Higher-dimension invariants of $\mathcal{N}=(1,1)$ $6D$ SYM from Harmonic Superspace

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Maximally extended gauge theories (with 16 supersymmetries) are under intensive study for the last few years:

$$\mathcal{N} = 4,\ 4D \implies \mathcal{N} = (1,1),\ 6D \implies \mathcal{N} = (1,0),\ 10D\,.$$

- $\mathcal{N} = 4,\ 4D$ SYM is UV finite and perhaps completely integrable.
- $\mathcal{N} = (1,1),\ 6D$ SYM is not renormalizable by formal power counting (the coupling constant is dimensionful) but is expected to also possess various unique properties.
- In particular, it respects the so called "dual superconformal symmetry" like its $4D$ counterpart.
- It provides the effective theory descriptions of some particular low energy sectors of string theory, such as D5-brane dynamics.
- $\mathcal{N} = (1,1)$ SYM is anomaly free (Frampton, Kephart, 1983, *et al*), as distinct from $\mathcal{N} = (1,0)$ SYM.
- $\mathcal{N} = (1,1)$ and $\mathcal{N} = (1,0)$ SYM are analogs of $\mathcal{N} = 8$ supergravity (also formally non-renormalizable).

- The full effective action of D5-brane is expected to be of non-abelian Born-Infeld type, generalizing the $\mathcal{N} = (1,1)$ SYM action (Tseytlin, 1997).
- The newest perturbative explicit calculations in $\mathcal{N} = (1,1)$ SYM show a lot of cancelations of the UV divergencies which cannot be predicted in advance.
- The theory is UV-finite up to 2 loops, while at 3 loops only a single-trace counterterm of canonical dim 10 is required. The allowed double-trace counterterms do not appear (Bern *et al*, 2010, 2011; Berkovits *et al*, 2009; Bjornsson *et al*, 2011, 2012).
- New non-renormalization theorems? The maximally supersymmetric off-shell formulations are needed!
- Maximum that one can achieve - off-shell $\mathcal{N} = (1,0)$ SUSY. The most natural off-shell formulation of $\mathcal{N} = (1,0)$ SYM - in harmonic $\mathcal{N} = (1,0), 6D$ superspace (Howe *et al*, 1985; Zupnik, 1986) as a generalization of the $\mathcal{N} = 2, 4D$ HSS (Galperin *et al*, 1984).

- [$\mathcal{N} = (1,0)$ SYM + $6D$ hypermultiplet] = [$\mathcal{N} = (1,1)$ SYM], with the second hidden on-shell $\mathcal{N} = (0,1)$ SUSY.

- How to construct higher-dimension $\mathcal{N} = (1,1)$ invariants in terms of $\mathcal{N} = (1,0)$ superfields?

- The "brute-force" method: To start with the appropriate dimension $\mathcal{N} = (1,0)$ SYM invariant and then to complete it to $\mathcal{N} = (1,1)$ invariant by adding the proper hypermultiplet terms. Very cumbersome technically and actually works only for the lowest-order invariants.

- The situation is simplified by the fact that for finding all admissible superfield counterterms it is enough to stay on the mass shell.

- One of the main results of our work with Bossard and Smilga is developing of the new approach to constructing higher-dimension $\mathcal{N} = (1,1)$ invariants based on the concept of the on-shell $\mathcal{N} = (1,1)$ harmonic superspace with the double set of the harmonic variables $u_i^{\pm}, u_A^{\pm}, i = 1,2; A = 1,2$ (Bossard, Howe & Stelle, 2009) and solving the $\mathcal{N} = (1,1)$ SYM constraints in terms of $\mathcal{N} = (1,0)$ superfields.

- The $d = 8$ and $d = 10$ invariants were explicitly constructed and an essential difference between the single- and double-trace $d = 10$ invariants was established.

$6D$ superspaces

- The standard $\mathcal{N} = (1,0), 6D$ superspace:

$$z = (x^M, \theta^a_i)\,, \quad M = 0, \ldots, 5\,,\ a = 1, \ldots, 4\,,\ i = 1, 2\,,$$

with Grassmann pseudoreal θ^a_i.

- The harmonic $\mathcal{N} = (1,0), 6D$ superspace:

$$Z := (z, u) = (x^M, \theta^a_i, u^{\pm i})\,, \quad u^-_i = (u^+_i)^*, u^{+i}u^-_i = 1\,,\ u^{\pm i} \in SU(2)_R/U(1)\,.$$

- The *analytic* $\mathcal{N} = (1,0), 6D$ superspace:

$$\zeta := (x^M_{(\mathrm{an})}, \theta^{+a}, u^{\pm i}) \subset Z\,, \quad x^M_{(\mathrm{an})} = x^M + \frac{i}{2}\theta^a_k \gamma^M_{ab} \theta^b_l u^{+k} u^{-l}, \quad \theta^{\pm a} = \theta^a_i u^{\pm i}\,.$$

- Basic differential operators in the analytic basis:

$$D^+_a = \partial_{-a}\,, \quad D^-_a = -\partial_{+a} - 2i\theta^{-b}\partial_{ab}\,,$$
$$D^0 = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}} + \theta^{+a}\partial_{+a} - \theta^{-a}\partial_{-a}$$
$$D^{++} = \partial^{++} + i\theta^{+a}\theta^{+b}\partial_{ab} + \theta^{+a}\partial_{-a}\,,\ D^{--} = \partial^{--} + i\theta^{-a}\theta^{-b}\partial_{ab} + \theta^{-a}\partial_{+a}\,,$$

where $\partial_{\pm a}\theta^{\pm b} = \delta^b_a$ and $\partial^{++} = u^{+i}\frac{\partial}{\partial u^{-i}}, \quad \partial^{--} = u^{-i}\frac{\partial}{\partial u^{+i}}$.

Basic superfields

- Analytic gauge $\mathcal{N} = (1,0)$ SYM connection:

$$\nabla^{++} = D^{++} + V^{++}\,, \quad \delta V^{++} = -\nabla^{++}\Lambda\,, \quad \Lambda = \Lambda(\zeta)\,.$$

- Second harmonic (non-analytic) connection:

$$\nabla^{--} = D^{--} + V^{--}\,, \quad \delta V^{--} = -\nabla^{--}\Lambda\,.$$

- Related by the harmonic flatness condition

$$[\nabla^{++}, \nabla^{--}] = D^0 \;\Rightarrow\; D^{++}V^{--} - D^{--}V^{++} + [V^{++}, V^{--}] = 0$$
$$\Rightarrow\; V^{--} = V^{--}(V^{++}, u^{\pm})\,.$$

- Wess-Zumino gauge:

$$V^{++} \;=\; \theta^{+a}\theta^{+b}A_{ab} + 2(\theta^{+})^3_a\lambda^{-a} - 3(\theta^{+})^4\mathcal{D}^{--}\,.$$

Here A_{ab} is the gauge field, $\lambda^{-a} = \lambda^{ai}u^-_i$ is the gaugino and $\mathcal{D}^{--} = \mathcal{D}^{ik}u^-_i u^-_k$, where $\mathcal{D}^{ik} = \mathcal{D}^{ki}$, are the auxiliary fields.

- Covariant derivatives

$$\nabla^-_a = [\nabla^{--}, D^+_a] = D^-_a + \mathcal{A}^-_a\,, \qquad \nabla_{ab} = \frac{1}{2i}[D^+_a, \nabla^-_b] = \partial_{ab} + \mathcal{A}_{ab}\,,$$

$$\mathcal{A}^-_a(V) = -D^+_a V^{--}, \quad \mathcal{A}_{ab}(V) = \frac{i}{2} D^+_a D^+_b V^{--},$$

$$[\nabla^{++}, \nabla^-_a] = D^+_a\,, \quad [\nabla^{++}, D^+_a] = [\nabla^{--}, \nabla^-_a] = [\nabla^{\pm\pm}, \nabla_{ab}] = 0\,.$$

- Covariant superfield strengths

$$[D^+_a, \nabla_{bc}] \;=\; \frac{i}{2}\varepsilon_{abcd} W^{+d}\,, \qquad [\nabla^-_a, \nabla_{bc}] \;=\; \frac{i}{2}\varepsilon_{abcd} W^{-d}\,,$$

$$W^{+a} = -\frac{1}{6}\varepsilon^{abcd} D^+_b D^+_c D^+_d V^{--}\,, \quad W^{-a} := \nabla^{--} W^{+a}\,,$$

$$\nabla^{++} W^{+a} = \nabla^{--} W^{-a} \;=\; 0\,, \quad \nabla^{++} W^{-a} = W^{+a}\,,$$

$$D^+_b W^{+a} = \delta^a_b F^{++}\,, \quad F^{++} = \frac{1}{4} D^+_a W^{+a} = (D^+)^4 V^{--}\,,$$

$$\nabla^{++} F^{++} = 0\,, \quad D^+_a F^{++} = 0\,.$$

- Hypermultiplet

$$q^{+A}(\zeta) = q^{iA}(x) u^+_i - \theta^{+a}\psi^A_a(x) + \text{ An infinite tail of auxiliary fields}\,, \ \mathrm{A} = 1, 2\,.$$

$\mathcal{N}=(1,0)$ superfield actions

- The $\mathcal{N}=(1,0)$ SYM action (Zupnik, 1986):

$$S^{SYM}=\frac{1}{f^2}\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n}\mathrm{Tr}\int d^6x\,d^8\theta\,du_1\ldots du_n\frac{V^{++}(z,u_1)\ldots V^{++}(z,u_n)}{(u_1^+u_2^+)\ldots(u_n^+u_1^+)}\,,$$

$$\delta S^{SYM}=0\ \Rightarrow\ F^{++}=0\,.$$

- The hypermultiplet action

$$S^q=-\frac{1}{2f^2}\mathrm{Tr}\int d\zeta^{-4}q^{+A}\nabla^{++}q^+_A\,,\quad \nabla^{++}q^+_A=D^{++}q^+_A+[V^{++},q^+_A]\,,$$

$$\delta S^q=0\ \Rightarrow\ \nabla^{++}q^{+A}=0\,.$$

- The $\mathcal{N}=(1,0)$ superfield form of the $\mathcal{N}=(1,1)$ SYM action:

$$S^{(V+q)}=S^{SYM}+S^q=\frac{1}{f^2}\left(\int dZ\mathcal{L}^{\mathrm{SYM}}-\frac{1}{2}\mathrm{Tr}\int d\zeta^{-4}q^{+A}\nabla^{++}q^+_A\right),$$

$$\delta S^{(V+q)}=0\ \Rightarrow\ F^{++}+\frac{1}{2}[q^{+A},q^+_A]=0\,,\quad \nabla^{++}q^{+A}=0\,.$$

It is invariant under the second $\mathcal{N}=(0,1)$ supersymmetry:

$$\delta V^{++}=\epsilon^{+A}q^+_A\,,\quad \delta q^{+A}=-(D^+)^4(\epsilon^-_A V^{--})\,,\quad \epsilon^\pm_A=\epsilon_{aA}\theta^{\pm a}\,.$$

Higher-dimensional $\mathcal{N} = (1,0)$ and $\mathcal{N} = (1,1)$ invariants

- $\underline{d = 6}$: In the pure SYM case it is unique

$$S^{(6)}_{SYM} = \frac{1}{2g^2}\mathrm{Tr}\int d\zeta^{-4}du\ (F^{++})^2 \ \sim\ \mathrm{Tr}\int d^6x[(\nabla^M F_{ML})^2 + \ldots]$$

- Does its off-shell completion to an off-shell $\mathcal{N} = (1,1)$ invariant exist? The answer is NO, only an expression can be found whose $\mathcal{N} = (0,1)$ variation vanishes *on-shell*. It is unique up to a real parameter

$$\mathcal{L}^{d=6} = \frac{1}{2g^2}\mathrm{Tr}\int dud\zeta^{(-4)}\left(F^{++} + \frac{1}{2}[q^{+A}, q^+_A]\right)\left(F^{++} + 2\beta[q^{+A}, q^+_A]\right)$$

But it vanishes on-shell by itself! We have thus shown that the non-vanishing on-shell counterterms of canonical dimension 6 are absent, and this proves *one-loop finiteness* of $\mathcal{N} = (1,1)$ SYM.

- $\underline{d=8}$: All $\mathcal{N}=(1,0)$ superfield terms of such dimension in the pure $\mathcal{N}=(1,0)$ SYM theory prove to vanish on the gauge fields mass shell, in accord with the old statement (Howe & Stelle, 1984). Can adding the hypermultiplet terms change this? Our analysis showed that there exist NO $\mathcal{N}=(1,0)$ supersymmetric off-shell invariants of the dimension 8 which would respect the on-shell $\mathcal{N}=(1,1)$ invariance.
- Surprisingly, the $d=8$ superfield expression which is non-vanishing on shell and respects the on-shell $\mathcal{N}=(1,1)$ supersymmetry can be constructed by *giving up* the requirement of *off-shell* $\mathcal{N}=(1,0)$ supersymmetry.
- An example of such an invariant in $\mathcal{N}=(1,0)$ SYM is very simple

$$\tilde{S}_1^{(8)} \sim \mathrm{Tr}\int d\zeta^{-4}\, \varepsilon_{abcd} W^{+a}W^{+b}W^{+c}W^{+d}\,.$$

Indeed, $D_a^+ W^{+b} = \delta_a^b F^{++}$, which vanishes on shell. Thus, W^{+a} is an analytic superfield, when disregarding the terms proportional to the equations of motion, and the above action respects $\mathcal{N}=(1,0)$ supersymmetry on shell. Also, double-trace on-shell invariant exists

$$\tilde{S}_2^{(8)} \sim \int d\zeta^{-4}\, \varepsilon_{abcd}\mathrm{Tr}(W^{+a}W^{+b})\mathrm{Tr}(W^{+c}W^{+d})\,.$$

- Do these invariants admit $\mathcal{N}=(1,1)$ completions? YES, they do!

- By varying the pure $\mathcal{N}=(1,0)$ SYM action by the transformations of the second hidden $\mathcal{N}=(0,1)$ supersymmetry and picking the appropriate compensating hypermultiplet terms, after rather cumbersome computations we find

$$\mathcal{L}^{+4}_{(1,1)} = \mathrm{Tr}_{(S)}\Big\{\frac{1}{4}\varepsilon_{abcd}W^{+a}W^{+b}W^{+c}W^{+d} + 3iq^{+A}\nabla_{ab}q^{+}_{A}W^{+a}W^{+b}$$
$$- q^{+A}\nabla_{ab}q^{+}_{A}\, q^{+B}\nabla^{ab}q^{+}_{B} - W^{+a}[D^{+}_{a}q^{-}_{A}, q^{+}_{B}]q^{+A}q^{+B}$$
$$- \frac{1}{2}[q^{+C}, q^{+}_{C}][q^{-}_{A}, q^{+}_{B}]q^{+A}q^{+B}\Big\}.$$

It is analytic, $D^{+}_{a}\mathcal{L}^{+4}_{(1,1)} = 0$, on the full shell $F^{++} + \frac{1}{2}[q^{+A}, q^{+}_{A}] = 0$, $\nabla^{++}q^{+A} = 0$, and also $\mathcal{N}=(1,1)$ supersymmetric. $\mathrm{Tr}_{(S)}$ stands for the *symmetrized* trace. Also, it is possible to extend the double-trace $d=8$ invariant in a similar way.

- Though the nontrivial on-shell $d=8$ invariants exist, the perturbative expansion for the amplitudes for the $\mathcal{N}=(1,1)$ SYM theory does not involve divergences at the two-loop level. The matter is that these invariants do *not* possess the full off-shell $\mathcal{N}=(1,0)$ supersymmetry which the physically relevant counterterms should obey. Indeed, we have at hand the harmonic off-shell $\mathcal{N}=(1,0)$ superfields. On the basis of that, one can construct the $\mathcal{N}=(1,0)$ gauge-covariant supergraph technique such that all the amplitudes and the counterterms would enjoy $\mathcal{N}=(1,0)$ supersymmetry *off shell*.

$\mathcal{N} = (1,1)$ on-shell harmonic superspace

- Though such $d = 8$ terms cannot appear as counterterms in $\mathcal{N} = (1,1)$ SYM theory, they can appear, e.g., as quantum corrections to the effective Wilsonian action. For the pure $\mathcal{N} = (1,0)$ SYM theory this was recently observed in Buchbinder & Pletnev, 2015. It would be desirable to work out some simple and systematic way of constructing such higher-order on-shell $\mathcal{N} = (1,1)$ invariants. This becomes possible in the framework of the on-shell harmonic $\mathcal{N} = (1,1)$ superspace.
- As the first step, extend the $\mathcal{N} = (1,0)$ superspace to the $\mathcal{N} = (1,1)$ one,

$$z = (x^{ab}, \theta^a_i) \;\Rightarrow\; \hat{z} = (x^{ab}, \theta^a_i, \hat{\theta}^A_a).$$

- Then we define the covariant spinor derivatives,

$$\nabla^i_a = \frac{\partial}{\partial \theta^a_i} - i\theta^{bi}\partial_{ab} + \mathcal{A}^i_a \,, \hat{\nabla}^{aA} = \frac{\partial}{\partial \hat{\theta}_{Aa}} - i\hat{\theta}^A_b \partial^{ab} + \hat{\mathcal{A}}^{aA} \,.$$

- The constraints defining the $\mathcal{N} = (1,1)$ SYM theory are as follows (Howe, Sierra, Townsend, 1983; Howe, Stelle, 1984):

$$\{\nabla^{(i}_a, \nabla^{j)}_b\} = \{\hat{\nabla}^{a(A}, \hat{\nabla}^{bB)}\} = 0\,, \quad \{\nabla^i_a, \hat{\nabla}^{bA}\} = \delta^b_a \phi^{iA}$$

$$\Rightarrow \quad \nabla^{(i}_a \phi^{j)A} = \hat{\nabla}^{a(A} \phi^{B)i} = 0 \quad (\text{By Bianchis})\,.$$

- As the next step, we define the $\mathcal{N}=(1,1)$ HSS with the double set of harmonics (Bossard, Howe & Stelle, 2009):

$$Z=(x^{ab},\theta^a_i,u^\pm_k)\ \Rightarrow\ \hat{Z}=(x^{ab},\theta^a_i,\hat{\theta}^A_b,u^\pm_k,u^{\hat{\pm}}_A)$$

- Then we pass to the analytic basis and choose the "hatted" spinor derivatives short, $\nabla^{\hat{+}a}=D^{\hat{+}a}=\frac{\partial}{\partial\theta^{\hat{-}}_a}$. The set of constraints in the ordinary $\mathcal{N}=(1,1)$ superspace amounts to the following set in the $\mathcal{N}=(1,1)$ HSS

$$\{\nabla^+_a,\nabla^+_b\}=0\,,\quad \{D^{\hat{+}a},D^{\hat{+}b}\}=0\,,\quad \{\nabla^+_a,D^{\hat{+}b}\}=\delta^b_a\phi^{+\hat{+}}\,,$$
$$[\nabla^{\hat{+}\hat{+}},\nabla^+_a]=0\,,\ [\tilde{\nabla}^{++},\nabla^+_a]=0\,,\ [\nabla^{\hat{+}\hat{+}},D^{a\hat{+}}]=0\,,\ [\tilde{\nabla}^{++},D^{a\hat{+}}]=0\,,$$
$$[\tilde{\nabla}^{++},\nabla^{\hat{+}\hat{+}}]=0\,.$$

- Here

$$\nabla^+_a=D^+_a+\mathcal{A}^+_a(\hat{Z})\,,\ \tilde{\nabla}^{++}=D^{++}+\tilde{V}^{++}(\hat{\zeta})\,,\quad \nabla^{\hat{+}\hat{+}}=D^{\hat{+}\hat{+}}+V^{\hat{+}\hat{+}}(\hat{\zeta})\,,$$
$$\hat{\zeta}=(x^{ab}_{\rm an},\theta^{\pm a},\theta^{\hat{+}}_c,u^\pm_i,u^{\hat{\pm}}_A)\,.$$

Solving $\mathcal{N} = (1,1)$ SYM constraints

- The starting point of our analysis was to fix, using the $\Lambda(\hat{\zeta})$ gauge freedom, the WZ gauge for the second harmonic connection $V^{\hat{+}\hat{+}}(\hat{\zeta})$ as

  $$V^{\hat{+}\hat{+}} = i\theta^{\hat{+}}_a\theta^{\hat{+}}_b\hat{\mathcal{A}}^{ab} + \varepsilon^{abcd}\theta^{\hat{+}}_a\theta^{\hat{+}}_b\theta^{\hat{+}}_c\varphi^A_d u^{\hat{-}}_A + \varepsilon^{abcd}\theta^{\hat{+}}_a\theta^{\hat{+}}_b\theta^{\hat{+}}_c\theta^{\hat{+}}_d\mathcal{D}^{AB}u^{\hat{-}}_A u^{\hat{-}}_B$$

  where $\hat{\mathcal{A}}^{ab}, \varphi^A_d$ and $\mathcal{D}^{(AB)}$ are some $\mathcal{N} = (1,0)$ harmonic superfields, still arbitrary at this step.
- Then the above constraints are reduced to some sets of harmonic equations which we have explicitly solved. The crucial point was the requirement that the vector $6D$ connections in the sectors of hatted and unhatted variables are identical to each other.
- As the result, we have obtained that the first harmonic connection V^{++} coincides precisely with the previous $\mathcal{N} = (1,0)$ one, $V^{++} = V^{++}(\zeta)$, while the dependence of all other geometric $\mathcal{N} = (1,1)$ objects on the variables with "hat" is strictly fixed

$$V^{\hat{+}\hat{+}} = i\theta^{\hat{+}}_a\theta^{\hat{+}}_b\mathcal{A}^{ab} - \frac{1}{3}\epsilon^{abcd}\theta^{\hat{+}}_a\theta^{\hat{+}}_b\theta^{\hat{+}}_c D^+_d q^{-\hat{-}} + \frac{1}{8}\epsilon^{abcd}\theta^{\hat{+}}_a\theta^{\hat{+}}_b\theta^{\hat{+}}_c\theta^{\hat{+}}_d[q^{+\hat{-}}, q^{-\hat{-}}]$$

$$\phi^{+\hat{+}} = q^{+\hat{+}} - \theta^{\hat{+}}_a W^{+a} - i\theta^{\hat{+}}_a\theta^{\hat{+}}_b\nabla^{ab}q^{+\hat{-}} + \frac{1}{6}\varepsilon^{abcd}\theta^{\hat{+}}_a\theta^{\hat{+}}_b\theta^{\hat{+}}_c[D^+_d q^{-\hat{-}}, q^{+\hat{-}}]$$

$$+ \frac{1}{24}\varepsilon^{abcd}\theta^{\hat{+}}_a\theta^{\hat{+}}_b\theta^{\hat{+}}_c\theta^{\hat{+}}_d[q^{+\hat{-}}, [q^{+\hat{-}}, q^{-\hat{-}}]]\,.$$

- Here, $q^{+\hat{\pm}} = q^{+A}(\zeta)u_A^{\hat{\pm}}$, $q^{-\hat{\pm}} = q^{-A}(\zeta)u_A^{\hat{\pm}}$ and $W^{+a}, q^{\pm A}$ are just the $\mathcal{N} = (1,0)$ superfields explored previously. In the process of solving the constraints, there appeared the analyticity conditions for q^{+A}, as well as the full set of the superfield equations of motion

$$\nabla^{++}q^{+A} = 0\,, \quad F^{++} = \frac{1}{4}D_a^{+}W^{+a} = -\frac{1}{2}[q^{+A}, q_A^{+}]\,.$$

- Also, the structure of the spinor covariant derivatives was fully fixed

$$\nabla_a^{+} = D_a^{+} - \theta_a^{\hat{+}}q^{+\hat{-}} + \theta_a^{\hat{-}}\phi^{+\hat{+}}\,,$$
$$\nabla_a^{-} = D_a^{-} - D_a^{+}V^{--} - \theta_a^{\hat{+}}q^{-\hat{-}} + \theta_a^{\hat{-}}\phi^{-\hat{+}}\,, \quad \phi^{-\hat{+}} = \nabla^{--}\phi^{+\hat{+}}\,.$$

- The basic advantage of using the constrained $\mathcal{N} = (1,1)$ strengths $\phi^{\pm\hat{+}}$ for constructing various invariants is their extremely simple transformation rules under the hidden $\mathcal{N} = (0,1)$ supersymmetry

$$\delta\phi^{\pm\hat{+}} = -\epsilon_a^{\hat{+}}\frac{\partial}{\partial\theta_a^{\hat{+}}}\phi^{\pm\hat{+}} - 2i\epsilon_a^{\hat{-}}\theta_b^{\hat{+}}\partial^{ab}\phi^{\pm\hat{+}} - [\Lambda^{(comp)}, \phi^{\pm\hat{+}}]\,,$$

where $\Lambda^{(comp)}$ is some common composite gauge parameter which does not contribute under Tr.

Invariants in $\mathcal{N} = (1,1)$ superspace

- The previous single-trace $d = 8$ invariant Lagrangian admits a simple rewriting in $\mathcal{N} = (1,1)$ superspace

$$S_{(1,1)} = \int du d\zeta^{(-4)} \mathcal{L}^{+4}_{(1,1)}\,, \;\; \mathcal{L}^{+4}_{(1,1)} = -\mathrm{Tr}\,\frac{1}{4}\int d\hat{\zeta}^{(-4)} d\hat{u}\,(\phi^{+\hat{+}})^4, \;\; d\hat{\zeta}^{(-4)} \sim (D^{\hat{-}})^4$$

$$\delta\mathcal{L}^{+4}_{(1,1)} = -2i\partial^{ab}\mathrm{Tr}\int d\hat{\zeta}^{(-4)} d\hat{u}\left[\epsilon^{\hat{-}}_{a}\theta^{\hat{+}}_{b}\frac{1}{4}(\phi^{+\hat{+}})^4\right].$$

- The double-trace $d = 8$ invariant is given by

$$\hat{\mathcal{L}}^{+4}_{(1,1)} = -\frac{1}{4}\int d\hat{\zeta}^{(-4)} d\hat{u}\,\mathrm{Tr}\,(\phi^{+\hat{+}})^2\,\mathrm{Tr}\,(\phi^{+\hat{+}})^2.$$

- Now it is easy to construct the single- and double-trace $d = 10$ invariants possibly responsible for the 3-loop counterterms

$$S^{(10)}_1 = \mathrm{Tr}\int dZ d\hat{\zeta}^{(-4)} d\hat{u}\,(\phi^{+\hat{+}})^2(\phi^{-\hat{+}})^2, \quad \phi^{-\hat{+}} = \nabla^{--}\phi^{+\hat{+}}\,,$$

$$S^{(10)}_2 = -\int dZ d\hat{\zeta}^{(-4)} d\hat{u}\,\mathrm{Tr}\Big(\phi^{+\hat{+}}\phi^{-\hat{+}}\Big)\,\mathrm{Tr}\Big(\phi^{+\hat{+}}\phi^{-\hat{+}}\Big)\,.$$

- These are $\mathcal{N} = (1,1)$ extensions of the pure $\mathcal{N} = (1,0)$ SYM invariants $\sim \varepsilon_{abcd}\mathrm{Tr}\,(W^{+a}W^{-b}W^{+c}W^{-d}),\ \sim \varepsilon_{abcd}\,\mathrm{Tr}\,(W^{+a}W^{-b})\,\mathrm{Tr}\,(W^{+c}W^{-d})$.

- It is notable that the single-trace $d = 10$ invariant admits a representation as an integral over the *full* $\mathcal{N} = (1, 1)$ superspace

$$S_1^{(10)} \sim \mathrm{Tr} \int dZd\hat{Z}\, \phi^{+\hat{+}}\phi^{-\hat{-}}\,, \quad \phi^{-\hat{-}} = \nabla^{\hat{-}\hat{-}}\phi^{-\hat{+}}\,.$$

- On the other hand, the double-trace $d = 10$ invariant *cannot* be written as the full integral and so looks as being *UV* protected.

- This could explain why in the perturbative calculations of the amplitudes in the $\mathcal{N} = (1, 1)$ SYM single-trace 3-loop divergence is seen, while no double-trace structures at the same order were observed (Berkovits *et al* 2009; Bjornsson & Green, 2010; Bjornsson, 2011).

- However, this does not seem to be like the standard non-renormalization theorems because the quantum calculation of $\mathcal{N} = (1, 0)$ supergraphs should give some invariants in the off-shell $\mathcal{N} = (1, 0)$ superspace, not in the on-shell $\mathcal{N} = (1, 1)$ superspace. So the above property seems not enough to explain the absence of the double-trace divergences and some additional piece of reasoning is needed.

- Now there exist new methods in the $6D$ $\mathcal{N} = (1, 1)$ SYM perturbative calculations based on the notion of the so called on-shell harmonic momentum superspace (Dennen *et al*, 2010). It also involves two sets of harmonic coordinates. Perhaps it is closely related to the x-space harmonic $\mathcal{N} = (1, 1)$ superspace approach and would help to prove that all divergent quantum corrections to $\mathcal{N} = (1, 1)$ SYM action arise just as integrals over the whole $\mathcal{N} = (1, 1)$ harmonic superspace.

Summary and outlook

- We applied the off-shell $\mathcal{N} = (1,0)$ and on-shell harmonic $\mathcal{N} = (1,1)$ superspaces for constructing higher-dimensional invariants in the $\mathcal{N} = (1,0)$ SYM and $\mathcal{N} = (1,1)$ SYM theories.
- The $\mathcal{N} = (1,1)$ SYM constraints were solved in terms of harmonic $\mathcal{N} = (1,0)$ superfields. This allowed us to explicitly construct the full set of the superfield dimensions $d = 8$ and $d = 10$ invariants possessing $\mathcal{N} = (1,1)$ on-shell supersymmetry.
- All possible $d = 6$ $\mathcal{N} = (1,1)$ invariants were shown to be on-shell vanishing, proving the UV finiteness of $\mathcal{N} = (1,1)$ SYM at one loop.
- The off-shell $d = 8$ invariants are absent. The on-shell ones are integrals over the analytic $\mathcal{N} = (1,0)$ subspace. Assuming that the $\mathcal{N} = (1,0)$ supergraphs yield integrals over the full $\mathcal{N} = (1,0)$ harmonic superspace, this means the absence of two-loop counterterms.
- Two $d = 10$ invariants were explicitly constructed as integrals over the whole $\mathcal{N} = (1,0)$ harmonic superspace. The single-trace invariant can be rewritten as an integral over the $\mathcal{N} = (1,1)$ superspace, while the double-trace one cannot. This property combined with an additional reasoning could explain why the double-trace invariant is UV protected.

- *Some further lines of development*:

(a) To construct the next $d \geq 12$ invariants in the $\mathcal{N} = (1, 1)$ SYM theory with the help of the on-shell $\mathcal{N} = (1, 1)$ harmonic superspace techniques.

(b) To apply the same method for constructing the Born-Infeld action with the manifest off-shell $\mathcal{N} = (1, 0)$ and hidden on-shell $\mathcal{N} = (0, 1)$ supersymmetries. To check the hypothesis that such an action should coincide with the full quantum effective action of the $\mathcal{N} = (1, 1)$ SYM theory.

(c) To develop an analogous on-shell harmonic $\mathcal{N} = 4, 4D$ superspace approach to the $\mathcal{N} = 4, 4D$ SYM theory in the $\mathcal{N} = 2$ superfield formulation and apply it to the problem of constructing the relevant effective action.

(d) Applications in supergravity?

THANK YOU!