ \nabla_a = [\nabla^-, D_a^+] = D_a^- + A_a^-, \quad \nabla_{ab} = \frac{1}{2i}[D_a^+, \nabla^-] = \partial_{ab} + A_{ab}, \]

\[ A_a^-(V) = -D_a^+ V^-, \quad A_{ab}(V) = \frac{i}{2}D_a^+ D_b^+ V^-, \]

\[ [\nabla^{++}, \nabla^-] = D_a^+, \quad [\nabla^{++}, D_a^+] = [\nabla^-, \nabla^-] = [\nabla^{\pm\pm}, \nabla_{ab}] = 0. \]

Covariant superfield strengths

\[ [D_a^+, \nabla_{bc}] = \frac{i}{2}\varepsilon_{abcd} W^{+d}, \quad [\nabla^-, \nabla_{bc}] = \frac{i}{2}\varepsilon_{abcd} W^{-d}, \]

\[ W^{+a} = -\frac{1}{6}\varepsilon^{abcd} D_b^+ D_c^+ D_d^+ V^-, \quad W^{-a} := \nabla^- W^{+a}, \]

\[ \nabla^{++} W^{+a} = \nabla^- W^{-a} = 0, \quad \nabla^{++} W^{-a} = W^{+a}, \]

\[ D_b^+ W^{+a} = \delta^a_b F^{++}, \quad F^{++} = \frac{1}{4}D_a^+ W^{+a} = (D^+)^4 V^-, \]

\[ \nabla^{++} F^{++} = 0, \quad D_a^+ F^{++} = 0. \]

Hypermultiplet

\[ q^{+A}(\zeta) = q^{iA}(x) u_i^+ - \theta^+ a \psi^A_a(x) + \text{An infinite tail of auxiliary fields, } A = 1, 2. \]
\( \mathcal{N} = (1, 0) \) superfield actions

- The \( \mathcal{N} = (1, 0) \) SYM action (Zupnik, 1986):

  \[
  S^{\text{SYM}} = \frac{1}{f^2} \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n} \text{Tr} \int d^6x \, d^8\theta \, du_1 \ldots du_n \frac{V^{++}(z, u_1) \ldots V^{++}(z, u_n)}{(u_1^+ u_2^+) \ldots (u_n^+ u_1^+)} ,
  \]

  \[
  \delta S^{\text{SYM}} = 0 \Rightarrow F^{++} = 0 .
  \]

- The hypermultiplet action

  \[
  S^q = -\frac{1}{2f^2} \text{Tr} \int d\zeta^{-4} q^A \nabla^{++} q_A^+ , \quad \nabla^{++} q_A^+ = D^{++} q_A^+ + [V^{++}, q_A^+] ,
  \]

  \[
  \delta S^q = 0 \Rightarrow \nabla^{++} q^A = 0 .
  \]

- The \( \mathcal{N} = (1, 0) \) superfield form of the \( \mathcal{N} = (1, 1) \) SYM action:

  \[
  S^{(V+q)} = S^{\text{SYM}} + S^q = \frac{1}{f^2} \left( \int dZ L^{\text{SYM}} - \frac{1}{2} \text{Tr} \int d\zeta^{-4} q^A \nabla^{++} q_A^+ \right) ,
  \]

  \[
  \delta S^{(V+q)} = 0 \Rightarrow F^{++} + \frac{1}{2} [q^A, q_A^+] = 0 , \quad \nabla^{++} q^A = 0 .
  \]

  It is invariant under the second \( \mathcal{N} = (0, 1) \) supersymmetry:

  \[
  \delta V^{++} = \epsilon^{++} q_A^+ , \quad \delta q^A = -(D^+)^4 (\epsilon_A^- V^{--}) , \quad \epsilon_A^{\pm} = \epsilon_{aA} \theta^{\pm a} .
  \]
Higher-dimensional $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (1, 1)$ invariants

- $d = 6$: In the pure SYM case it is unique

$$S^{(6)}_{SYM} = \frac{1}{2g^2} \Tr \int d\zeta^{-4} du \ (F^{++})^2 \sim \Tr \int d^6x [(\nabla^M F_{ML})^2 + \ldots]$$

- Does its off-shell completion to an off-shell $\mathcal{N} = (1, 1)$ invariant exist? The answer is NO, only an expression can be found whose $\mathcal{N} = (0, 1)$ variation vanishes on-shell. It is unique up to a real parameter

$$\mathcal{L}^{d=6} = \frac{1}{2g^2} \Tr \int du d\zeta^{(-4)} \left( F^{++} + \frac{1}{2} [q^+ A, q^+_A] \right) \left( F^{++} + 2\beta[q^+ A, q^+_A] \right)$$

But it vanishes on-shell by itself! We have thus shown that the non-vanishing on-shell counterterms of canonical dimension 6 are absent, and this proves one-loop finiteness of $\mathcal{N} = (1, 1)$ SYM.
\( d = 8 \): All \( \mathcal{N} = (1, 0) \) superfield terms of such dimension in the pure \( \mathcal{N} = (1, 0) \) SYM theory prove to vanish on the gauge fields mass shell, in accord with the old statement (Howe & Stelle, 1984). Can adding the hypermultiplet terms change this? Our analysis showed that there exist NO \( \mathcal{N} = (1, 0) \) supersymmetric off-shell invariants of the dimension 8 which would respect the on-shell \( \mathcal{N} = (1, 1) \) invariance.

Surprisingly, the \( d = 8 \) superfield expression which is non-vanishing on shell and respects the on-shell \( \mathcal{N} = (1, 1) \) supersymmetry can be constructed by giving up the requirement of off-shell \( \mathcal{N} = (1, 0) \) supersymmetry.

An example of such an invariant in \( \mathcal{N} = (1, 0) \) SYM is very simple

\[
\tilde{S}^{(8)}_1 \sim \text{Tr} \int d\zeta^{-4} \varepsilon_{abcd} W^{+a} W^{+b} W^{+c} W^{+d}.
\]

Indeed, \( D_a^+ W^{+b} = \delta^b_a F^{++} \), which vanishes on shell. Thus, \( W^{+a} \) is an analytic superfield, when disregarding the terms proportional to the equations of motion, and the above action respects \( \mathcal{N} = (1, 0) \) supersymmetry on shell. Also, double-trace on-shell invariant exists

\[
\tilde{S}^{(8)}_2 \sim \int d\zeta^{-4} \varepsilon_{abcd} \text{Tr}(W^{+a}W^{+b})\text{Tr}(W^{+c}W^{+d}).
\]

Do these invariants admit \( \mathcal{N} = (1, 1) \) completions? YES, they do!
By varying the pure $\mathcal{N} = (1, 0)$ SYM action by the transformations of the second hidden $\mathcal{N} = (0, 1)$ supersymmetry and picking the appropriate compensating hypermultiplet terms, after rather cumbersome computations we find

\[
\mathcal{L}^{+4}_{(1,1)} = \text{Tr}(S) \left\{ \frac{1}{4} \varepsilon_{abcd} W^{+a} W^{+b} W^{+c} W^{+d} + 3i q^+A \nabla_{ab} q^+_A W^{+a} W^{+b} - q^+A \nabla_{ab} q^+_A q^+_B \nabla^{ab} q^+_B - W^{+a} [D^+_a q^-_B, q^+_B] q^+A q^+B - \frac{1}{2} [q^+_C, q^+_C] [q^-_A, q^+_B] q^+A q^+B \right\}.
\]

It is analytic, $D^+_a \mathcal{L}^{+4}_{(1,1)} = 0$, on the full shell $F^{++} + \frac{1}{2} [q^+A, q^+_A] = 0$, $\nabla^{++} q^+A = 0$, and also $\mathcal{N} = (1, 1)$ supersymmetric. $\text{Tr}(S)$ stands for the symmetrized trace. Also, it is possible to extend the double-trace $d = 8$ invariant in a similar way.

Though the nontrivial on-shell $d = 8$ invariants exist, the perturbative expansion for the amplitudes for the $\mathcal{N} = (1, 1)$ SYM theory does not involve divergences at the two-loop level. The matter is that these invariants do not possess the full off-shell $\mathcal{N} = (1, 0)$ supersymmetry which the physically relevant counterterms should obey. Indeed, we have at hand the harmonic off-shell $\mathcal{N} = (1, 0)$ superfields. On the basis of that, one can construct the $\mathcal{N} = (1, 0)$ gauge-covariant supergraph technique such that all the amplitudes and the counterterms would enjoy $\mathcal{N} = (1, 0)$ supersymmetry off shell.
\( \mathcal{N} = (1, 1) \) on-shell harmonic superspace

- Though such \( d = 8 \) terms cannot appear as counterterms in \( \mathcal{N} = (1, 1) \) SYM theory, they can appear, e.g., as quantum corrections to the effective Wilsonian action. For the pure \( \mathcal{N} = (1, 0) \) SYM theory this was recently observed in Buchbinder & Pletnev, 2015. It would be desirable to work out some simple and systematic way of constructing such higher-order on-shell \( \mathcal{N} = (1, 1) \) invariants. This becomes possible in the framework of the on-shell harmonic \( \mathcal{N} = (1, 1) \) superspace.

- As the first step, extend the \( \mathcal{N} = (1, 0) \) superspace to the \( \mathcal{N} = (1, 1) \) one,

\[
\mathbf{z} = (x^{ab}, \theta^a_i) \Rightarrow \hat{\mathbf{z}} = (x^{ab}, \theta^a_i, \hat{\theta}^A_a).
\]

- Then we define the covariant spinor derivatives,

\[
\nabla^i_a = \frac{\partial}{\partial \theta^a_i} - i \theta^{bi} \partial_{ab} + \mathcal{A}^i_a, \quad \hat{\nabla}^A_a = \frac{\partial}{\partial \hat{\theta}^A_a} - i \hat{\theta}^{Aa} \partial^{ab} + \hat{\mathcal{A}}^{Aa}.
\]

- The constraints defining the \( \mathcal{N} = (1, 1) \) SYM theory are as follows (Howe, Sierra, Townsend, 1983; Howe, Stelle, 1984):

\[
\{ \nabla^i_a, \nabla^j_b \} = \{ \hat{\nabla}^A_a, \hat{\nabla}^{bB} \} = 0, \quad \{ \nabla^i_a, \hat{\nabla}^{bA} \} = \delta^b_a \phi^{iA}
\]

\[
\Rightarrow \quad \nabla^i_a \phi^j = \hat{\nabla}^A_a \phi^{Bj} = 0 \quad \text{(By Bianchis)}.
\]
As the next step, we define the $\mathcal{N} = (1, 1)$ HSS with the double set of harmonics (Bossard, Howe & Stelle, 2009):

$$Z = (x^{ab}, \theta^a_i, u^\pm_k) \Rightarrow \hat{Z} = (x^{ab}, \theta^a_i, \hat{\theta}^A_b, u^\pm_k, u^\pm_A)$$

Then we pass to the analytic basis and choose the “hatted” spinor derivatives short, $\nabla^{\hat{+}a} = D^{\hat{+}a} = \frac{\partial}{\partial \hat{\theta}^a}$. The set of constraints in the ordinary $\mathcal{N} = (1, 1)$ superspace amounts to the following set in the $\mathcal{N} = (1, 1)$ HSS

$$\{\nabla^{+}_a, \nabla^{+}_b\} = 0, \quad \{D^{\hat{+}a}, D^{\hat{+}b}\} = 0, \quad \{\nabla^{+}_a, D^{\hat{+}b}\} = \delta^b_a \phi^{\hat{+}\hat{+}},$$

$$[\nabla^{\hat{+}\hat{+}}, \nabla^{+}_a] = 0, \quad [\tilde{\nabla}^{++}, \nabla^{+}_a] = 0, \quad [\nabla^{\hat{+}\hat{+}}, D^{\hat{a}\hat{+}}] = 0, \quad [\tilde{\nabla}^{++}, D^{\hat{a}\hat{+}}] = 0, \quad [\tilde{\nabla}^{++}, \nabla^{\hat{+}\hat{+}}] = 0.$$

Here

$$\nabla^{+}_a = D^{+}_a + A^{+}_a(\hat{Z}), \quad \tilde{\nabla}^{++} = D^{++} + \tilde{V}^{++}(\hat{\zeta}), \quad \nabla^{\hat{+}\hat{+}} = D^{\hat{+}\hat{+}} + V^{\hat{+}\hat{+}}(\hat{\zeta}),$$

$$\hat{\zeta} = (x^{ab}_a, \theta^{\pm a}_i, \theta^{\hat{+}}_c, u^\pm_i, u^\hat{\pm}_A).$$
Solving $\mathcal{N} = (1, 1)$ SYM constraints

- The starting point of our analysis was to fix, using the $\Lambda(\hat{\zeta})$ gauge freedom, the WZ gauge for the second harmonic connection $V^{++}(\hat{\zeta})$ as

$$V^{++} = i\theta_a^\dagger \theta_b^\dagger \hat{A}^{ab} + \epsilon^{abcd} \theta_a^\dagger \theta_b^\dagger \theta_c^\dagger \theta_d^\dagger \varphi_d u_A + \epsilon^{abcd} \theta_a^\dagger \theta_b^\dagger \theta_c^\dagger \theta_d^\dagger \mathcal{D}^{AB} u_A u_B$$

where $\hat{A}^{ab}$, $\varphi_d^A$ and $\mathcal{D}^{(AB)}$ are some $\mathcal{N} = (1, 0)$ harmonic superfields, still arbitrary at this step.

- Then the above constraints are reduced to some sets of harmonic equations which we have explicitly solved. The crucial point was the requirement that the vector 6$\mathcal{D}$ connections in the sectors of hatted and unhatted variables are identical to each other.

- As the result, we have obtained that the first harmonic connection $V^{++}$ coincides precisely with the previous $\mathcal{N} = (1, 0)$ one, $V^{++} = V^{++}(\hat{\zeta})$, while the dependence of all other geometric $\mathcal{N} = (1, 1)$ objects on the variables with “hat” is strictly fixed

$$V^{++} = i\theta_a^\dagger \theta_b^\dagger \hat{A}^{ab} - \frac{1}{3} \epsilon^{abcd} \theta_a^\dagger \theta_b^\dagger \theta_c^\dagger \theta_d^\dagger D_d q^{--} + \frac{1}{8} \epsilon^{abcd} \theta_a^\dagger \theta_b^\dagger \theta_c^\dagger \theta_d^\dagger [q^{++}, q^{--}]$$

$$\phi^{++} = q^{++} - \theta_a^\dagger W^{+a} - i\theta_a^\dagger \nabla^{ab} q^{+-} + \frac{1}{6} \epsilon^{abcd} \theta_a^\dagger \theta_b^\dagger \theta_c^\dagger \theta_d^\dagger [D_d q^{--}, q^{++}] + \frac{1}{24} \epsilon^{abcd} \theta_a^\dagger \theta_b^\dagger \theta_c^\dagger \theta_d^\dagger [q^{+-}, [q^{+-}, q^{--}]].$$
Here, $q^{\pm A}(\zeta) u^\pm_A$, $q^{-A}(\zeta) u^\pm_A$ and $W^+, q^{\pm A}$ are just the $\mathcal{N} = (1, 0)$ superfields explored previously. In the process of solving the constraints, there appeared the analyticity conditions for $q^{+A}$, as well as the full set of the superfield equations of motion

$$\nabla^{++} q^{+A} = 0, \quad F^{++} = \frac{1}{4} D_a^+ W^+ = -\frac{1}{2}[q^{+A}, q^{+A}].$$

Also, the structure of the spinor covariant derivatives was fully fixed

$$\nabla^+_a = D^+_a - \theta^+_a q^{^+_^\pm} + \theta^+_a \phi^{^+_^\pm},$$
$$\nabla^-_a = D^-_a - D^+_a V^- - \theta^+_a q^{^-_^\pm} + \theta^+_a \phi^{^-_^\pm}, \quad \phi^{^-_^\pm} = \nabla^{--} \phi^{^+_^\pm}. $$

The basic advantage of using the constrained $\mathcal{N} = (1, 1)$ strengths $\phi^{\pm_\pm}$ for constructing various invariants is their extremely simple transformation rules under the hidden $\mathcal{N} = (0, 1)$ supersymmetry

$$\delta \phi^{\pm_\pm} = -\epsilon^{\pm_\pm}_a \frac{\partial}{\partial \theta^+_a} \phi^{\pm_\pm} - 2i \epsilon^{\pm_\pm}_a \theta^+_a \partial^{ab} \phi^{\pm_\pm} - [\Lambda^{(comp)}, \phi^{\pm_\pm}],$$

where $\Lambda^{(comp)}$ is some common composite gauge parameter which does not contribute under $Tr$. 
Invariants in $\mathcal{N} = (1, 1)$ superspace

- The previous single-trace $d = 8$ invariant Lagrangian admits a simple rewriting in $\mathcal{N} = (1, 1)$ superspace

$$S_{(1,1)} = \int dud\zeta^{(-4)} \mathcal{L}_{(1,1)}^{+4}, \quad \mathcal{L}_{(1,1)}^{+4} = -\text{Tr} \frac{1}{4} \int d\zeta^{(-4)} d\hat{u} (\phi^{++})^4, \quad d\zeta^{(-4)} \sim (D^-)^4$$

$$\delta \mathcal{L}_{(1,1)}^{+4} = -2i\partial^{ab}\text{Tr} \int d\zeta^{(-4)} d\hat{u} \left[ \epsilon_a^\hat{\theta}^b \frac{1}{4} (\phi^{++})^4 \right].$$

- The double-trace $d = 8$ invariant is given by

$$\hat{\mathcal{L}}_{(1,1)}^{+4} = -\frac{1}{4} \int d\zeta^{(-4)} d\hat{u} \text{Tr} (\phi^{++})^2 \text{Tr} (\phi^{+-})^2.$$

- Now it is easy to construct the single- and double-trace $d = 10$ invariants possibly responsible for the 3-loop counterterms

$$S_{1}^{(10)} = \text{Tr} \int dZ d\zeta^{(-4)} d\hat{u} (\phi^{++})^2 (\phi^{-+})^2, \quad \phi^{-+} = \nabla^{--} \phi^{++},$$

$$S_{2}^{(10)} = -\int dZ d\zeta^{(-4)} d\hat{u} \text{Tr}(\phi^{++} \phi^{-+}) \text{Tr}(\phi^{++} \phi^{-+}).$$

- These are $\mathcal{N} = (1, 1)$ extensions of the pure $\mathcal{N} = (1, 0)$ SYM invariants

$$\sim \epsilon_{abcd} \text{Tr} (W^{+a} W^{-b} W^{+c} W^{-d}), \sim \epsilon_{abcd} \text{Tr} (W^{+a} W^{-b}) \text{Tr} (W^{+c} W^{-d}).$$
It is notable that the single-trace $d = 10$ invariant admits a representation as an integral over the full $\mathcal{N} = (1, 1)$ superspace

$$S_1^{(10)} \sim \text{Tr} \int dZ d\hat{Z} \phi^{++} \phi^{--}, \quad \phi^{--} = \nabla^{--} \phi^{-+}.$$ 

On the other hand, the double-trace $d = 10$ invariant cannot be written as the full integral and so looks as being UV protected.

This could explain why in the perturbative calculations of the amplitudes in the $\mathcal{N} = (1, 1)$ SYM single-trace 3-loop divergence is seen, while no double-trace structures at the same order were observed (Berkovits et al 2009; Bjornsson & Green, 2010; Bjornsson, 2011).
However, this does not seem to be like the standard non-renormalization theorems because the quantum calculation of $\mathcal{N} = (1, 0)$ supergraphs should give some invariants in the off-shell $\mathcal{N} = (1, 0)$ superspace, not in the on-shell $\mathcal{N} = (1, 1)$ superspace. So the above property seems not enough to explain the absence of the double-trace divergences and some additional piece of reasoning is needed.

Now there exist new methods in the $6D \mathcal{N} = (1, 1)$ SYM perturbative calculations based on the notion of the so called on-shell harmonic momentum superspace (Dennen et al, 2010). It also involves two sets of harmonic coordinates. Perhaps it is closely related to the $x$-space harmonic $\mathcal{N} = (1, 1)$ superspace approach and would help to prove that all divergent quantum corrections to $\mathcal{N} = (1, 1)$ SYM action arise just as integrals over the whole $\mathcal{N} = (1, 1)$ harmonic superspace.
Summary and outlook

▶ We applied the off-shell $\mathcal{N} = (1, 0)$ and on-shell harmonic $\mathcal{N} = (1, 1)$ superspaces for constructing higher-dimensional invariants in the $\mathcal{N} = (1, 0)$ SYM and $\mathcal{N} = (1, 1)$ SYM theories.

▶ The $\mathcal{N} = (1, 1)$ SYM constraints were solved in terms of harmonic $\mathcal{N} = (1, 0)$ superfields. This allowed us to explicitly construct the full set of the superfield dimensions $d = 8$ and $d = 10$ invariants possessing $\mathcal{N} = (1, 1)$ on-shell supersymmetry.

▶ All possible $d = 6$ $\mathcal{N} = (1, 1)$ invariants were shown to be on-shell vanishing, proving the UV finiteness of $\mathcal{N} = (1, 1)$ SYM at one loop.

▶ The off-shell $d = 8$ invariants are absent. The on-shell ones are integrals over the analytic $\mathcal{N} = (1, 0)$ subspace. Assuming that the $\mathcal{N} = (1, 0)$ supergraphs yield integrals over the full $\mathcal{N} = (1, 0)$ harmonic superspace, this means the absence of two-loop counterterms.

▶ Two $d = 10$ invariants were explicitly constructed as integrals over the whole $\mathcal{N} = (1, 0)$ harmonic superspace. The single-trace invariant can be rewritten as an integral over the $\mathcal{N} = (1, 1)$ superspace, while the double-trace one cannot. This property combined with an additional reasoning could explain why the double-trace invariant is UV protected.
Some further lines of development:

(a) To construct the next $d \geq 12$ invariants in the $\mathcal{N} = (1, 1)$ SYM theory with the help of the on-shell $\mathcal{N} = (1, 1)$ harmonic superspace techniques.

(b) To apply the same method for constructing the Born-Infeld action with the manifest off-shell $\mathcal{N} = (1, 0)$ and hidden on-shell $\mathcal{N} = (0, 1)$ supersymmetries. To check the hypothesis that such an action should coincide with the full quantum effective action of the $\mathcal{N} = (1, 1)$ SYM theory.

(c) To develop an analogous on-shell harmonic $\mathcal{N} = 4, 4D$ superspace approach to the $\mathcal{N} = 4, 4D$ SYM theory in the $\mathcal{N} = 2$ superfield formulation and apply it to the problem of constructing the relevant effective action.

(d) Applications in supergravity?
THANK YOU!